Finite Element Analysis (FEA) : 유한요소해석

Why do we need the "Finite Element Analysis"?

We are already accustomed to the matrix method for the numerical analysis of frames with line elements: columns and beams.

$\underline{P} = \underline{K} \cdot \underline{U}$

P = nodal forces, $\underline{K}$ = stiffness matrix, $\underline{U}$ = nodal displacements

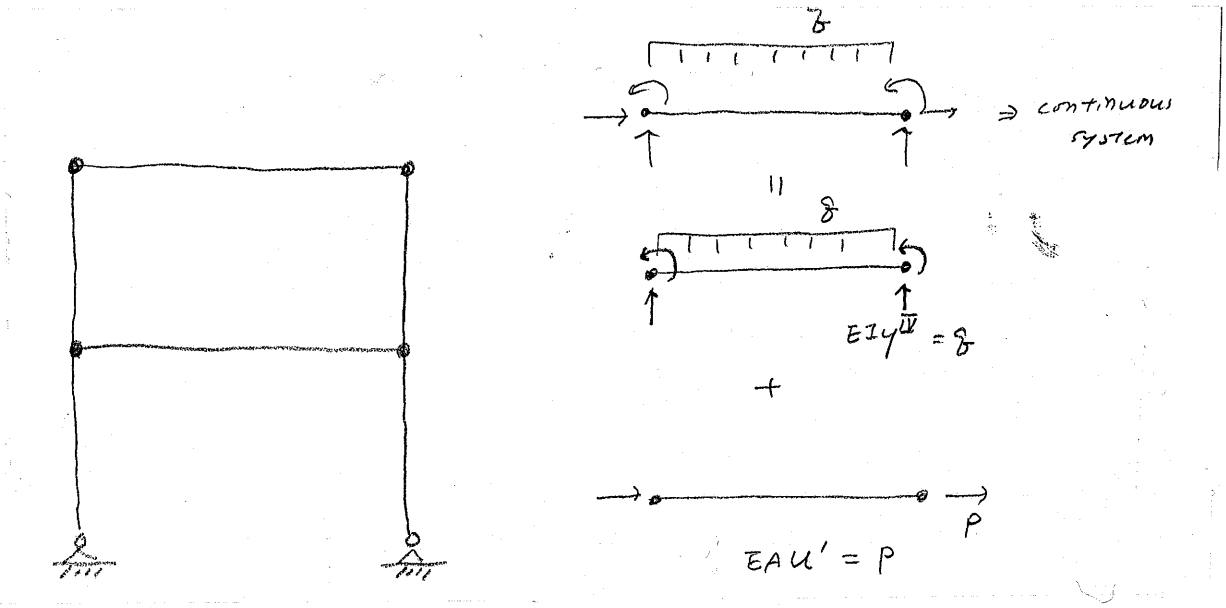
All structural analysis should satisfy three conditions:

1) force-equilibrium,

2) displacement-compatibility,

3) constitutive relationship between force and displacement (or stress and strain)

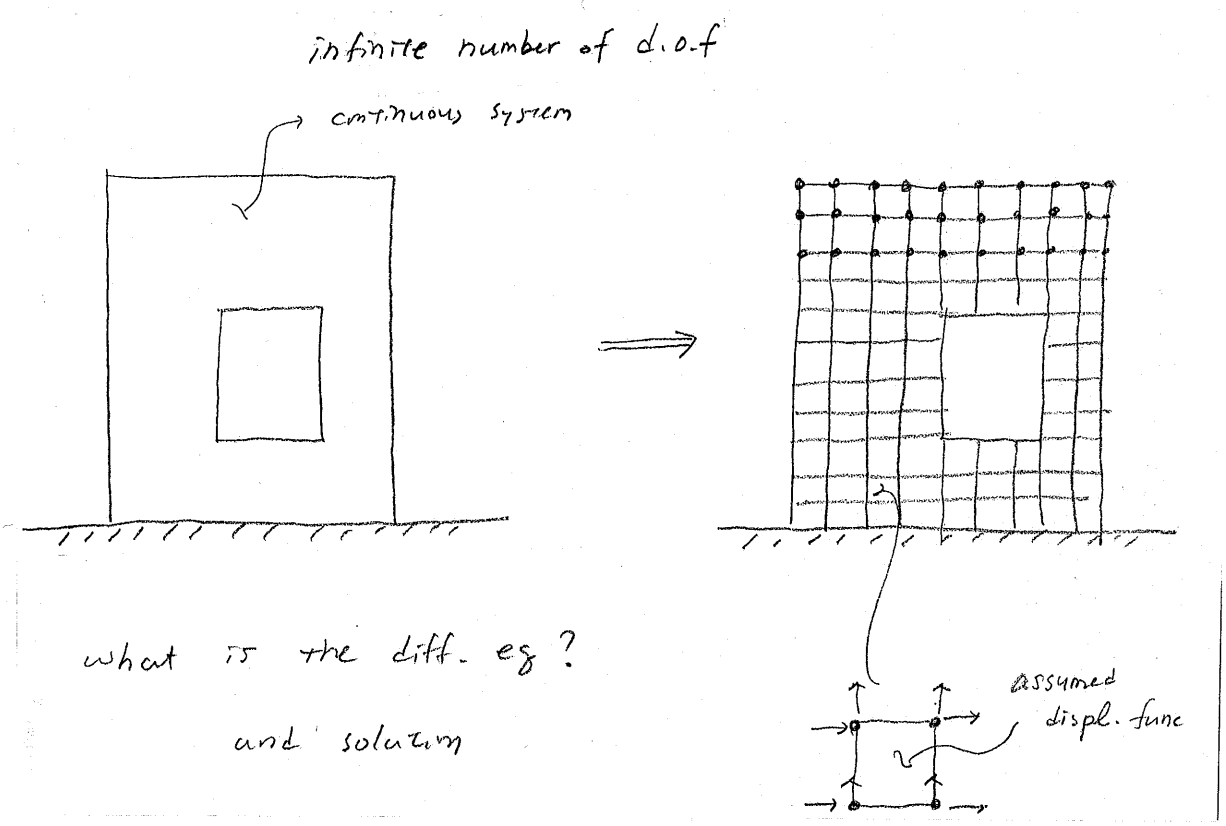
Frame analysis (with line elements)



