INVISCID FLOW
Week 14

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Propagation of Surface Waves

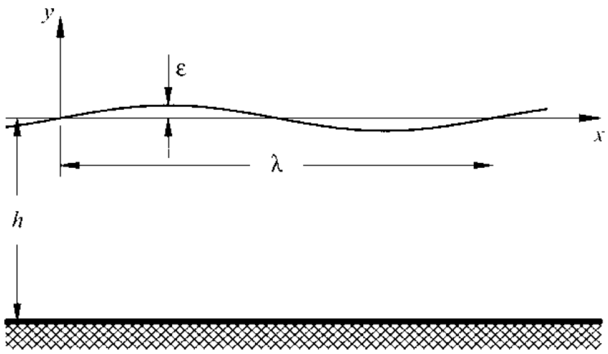
- A small-amplitude plane wave is traveling along the liquid surface with velocity *c*.
  - Sinusoidal wave $\eta(x,t) = \varepsilon \sin \frac{2\pi}{\lambda}(x-ct)$
  - Given ε, λ, and *h*, what will be the propagation speed *c*?
  - For the time being, surface-tension effect is ignored → constant *P*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial \phi}{\partial y}(x,0,t) = -\varepsilon \frac{2\pi c}{\lambda} \cos \frac{2\pi}{\lambda}(x-ct)$$

$$\frac{\partial^2 \phi}{\partial t^2}(x,0,t) + g\frac{\partial \phi}{\partial y}(x,0,t) = 0$$

$$\frac{\partial \phi}{\partial y}(x,-h,t) = 0$$



Propagation of Surface Waves

– Appropriate form of solution to the Laplace equation is

$$\phi(x,y,t)=\cos\frac{2\pi}{\lambda}(x-ct)\left(C_1\sinh\frac{2\pi y}{\lambda}+C_2\cosh\frac{2\pi y}{\lambda}\right)$$

– Applying the boundary condition @ $y = -h$;

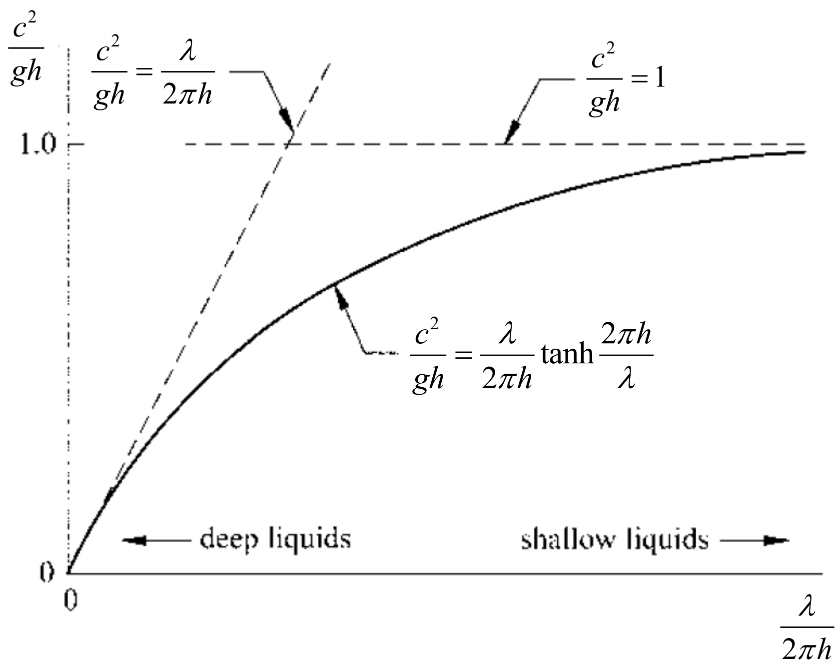
$$C_1=C_2\tanh\frac{2\pi h}{\lambda}$$

– Applying the dynamic boundary condition;

$$\frac{c^2}{gh}=\frac{\lambda}{2\pi h}\tanh\frac{2\pi h}{\lambda}$$

- for deep liquids ($h >> \lambda$) $\quad \dfrac{c^2}{gh}=\dfrac{\lambda}{2\pi h} \quad (\varepsilon \ll \lambda \ll h)$
- for shallow liquids ($h << \lambda$) $\quad \dfrac{c^2}{gh}=1 \quad (\varepsilon \ll h \ll \lambda)$

