INVISCID FLOW
Week 15

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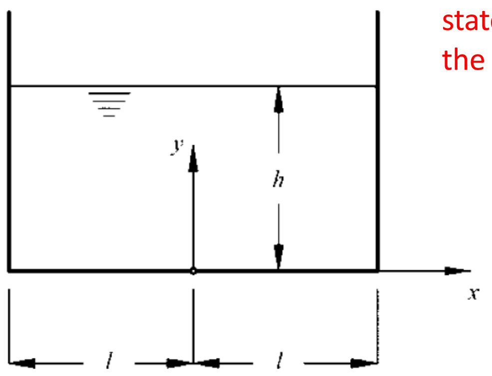
Waves in Rectangular Vessels

- One may ask whether standing waves may exist on the surface of a liquid that is contained in a vessel of finite extent.
- Rectangular vessels will be considered, and it will be shown that only standing waves whose wavelengths coincide with a discrete spectrum of values may exist on such liquid surfaces.



What type of steady-state or pseudo-steady-state waves, if any, may exist on the surface of the liquid?

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(a)}$$

$$\frac{\partial^2 \phi}{\partial t^2}(x,h,t) + g\frac{\partial \phi}{\partial y}(x,h,t) = 0 \qquad \text{(b)}$$

$$\frac{\partial \phi}{\partial y}(x,0,t) = 0 \qquad \text{(c)}$$

$$\frac{\partial \phi}{\partial x}(\pm l,y,t) = 0 \qquad \text{(d)}$$

Waves in Rectangular Vessels

- Since a steady-state wave solution is being sought, the velocity potential should have a trigonometric time dependence.

$$\phi(x,y,t)=\left(A_1\sin\frac{2\pi x}{\lambda}+A_2\cos\frac{2\pi x}{\lambda}\right)\left(B_1\sinh\frac{2\pi y}{\lambda}+B_2\cosh\frac{2\pi y}{\lambda}\right)\sin\frac{2\pi ct}{\lambda}$$

  - The *x* dependence: trigonometric due to homogeneous boundary conditions at $x = \pm l$.
  - To satisfy Laplace's equation, the *y* dependence must be exponential or hyperbolic.

– By applying (c); $B_1 = 0$

$$\phi(x,y,t)=\left(D_1\sin\frac{2\pi x}{\lambda}+D_2\cos\frac{2\pi x}{\lambda}\right)\cosh\frac{2\pi y}{\lambda}\sin\frac{2\pi ct}{\lambda}$$

– Pressure condition (b);

$$\frac{c^2}{gh}=\frac{\lambda}{2\pi h}\tanh\frac{2\pi h}{\lambda}$$

Waves in Rectangular Vessels

– Final boundary condition at the side walls (d);

$$D_1\cos\frac{2\pi l}{\lambda}=\pm D_2\sin\frac{2\pi l}{\lambda}$$

- If $D_1 = D_2 = 0$ → Trivial solution
- For non-trivial solution, suppose $D_1 \neq 0$ and $D_2 = 0$

$$\cos\frac{2\pi l}{\lambda_n}=0\rightarrow\lambda_n=\frac{4l}{2n+1}$$

$$\phi_n(x,y,t)=D_{1n}\sin\frac{(2n+1)\pi x}{2l}\cosh\frac{(2n+1)\pi y}{2l}\sin\frac{(2n+1)\pi c_n t}{2l}$$

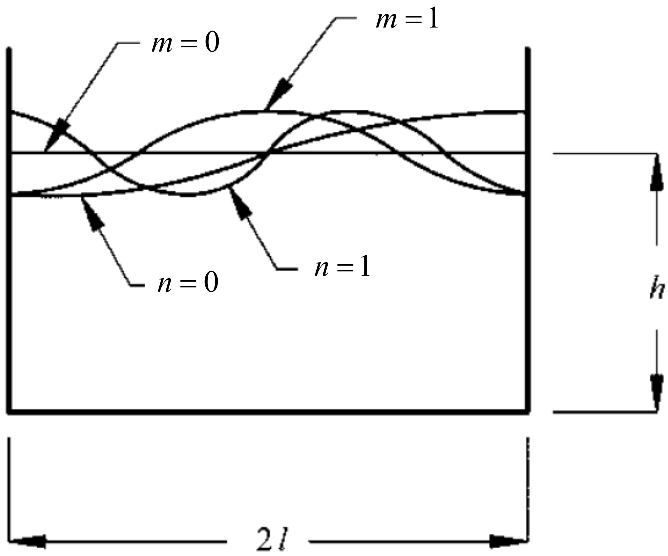
- $D_2 \neq 0$ and $D_1 = 0$

$$\sin\frac{2\pi l}{\lambda_m}=0\rightarrow\lambda_m=\frac{2l}{m}$$

$$\phi_m(x,y,t)=D_{2m}\cos\frac{m\pi x}{l}\cosh\frac{m\pi y}{l}\sin\frac{m\pi c_m t}{l}$$