In general, the problem is very simple: each element is a line element with limited number of degree of freedom (DOF) at the end nodes.

Thus, the differential equation can be easily derived considering the three conditions. The exact solution can be obtained.

Continuum structure (plate, box, shell, etc)



On the other hand, behavior of the multi-dimensional continuum structure is very complex and diverse according to the shape, geometry, loading, support conditions.

Thus, in general, it is impossible to standardize the forms of the equation and solution.

For this reason, approximate formulations and equations are required to obtain the solutions under various loading and geometry conditions, even though the solutions are not exact.

The characteristics of FEA :

1) A continuum structure is simplified with finite number of elements and nodes.

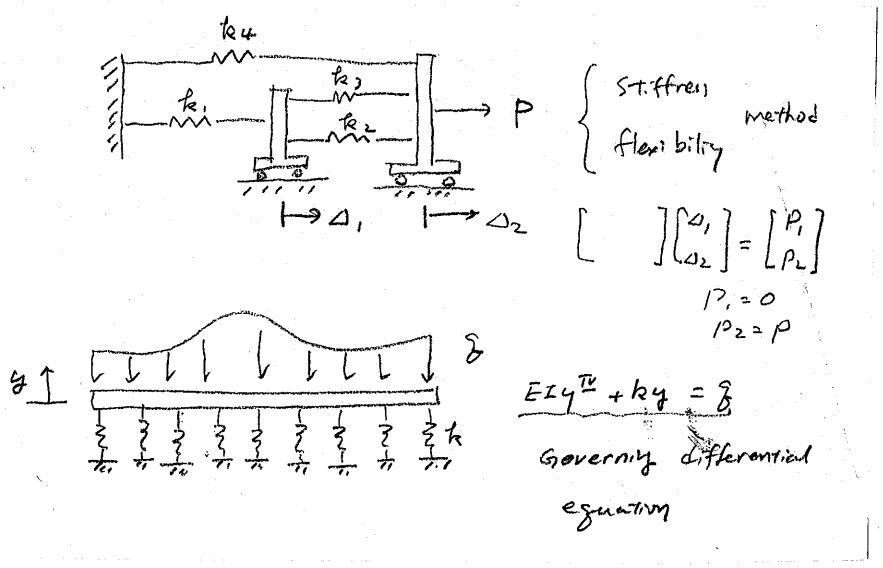
2) In each element, a simplified displacement function is assumed though it is not exact.

3) The accuracy of the solution can be enhanced using a large number of elements each of which occupies only a small region with small variations of displacement, strains, and stress (As the area of an element decreases, the displacement variation becomes simpler. Thus, a simplified displacement function can represent the actual displacement field).

Structural system

discrete system with limited number of dofs

continuous system with infinite number of dofs



Solution of continuous system

Classical approach

In case of beam

Differential governing equation : force equilibrium $\frac{dM}{dx} = V$ $\frac{dV}{dx} = g - ky$

Constitutive relationship $\frac{M}{EI} = y''$ $\Rightarrow$

Displacement compatibility $EI\,y^{IV} + k y = g$

Formulation (derivation of governing equation)

Direct method – direct derivation considering the three equations.

Variational method (energy method) – instead of three conditions, energy condition + two conditions (usually constitutive relationship and displ. Compatibility) are used.