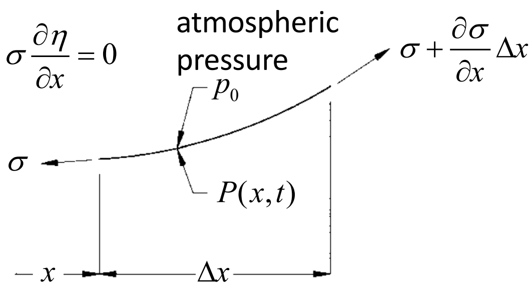
Propagation of Surface Waves



Effect of Surface Tension

- Surface-tensions effects
  - pressure along the liquid surface will be different from the pressure outside the liquid unless the surface is flat.
  - vertical equilibrium of the element;

$$(P-p_0)\Delta x+\left(\sigma+\frac{\partial\sigma}{\partial x}\Delta x\right)\left(\frac{\partial\eta}{\partial x}+\frac{\partial^2\eta}{\partial x^2}\Delta x\right)-\sigma\frac{\partial\eta}{\partial x}=0$$



$$(P-p_0)+\sigma\frac{\partial^2\eta}{\partial x^2}+\frac{\partial\sigma}{\partial x}\frac{\partial\eta}{\partial x}=0$$

$$P(x,t)=p_0-\sigma\frac{\partial^2\eta}{\partial x^2}$$

$$\frac{\partial P}{\partial t}=-\sigma\frac{\partial^2}{\partial x^2}\left(\frac{\partial\eta}{\partial t}\right)=-\sigma\frac{\partial^3\phi}{\partial x^2\partial y}(x,0,t)$$

$$\frac{\partial^2\phi}{\partial t^2}(x,0,t)-\frac{\sigma}{\rho}\frac{\partial^3\phi}{\partial x^2\partial y}(x,0,t)+g\frac{\partial\phi}{\partial y}(x,0,t)=0$$

modified dynamic condition

Effect of Surface Tension

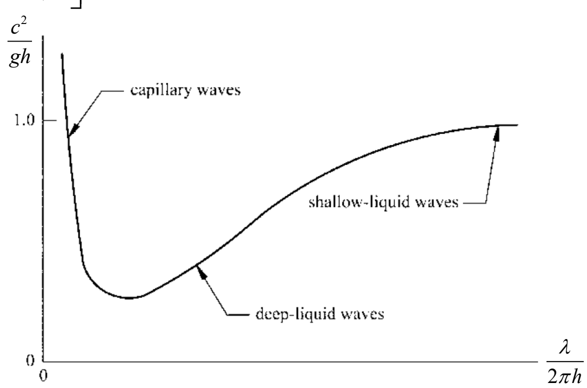
- Re-evaluate the propagation speed (*c*) under the effect of surface tension

$$\phi(x,y,t)=C_2\cos\frac{2\pi}{\lambda}(x-ct)\left(\tanh\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}+\cosh\frac{2\pi y}{\lambda}\right)$$

$$\frac{c^2}{gh}=\frac{\lambda}{2\pi h}\left[1+\frac{\sigma}{\rho g}\left(\frac{2\pi}{\lambda}\right)^2\right]\tanh\frac{2\pi h}{\lambda}$$

  - for deep liquids $\frac{c^2}{gh}=\frac{\lambda}{2\pi h}\left[1+\frac{\sigma}{\rho g}\left(\frac{2\pi}{\lambda}\right)^2\right]$
    - If $\frac{\sigma}{\rho g}\left(\frac{2\pi}{\lambda}\right)^2\gg 1,\ \frac{c^2}{gh}=\frac{2\pi\sigma}{\rho g\lambda h}$

Capillary waves



Complex Potential for Traveling Waves

- Small-amplitude surface wave in a fluid of arbitrary depth. For the sinusoidal wave form,

$$\eta(x,t) = \varepsilon \sin\frac{2\pi}{\lambda}(x-ct)$$

- the velocity potential is

$$\phi(x,y,t) = C_2 \cos\frac{2\pi}{\lambda}(x-ct)\left(\tanh\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda} + \cosh\frac{2\pi y}{\lambda}\right)$$