Waves in Rectangular Vessels

– First two modes of ϕ_n and ϕ_m



Waves in Rectangular Vessels

– General form of solutions

$$\phi(x,y,t)=\sum_{n=0}^{\infty}\phi_n+\sum_{m=0}^{\infty}\phi_m$$

$$=\sum_{n=0}^{\infty}D_{1n}\sin\frac{(2n+1)\pi x}{2l}\cosh\frac{(2n+1)\pi y}{2l}\sin\frac{(2n+1)\pi c_n t}{2l}+\sum_{m=0}^{\infty}D_{2m}\cos\frac{m\pi x}{l}\cosh\frac{m\pi y}{l}\sin\frac{m\pi c_m t}{l}$$

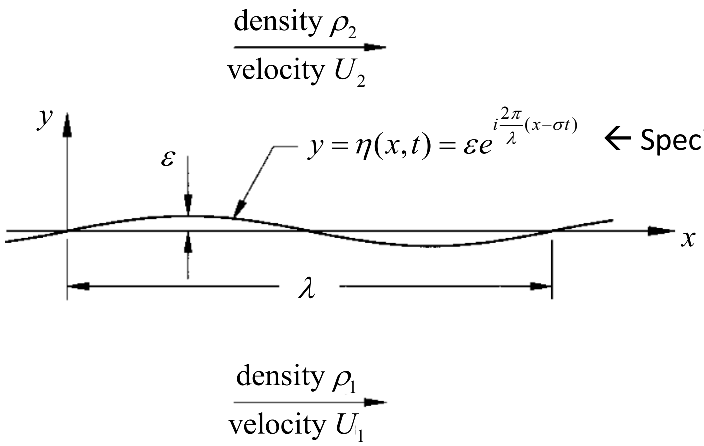
$$\frac{c_n^2}{gh}=\frac{2l}{(2n+1)\pi h}\tanh\frac{(2n+1)\pi h}{2l}$$

$$\frac{c_m^2}{gh}=\frac{l}{m\pi h}\tanh\frac{m\pi h}{l}$$

D_{1n} and D_{1m} may be determined if initial shape and velocity of the free surface are specified.

Propagation of Waves at an Interface

- Behavior of propagating waves at the separation of two dissimilar fluids



← Specified wave profile

  – Using a perturbation analysis, one may derive new boundary conditions for this problem.

$$u_i = U_i \hat{e}_x + \nabla \phi_i \qquad (i = 1, 2)$$

velocity potential for the perturbation to the uniform flow caused by the waves at the interface

Propagation of Waves at an Interface

- Governing equations and boundary conditions

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0, \ \left|\nabla \phi_1\right| = \text{ finite}$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0, \ \left|\nabla \phi_2\right| = \text{ finite}$$

$$\frac{\partial \phi_i}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t) + U_i \frac{\partial \eta}{\partial x}(x,t)$$

$$\rho_i \frac{\partial \phi_i}{\partial t}(x,0,t) + \rho_i U_i \frac{\partial \phi_i}{\partial x}(x,0,t) + \rho_i g \eta(x,t) = \text{ constant}$$

- Solutions $\phi_1(x,y,t) = -i\varepsilon\left(\sigma - U_1\right) e^{(2\pi/\lambda)y} e^{i(2\pi/\lambda)(x-\sigma t)}$

$$\phi_2(x,y,t) = i\varepsilon\left(\sigma - U_2\right) e^{-(2\pi/\lambda)y} e^{i(2\pi/\lambda)(x-\sigma t)}$$

  – Still, these should satisfy the pressure condition at the interface.

Propagation of Waves at an Interface

– Pressure condition

$$-\rho_1 \frac{2\pi}{\lambda}(\sigma - U_1)^2 + \rho_1 g = \rho_2 \frac{2\pi}{\lambda}(\sigma - U_2)^2 + \rho_2 g$$

$$\sigma = \frac{\rho_2 U_2 + \rho_1 U_1}{\rho_2 + \rho_1} \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_2 + \rho_1}\right)\frac{g\lambda}{2\pi} - \frac{\rho_1 \rho_2}{(\rho_2 + \rho_1)^2}(U_2 - U_1)^2}$$

- $U_1 = U_2 = 0$, $\rho_2 = 0$ (stationary gas over stationary liquid) $\sigma = \pm\sqrt{\frac{g\lambda}{2\pi}}$
  – for real σ, the waves at the interface will propagate, so that the surface of separation will remain intact. Stable.

- $\rho_2 = 0$ (gas blowing over a liquid surface) $\sigma = U_1 \pm \sqrt{\frac{g\lambda}{2\pi}}$ (Stable)

- $\rho_1 = \rho_2$ $\sigma = \frac{U_1 + U_2}{2} \pm i \frac{U_2 - U_1}{2}$

Interfacial wave growing exponentially with time. Unstable. Helmholtz (Rayleigh) instability

Propagation of Waves at an Interface

- $U_1 = U_2 = 0$, $\rho_1 \neq \rho_2$

$$\sigma = \pm\sqrt{\frac{g\lambda}{2\pi}\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)} \rightarrow \begin{cases} \rho_1 > \rho_2 : & \sigma \text{ is real. Stable} \\ \rho_1 < \rho_2 : & \sigma \text{ is imaginary. Unstable. Taylor instability} \end{cases}$$