Disadvantages

For complex problem, it is difficult to derive the governing equation.

Even if it can be derived, the solution is difficult to obtain.

Approximate solution methods

- weighted residuals : galerkin method
  - Collocation
  - Subdomains
  - Moments
  - Least squares
- Ritz method – energy method
- finite difference method
- finite element method
- boundary element method – integral method

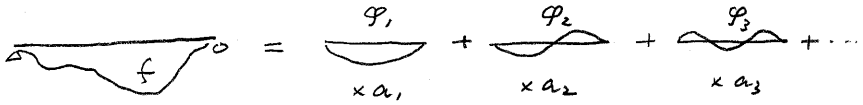
Weighted Residuals

General form of governing equations

$L_{2m}(f) = q$: differential equation

for example of beams, $EIy^{IV} = q$: $L_{2m} = \mathrm{EI}\dfrac{d^4}{dx^4}$, $f = y$

Assumed function $f = \sum a_j \varphi_j$



φ_j = selected functions

a_j = coefficients of functions = variables to be determined.

To determine the variables a_j, conditions are required, and the number of conditions should be *j*.

The condition should be the force-equilibrium (internal force = external force).

Basically, the force-equilibrium should be satisfied at all locations inside the structure to be analyzed. However, with the arbitrarily selected functions, the force-equilibrium cannot be satisfied. Instead, selected conditions are given to determine the variables. The number of the conditions should be the same as the number of the variables.

The types of weighted residual methods are classified according to the types of the conditions given to determine the variables.

Let's set $R = L_{2m}(f) - q$ where R = residuals.

- collocation method : $R(\mathrm{x}_i) = 0$, i = 1,2,----, n
- subdomains : $\int_{Di} R dD_i = 0$
- moments: $\int_{Di} x^{i-1} R dD_i = 0$
- least squares: $\int_D R \frac{\partial R}{\partial a_i} = 0$ $\int_D R^2 =$ minimum
- gelerkin : $\int_D R\varphi_i = 0$

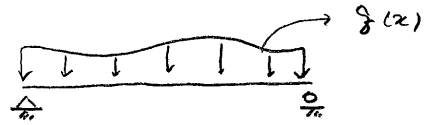
General Form $\int R \cdot w_i = 0 \qquad i = 1, \cdots, ?$

R : residuals

w_i : weighting functions

Collocation

$R(x_i) = 0$



$\varphi_0 = 0$: homogeneous B/C

$y(0) = 0 \qquad y(L) = 0$

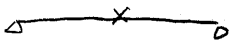
$y''(0) = 0 \qquad y''(L) = 0$

$R_j = L_{2m}(f) - g$

$f = \varphi_0^{\circ} + \Sigma a_j \varphi_j$

$\varphi_j(x) = \sin j\frac{\pi x}{L}$

$f = \Sigma a_j \sin j\frac{\pi x}{L} \qquad g(x) = g$ constant



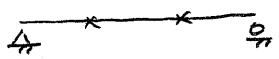
1 point $\left(x_1 = \frac{L}{2}\right)$

$y = a_1 \sin\frac{\pi x}{L}, \quad y^{IV} = a_1 \frac{\pi^4}{L^4} \sin\frac{\pi x}{L}$

$a_1 EI \frac{\pi^4}{L^4} \sin\frac{\pi}{2} = g$

$a_1 = \frac{gL^4}{\pi^4 EI}$

$y = \frac{gL^4}{\pi^4 EI} \sin\frac{\pi x}{L}$

2 points  $x_1 = \frac{L}{3}$ $x_2 = \frac{2L}{3}$

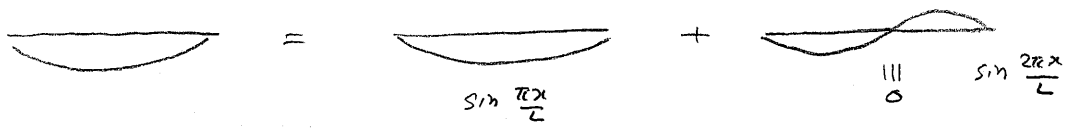
$$y = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L}$$

$$y^{IV} = a_1 \frac{\pi^4}{L^4}\sin\frac{\pi x}{L} + a_2 \frac{16\pi^4}{L^4}\sin\frac{2\pi x}{L}$$

$$\left(EI\frac{\pi^4}{L^4}\sin\frac{\pi}{3}\right)a_1 + \left(EI\frac{16\pi^4}{L^4}\sin\frac{2\pi}{3}\right)a_2 = q$$

$$\left(EI\frac{\pi^4}{L^4}\sin\frac{2\pi}{3}\right)a_1 + \left(EI\frac{16\pi^4}{L^4}\sin\frac{4\pi}{3}\right)a_2 = q$$