- C_2 is determined by imposing the kinematic boundary condition.

$$\frac{\partial \eta}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x,0,t)$$

$$C_2 \frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(x-ct)\tanh\frac{2\pi h}{\lambda} = -\varepsilon\frac{2\pi c}{\lambda}\cos\frac{2\pi}{\lambda}(x-ct)$$

$$C_2 = -\frac{c\varepsilon}{\tanh(2\pi h/\lambda)}$$

$$\therefore \phi(x,y,t) = -c\varepsilon\cos\frac{2\pi}{\lambda}(x-ct)\left(\sinh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\cosh\frac{2\pi y}{\lambda}\right)$$

Complex Potential for Traveling Waves

- Stream function

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = \frac{2\pi c}{\lambda}\varepsilon\sin\frac{2\pi}{\lambda}(x-ct)\left(\sinh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\cosh\frac{2\pi y}{\lambda}\right)$$

$$\psi(x,y,t) = c\varepsilon\sin\frac{2\pi}{\lambda}(x-ct)\left(\cosh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}\right) + F(x)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial \psi}{\partial x} = \frac{2\pi c}{\lambda}\varepsilon\cos\frac{2\pi}{\lambda}(x-ct)\left(\cosh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}\right)$$

$$\psi(x,y,t) = c\varepsilon\sin\frac{2\pi}{\lambda}(x-ct)\left(\cosh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}\right) + G(y)$$

$$\therefore \psi(x,y,t) = c\varepsilon\sin\frac{2\pi}{\lambda}(x-ct)\left(\cosh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}\right)$$

Complex Potential for Traveling Waves

- Complex potential

$$F(z,t) = \phi(x,y,t) + i\psi(x,y,t)$$

$$= -c\varepsilon \cos\frac{2\pi}{\lambda}(x-ct)\left(\sinh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\cosh\frac{2\pi y}{\lambda}\right)$$

$$+ic\varepsilon \sin\frac{2\pi}{\lambda}(x-ct)\left(\cosh\frac{2\pi y}{\lambda} + \coth\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda}\right)$$

$$= -\frac{c\varepsilon}{\sinh(2\pi h/\lambda)}\left[\cosh\frac{2\pi h}{\lambda}\cos\frac{2\pi}{\lambda}(x-ct) - i\sinh\frac{2\pi h}{\lambda}\sin\frac{2\pi}{\lambda}(x-ct)\right]$$

$$= -\frac{c\varepsilon}{\sinh(2\pi h/\lambda)}\left[\cos\left(i\frac{2\pi h}{\lambda}\right)\cos\frac{2\pi}{\lambda}(z-ct) - \sin\left(i\frac{2\pi h}{\lambda}\right)\sin\frac{2\pi}{\lambda}(z-ct)\right]$$

$$= -\frac{c\varepsilon}{\sinh(2\pi h/\lambda)}\cos\frac{2\pi}{\lambda}(z-ct+ih)$$

$x - ct + iy = z - ct$

Standing Waves

- Waves that remain stationary; the surface moves vertically only
  - Superimpose two identical traveling waves which are moving in opposite directions.

$$\eta_1(x,t) = \frac{1}{2}\varepsilon\sin\frac{2\pi}{\lambda}(x-ct)$$

$$\eta_2(x,t) = \frac{1}{2}\varepsilon\sin\frac{2\pi}{\lambda}(x+ct)$$

$$\eta(x,t) = \frac{1}{2}\varepsilon\left[\sin\frac{2\pi}{\lambda}(x-ct) + \sin\frac{2\pi}{\lambda}(x+ct)\right] = \frac{1}{2}\varepsilon\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi ct}{\lambda}$$

  - Complex potential

$$F(z,t) = \frac{c\varepsilon/2}{\sinh(2\pi h/\lambda)}\left[-\cos\frac{2\pi}{\lambda}(z-ct+ih) + \cos\frac{2\pi}{\lambda}(z+ct+ih)\right]$$

$$= -\frac{c\varepsilon}{\sinh(2\pi h/\lambda)}\sin\frac{2\pi}{\lambda}(z+ih)\sin\frac{2\pi ct}{\lambda}$$