$$\begin{cases} \frac{\sqrt{3}}{2}a_1 + 16\frac{\sqrt{3}}{2}a_2 = \frac{qL^4}{\pi^4 EI} \\ \frac{\sqrt{3}}{2}a_1 - 16\frac{\sqrt{3}}{2}a_2 = \frac{qL^4}{\pi^4 EI} \end{cases} \Rightarrow \begin{array}{l} a_2 = 0 \\ a_1 = \dfrac{2qL^4}{\sqrt{3}\,\pi^4 EI} \end{array}$$



we can choose other combinations of the points

Generally,

$$\begin{cases} L_{2m}(f)_{x=x_1} = q \\ L_{2m}(f)_{x=x_2} = q \\ \vdots \\ L_{2m}(f)_{x=x_n} = q \end{cases} \Rightarrow \begin{cases} \sum a_j L_{2m}(\varphi_j)_{x=x_1} = q(x_1) \\ \sum a_j L_{2m}(\varphi_j)_{x=x_2} = q(x_2) \\ \vdots \\ \sum a_j L_{2m}(\varphi_j)_{x=x_n} = q(x_n) \end{cases}$$

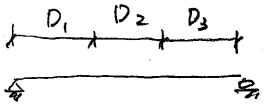
$$\underset{\sim}{L}\,\underset{\sim}{A} = \underset{\sim}{Q}$$

$$\underset{\sim}{L} = \begin{bmatrix} L_{2m}(\varphi_1)_{x_1} & L_{2m}(\varphi_2)_{x_1} & \cdots & L_{2m}(\varphi_n)_{x_1} \\ L_{2m}(\varphi_1)_{x_2} & L_{2m}(\varphi_2)_{x_2} & \cdots & L_{2m}(\varphi_n)_{x_2} \\ \vdots & & & \vdots \\ L_{2m}(\varphi_1)_{x_n} & L_{2m}(\varphi_2) & \cdots & L_{2m}(\varphi_n)_{x_n} \end{bmatrix} \quad \underset{\sim}{A} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$$

not symmetric

$$Q = \begin{Bmatrix} q(x_1) \\ \vdots \\ q(x_n) \end{Bmatrix}$$

Subdomains



$$\int_{D_1} R = 0 \qquad \sum a_j \left(\int_{D_1} L_{2m}(\varphi_j) \right) = \int_{D_1} f \, dx$$

$$\int_{D_2} R = 0 \qquad \sum a_j \left(\int_{D_2} L_{2m}(\varphi_j) \right) = \int_{D_2} f \, dx$$

$$\int_{D_3} R = 0 \qquad \sum a_j \left(\int_{D_3} L_{2m}(\varphi_j) \right) = \int_{D_3} f \, dx$$

$$\underset{\sim}{L} A = Q$$

$$\underset{\sim}{Q} = \begin{Bmatrix} \int_{D_1} f \, dx \\ \int_{D_2} f \, dx \\ \vdots \\ \int_{D_n} f \, dx \end{Bmatrix} \qquad \underset{\sim}{L} = \begin{bmatrix} \int_{D_1} L_{2m}(\varphi_1) & \int_{D_1} L_{2m}(\varphi_2) & \cdots & \int_{D_1} L_{2m}(\varphi_n) \\ \vdots & & & \vdots \\ \int_{D_n} L_{2m}(\varphi_1) & \int_{D_n} L_{2m}(\varphi_2) & \cdots & \int_{D_n} L_{2m}(\varphi_n) \end{bmatrix}$$

Least Squares

$$\int R^2 = \text{minimum} \qquad R = \sum a_j L_{2m}(\varphi_j) - f$$

$$\int R \frac{\partial R}{\partial a_1} = 0 \qquad \frac{\partial R}{\partial a_1} = L_{2m}(\varphi_1)$$

$$\int R \frac{\partial R}{\partial a_2} = 0 \qquad \frac{\partial R}{\partial a_2} = L_{2m}(\varphi_2)$$

$$\vdots \qquad\qquad \vdots$$

$$\int R \frac{\partial R}{\partial a_n} = 0 \qquad \frac{\partial R}{\partial a_n} = L_{2m}(\varphi_n)$$

$$\underset{\sim}{L} = \begin{bmatrix} \int L_{2m}(\varphi_1) L_{2m}(\varphi_1) \, dx & \cdots & \int L_{2m}(\varphi_1) L_{2m}(\varphi_n) \, dx \\ \vdots & & \vdots \\ \int L_{2m}(\varphi_n) L_{2m}(\varphi_1) \, dx & \cdots & \int L_{2m}(\varphi_n) L_{2m}(\varphi_n) \, dx \end{bmatrix} \qquad \underset{\sim}{Q} = \begin{bmatrix} \int f \, L_{2m}(\varphi_1) \, dx \\ \vdots \\ \int f \, L_{2m}(\varphi_n) \, dx \end{bmatrix}$$

Galerkin method

$$\int_0 R\,\varphi_i = 0 \qquad i = 1, \dots, n$$

$$\sum a_j \int L_{2m}(\varphi_j)\,\varphi_i = \int g\,\varphi_i$$

$$\underline{\underline{L}}\,\underline{A} = \underline{Q}$$

$$\underline{A} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} \qquad \underline{Q} = \begin{Bmatrix} \int g\,\varphi_1\,dx \\ \int g\,\varphi_2\,dx \\ \vdots \\ \int g\,\varphi_n\,dx \end{Bmatrix}$$

$$\underline{\underline{L}} = \begin{bmatrix} \int \varphi_1 L_{2m}(\varphi_1)\,dx & \cdots & \int \varphi_1 L_{2m}(\varphi_n)\,dx \\ \vdots & & \vdots \\ \int \varphi_n L_{2m}(\varphi_1)\,dx & \cdots & \int \varphi_n L_{2m}(\varphi_n)\,dx \end{bmatrix}$$

$$\left.\begin{array}{l} B_i(u) = B_i(v) = 0 \\ \int u\,L_{2m}(v) = \int v\,L_{2m}(u) \end{array}\right\} \Rightarrow \text{Symmetric}$$

$\Rightarrow$ Basis of Finite Element Method.

By using integration by parts, the matrix $\underline{\underline{L}}$ becomes the same as that produced by Rayleigh-Ritz method (Energy Method).

Weighted Residual Method

Advantages : Approximate solutions can be obtained

Disadvantages : 1) Differential Equations are required.

2) To solve accurately responses with large displ. or stress gradients, higher order functions may be needed.

3) no physical meaning on the variables (a_i's)

4) matrix is not symmetric in general

Rayleigh - Ritz ⇒ Principle of Stationary total potential energy

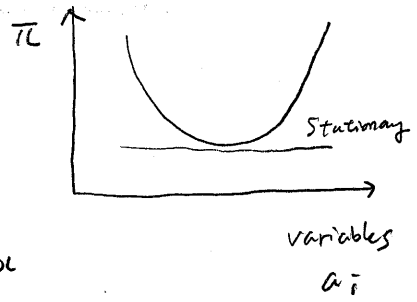
$\pi(f)$ = minimum (stationary) ⇒ $\delta\pi = 0$ ⇒ satisfy equilibrium (indirectly)

π = total potential Energy

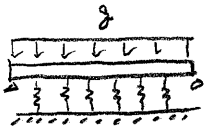
U = Strain Energy

V = change in potential energy

$\pi = U + V$



for a beam on elastic foundation (springs)



$$\pi = U + V = \frac{1}{2}\int EI y''^2 dx + \frac{1}{2}\int k y^2 dx - \int q y\, dx$$

$\delta\pi = 0$

$f = \sum a_j \varphi_j$ $\quad \delta\pi = 0 \Rightarrow \frac{\partial\pi}{\partial a_1} = 0,\ \frac{\partial\pi}{\partial a_2} = 0 \cdots \frac{\partial\pi}{\partial a_n} = 0$

$$y = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}$$

$$\pi = \frac{1}{2}\int EI \left(\frac{\pi^2}{L^2} a_1 \sin\frac{\pi x}{L} + \frac{9\pi^2}{L^2} a_2 \sin\frac{3\pi x}{L}\right)^2 dx$$

$$+ \frac{1}{2}\int k \left(a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}\right)^2 dx - \int q \left(a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}\right) dx$$

$$\frac{\partial\pi}{\partial a_1} = \int EI \left(\frac{\pi^2}{L^2} a_1 \sin\frac{\pi x}{L} + \frac{9\pi^2}{L^2} a_2 \sin\frac{3\pi x}{L}\right) \frac{\pi^2}{L^2} \sin\frac{\pi x}{L} dx$$

$$+ \int k \left(a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L}\right) \sin\frac{\pi x}{L} dx - \int q \sin\frac{\pi x}{L} dx = 0$$

$$\frac{\partial \pi}{\partial a_2} = \int EI \left(\frac{\pi^2}{L^2} a_1 \sin\frac{\pi x}{L} + \frac{4\pi^2}{L^2} a_2 \sin\frac{3\pi x}{L} \right) \frac{4\pi^2}{L^2} \sin\frac{3\pi x}{L} dx$$

$$+ \int k \left(a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{3\pi x}{L} \right) \sin\frac{3\pi x}{L} dx - \int q \sin\frac{3\pi x}{L} dx = 0$$