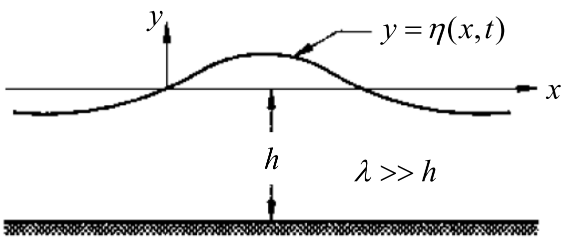
Standing sinusoidal wave of wavelength λ which is oscillating in time with frequency $2\pi c/\lambda$

Shallow-liquid Waves of Arbitrary Form

- Waves of arbitrary form will disperse if the liquid is deep.
  - Different propagation speeds of its Fourier components
  - Arbitrarily shaped wave will decompose unless the liquid depth (*h*) is small compared with the shortest wavelength (λ) of the various Fourier components
- We will verify that the shallow-liquid waves of small amplitude will not decompose.

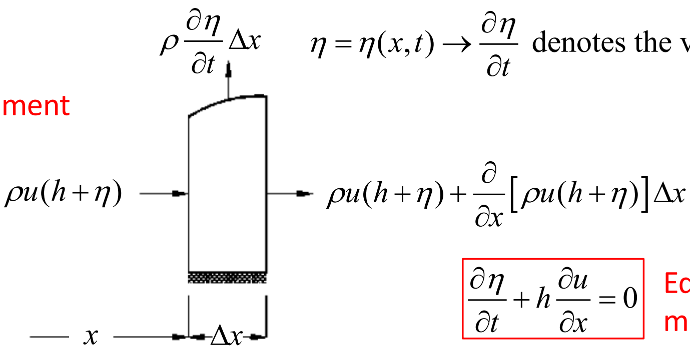
Shallow-liquid Waves of Arbitrary Form

- A liquid layer in which a surface wave of arbitrary form exists



  - One-dimensional approximation

mass-flow-rate balance for an element



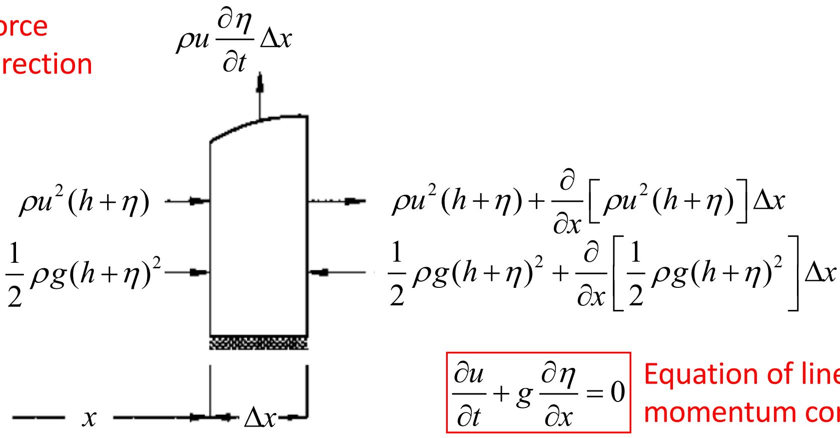
$\eta=\eta(x,t)\rightarrow\frac{\partial \eta}{\partial t}$ denotes the vertical velocity

$$\frac{\partial \eta}{\partial t}+h\frac{\partial u}{\partial x}=0$$

Equation of linearized mass conservation

Shallow-liquid Waves of Arbitrary Form

momentum and force balance in the x direction



$$\frac{\partial u}{\partial t}+g\frac{\partial \eta}{\partial x}=0$$

Equation of linearized momentum conservation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-gh\dfrac{\partial^2 u}{\partial x^2}=0 \\ \dfrac{\partial^2 \eta}{\partial t^2}-gh\dfrac{\partial^2 \eta}{\partial x^2}=0 \end{cases}$$

one-dimensional wave equation

$$u(x,t)=f_1(x-\sqrt{gh}t)+g_1(x+\sqrt{gh}t)$$

$$\eta(x,t)=f_2(x-\sqrt{gh}t)+g_2(x+\sqrt{gh}t),\ f_1, f_2, g_1, g_2\text{:any differentiable functions}$$