$$\underset{\sim}{\underset{\sim}{A}} = \underset{\sim}{Q}$$

Rayleigh - Ritz

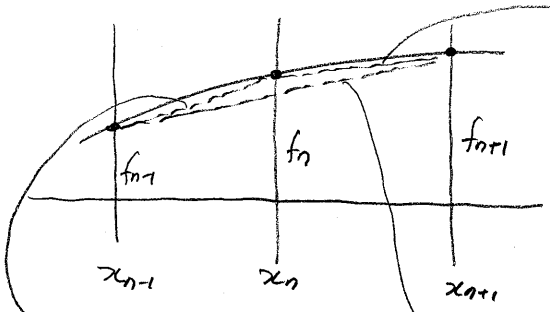
* the function is required to satisfy essential B/C. (displ. B/c).
  Natural B/c is automatically satisfied in Energy formulation.
  (force B/C) (and indirectly)
* Only if essential and natural B/c are satisfied, convergence is obtained.
* The result is the same as Galerkin's.

Advantage over weighted Residuals

Differential equations are not required. ⇒ Energy method
only displacement functions are used.

Finite Differences

$$EI\,y'' + ky = q.$$



Forward Difference

$$f'_n = \frac{f_{n+1} - f_n}{\Delta x} = \frac{\Delta f}{\Delta x}$$

Backward Difference

$$f'_n = \frac{f_n - f_{n-1}}{\Delta x}$$

central Difference

$$f'_n = \frac{f_{n+1} - f_{n-1}}{2\Delta x}$$

Lagrange Interpolation

$$f(x) = \frac{(x-x_n)(x-x_{n+1})}{(x_{n-1}-x_n)(x_{n-1}-x_{n+1})} y_{n-1} + \frac{(x-x_{n-1})(x-x_{n+1})}{(x_n-x_{n-1})(x_n-x_{n+1})} y_n + \frac{(x-x_{n-1})(x-x_n)}{(x_{n+1}-x_{n-1})(x_{n+1}-x_n)} y_{n+1}$$

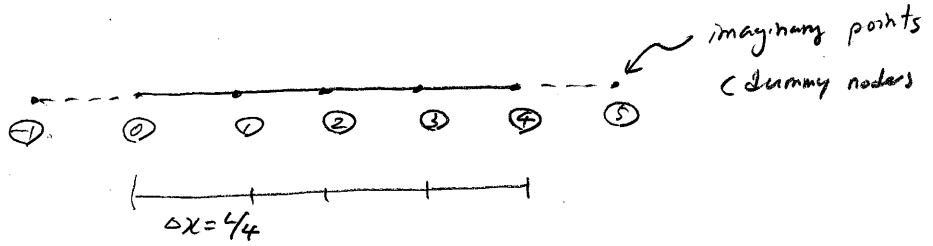
if $x_n - x_{n-1} = x_{n+1} - x_n = \Delta x$

$$f'(x_n) = \frac{1}{2\Delta x}(y_{n+1} - y_{n-1}) \Rightarrow \text{central difference}$$

$$f''(x_n) = \frac{1}{\Delta x^2}(y_{n-1} - 2y_n + y_{n+1})$$

$$f'''(x_n) = \frac{1}{2\Delta x^3}(-y_{n-2} + 2y_{n-1} - 2y_{n+1} + y_{n+2})$$

$$f^{IV}(x_n) = \frac{1}{\Delta x^4}(y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2})$$



at ① $\frac{EI}{\Delta x^4}\left[y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3\right] + ky_1 = f_1 = f(x_1)$

at ② $\frac{EI}{\Delta x^4}\left[y_0 - 4y_1 + 6y_2 - 4y_3 + y_4\right] + ky_2 = f_2 = f(x_2)$

at ③ $\frac{EI}{\Delta x^4}\left[y_1 - 4y_2 + 6y_3 - 4y_4 + y_5\right] + ky_3 = f_3 = f(x_3)$

7 unknowns, 3 eqns. $\Rightarrow$ 4 B/C are required

$y(0) = 0 \longrightarrow y_0 = 0$

$y''(0) = 0 \longrightarrow y_{-1} - 2y_0 + y_1 = 0 \longrightarrow y_{-1} = 2y_0 - y_1 = -y_1$

$y(L) = 0 \longrightarrow y_4 = 0$

$y''(L) = 0 \longrightarrow y_3 - 2y_4 + y_5 = 0 \longrightarrow y_5 = 2y_4 - y_3 = -y_3$

at ① $\frac{EI}{\Delta x^4}\left[+5y_1 - 4y_2 + y_3\right] + ky_1 = f_1$ — ①′

at ③ $\frac{EI}{\Delta x^4}\left[y_1 - 4y_2 + 5y_3\right] + ky_3 = f_3$ — ③′

$$\begin{matrix} ①' \text{ —} \\ ② \text{ —} \\ ③' \text{ —} \end{matrix} \quad \frac{EI}{\Delta x^4}\begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} k & & \\ & k & \\ & & k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

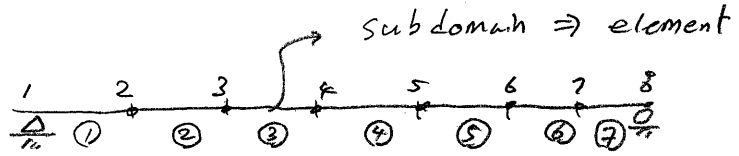
Disadvantages

1) Differential Equation is required.
2) According to the sparcity of the nodes and B/C, Adjustments of the modeling is required.

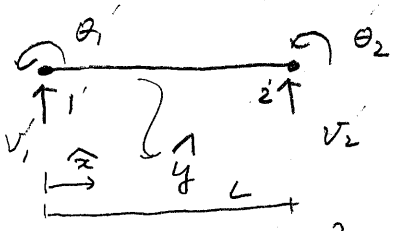
Advantages

1) Differential eqn $\xrightarrow{\text{transformed}}$ relationship between nodal disp's.
2) unknowns : nodal displacements

Finite Element Analysis



Element ①



$\hat{y} = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ for subdomain (simple displ. function)

$$\Rightarrow \begin{cases} \hat{y}(\hat{x}=0) = v_1 & \hat{y}'(0) = \theta_1 \\ \hat{y}(L) = v_2 & \hat{y}'(L) = \theta_2 \end{cases}$$ conditions

$$\hat{y} = f_1 v_1 + f_2 \theta_1 + f_3 v_2 + f_4 \theta_2$$

$f_1, f_2, f_3, f_4 \Rightarrow$ shape function ⇒ 3차 함수 (x의)

$\Rightarrow$ use Energy function $\delta \pi = 0$ for all elements

$$\underset{\sim}{K} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \vdots \\ v_8 \\ \theta_8 \end{bmatrix} = \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ \vdots \\ V_8 \\ M_8 \end{bmatrix}$$

$$\left[\pi = \frac{1}{2}\int EI y''^2 dx - \int q y dx \right]$$

variables ⇒ displacement

⇒ solve $\langle v_1 \theta_1 \cdots v_8 \theta_8 \rangle$

The advantages of FEA when compared to other weighted residual methods

1) Differential equation is not required. Energy method is used.

2) Simple displacement function is required, regardless of the type of the overall structure. Thus, it is easy to standardize the element stiffness and the procedure of the analysis method. The accuracy is enhanced by increasing the number of elements (or by decreasing the region that each element occupies).

3) Unknowns are nodal displacements which give direct physical meaning to us.

4) In general, the stiffness matrix is symmetric, which reduces computational time in analysis.

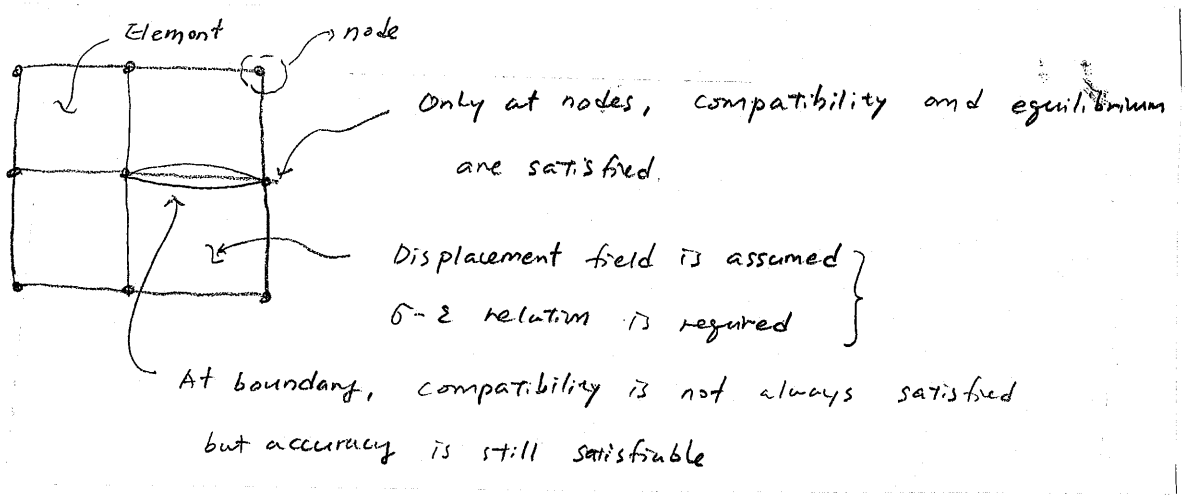
Finite Element Analysis

A. Concept

1) A continuum structure is transformed to subregions of a distretized continuum

2) By doing so, infinite number of dofs are simplified with finite number of dofs

3) computer oriented method. Large computational efforts are required.



B. Advantages

1) Differential equation is not required: Equilibrium condition is not required. Instead, energy condition is used, indirectly satisfying the force-equilibrium.

Energy = displacement x force.

Energy condition is the condition of scalar (not vector). Thus it is relatively very simple when compared to the condition of forces.

Thus, it is applicable to complex conditions of geometry, loading, and boundaries.

2) It is easy to standardize the procedure of constructing element stiffness (automation of process). Very simple (approximate) displacement function is assumed at each element, regardless of the complexity of the structure to be analyzed.

The assumed displacement function produce not exact but approximate solution.

However, this problem can be overcome by increasing the number of elements (i.e using a large number of elements). With fine discretization, the solution from FEA can converge to the exact solution.

3)Unknowns are nodal displacements which give direct physical meaning to us.

4)In general, the stiffness matrix is symmetric, which reduces computational time in analysis.

C. Disadvantage

A lot of computational efforts are required because of a large number of dof to get accurate solution. But, this problem can be easily eliminated with the development of high speed computer hardware.

D. wide range of application

Force – displacement

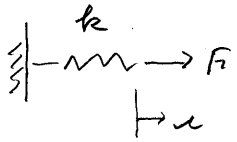
Heat – temperature (heat transfer)

Flow – Pressure (fluid mechanics)

Voltage – Intensity (electricity)

① Force - Displacement

$k\,u = F$ $\quad k$: stiffness $\quad u$: displacemt $\quad F$: force



② Heat - Temperature

$q = -k_t \Delta T$ $\quad k_t$: heat conductivity coeff.

ΔT : Temperature difference

q : heat flux

③ Flow - pressure

$Q = -k_p \Delta p$ $\quad k_p$: proportional constant

Δp : pressure difference

Q : flow rate

④ Voltage - Intensity

$V = R\,i$ $\quad R$: resistance

V : voltage

i : current flow

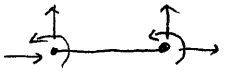
①	②	③	④
F	q	Q	V
k	k_t	k_p	R
u	ΔT	Δp	i

D. Formulation

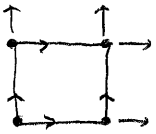
- Displacement Method
  - Stiffness method
  - Principle of total potential energy
  - Principle of virtual displacement
- Force Method
  - flexibility method
  - Principle of total complementary energy
  - Principle of virtual force
- Mixed (Hybrid) Formulation

E. Element Types

Line Element



2-D Element (plane stress, plane strain)



Isoparametric element (Formulation)

3-D Solid

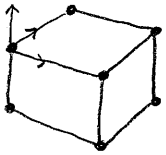
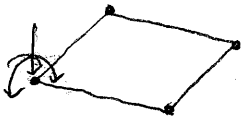


plate Bending



General shell

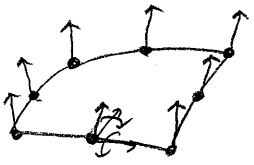
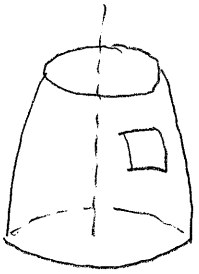


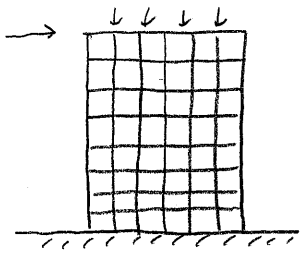
plate bending + plane stress

Axisymmetric shell



F. Solution procedure

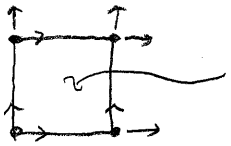
1) divide the continuum structure into a finite number of subregions.



2) provide input of geometric information, node and element numbering, material properties, loading conditions, boundary conditions.

3) construct element stiffness using energy principle (internal energy = external energy)

assumed displacement function



$\underset{\sim}{u} = \underset{\sim}{f}\,\underset{\sim}{q}$

$\underset{\sim}{u}$ = Generic displacements

$\underset{\sim}{f}$ = shape function

$\underset{\sim}{q}$ = nodal displacements

Stress-strain relationship $\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$

Strain-displacement relationship $\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q} = \underset{\sim}{B}\,\underset{\sim}{q}$

4) assemble overall structural stiffness using element stiffness

$\underset{\sim}{P} = \underset{\sim}{k}\,\underset{\sim}{q} \Rightarrow \underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{Q}$

5) modify the structural stiffness considering support conditions

$\underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{Q}$ $\underset{\sim}{K}$ ⇒ modified

6) solve the free displacements

solve Q

7) calculate the stresses and strains of each element from the displacement vector.

$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{B}\,\underset{\sim}{q}$

$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$