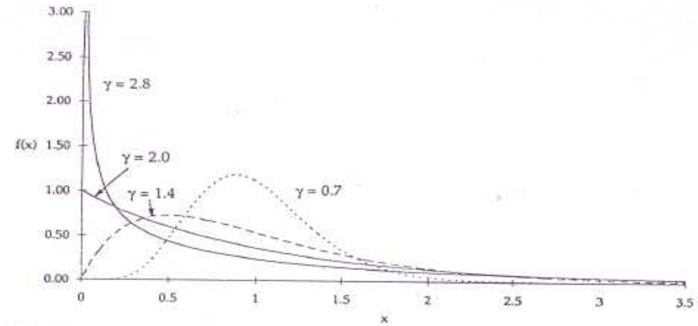

FIGURE 18.2.2 The probability density function for the Pearson type 3 (P3) distribution with lower bound $\xi = 0$, mean $\mu = 1$, and coefficients of skewness $\gamma = 0.7$, 1.4, 2.0, and 2.8 (corresponding to a gamma distribution and shape parameters $\alpha = 8$, 2, 1, and 0.5, respectively).

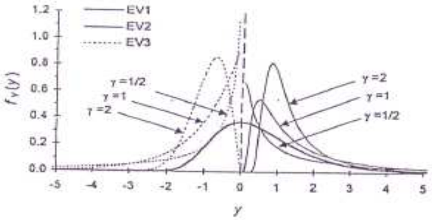

FIGURE 7.2.1
pdf's of asymptotic distributions for largest extreme values.

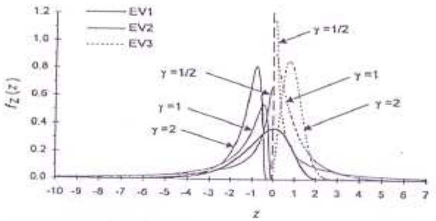

FIGURE 7.2.5
pdf's of asymptotic distributions for smallest extreme values.

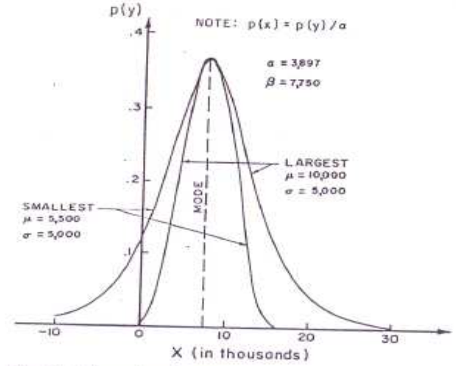

Fig. 6.5. Example of extreme value type I density curves.

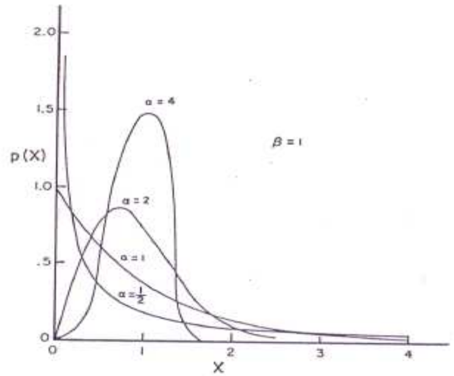

Fig. 6.6. Examples of extreme value type III minimum (Weibull) density curves.

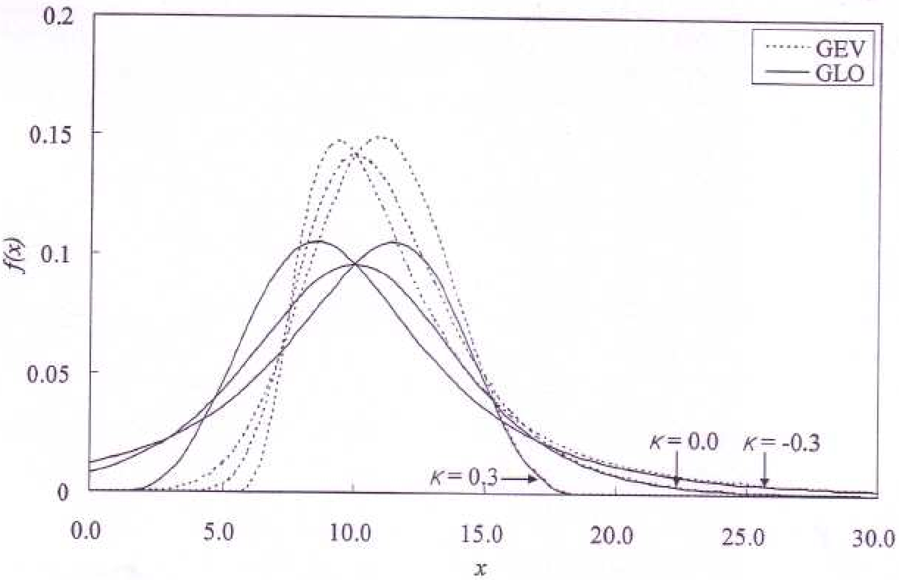

Figure 1. Probability density function of the GEV and GLO distributions for $\kappa = -0.3$, 0.0, and 0.3 (where $\xi = 10$ and $\alpha = 2.6$)

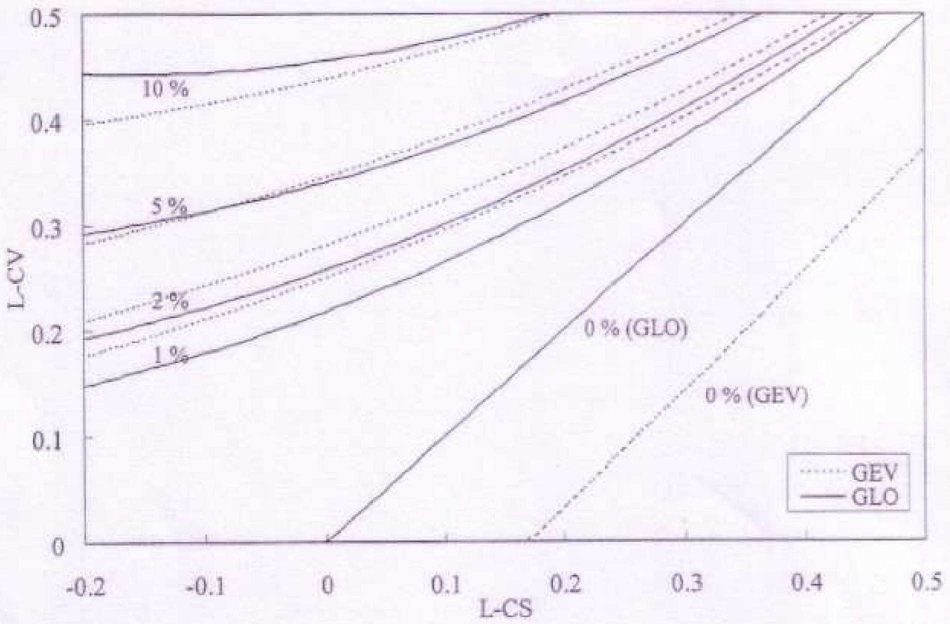

Figure 5. Probability of negative flows for GEV and GLO distributions with L-CV and L-CS combinations

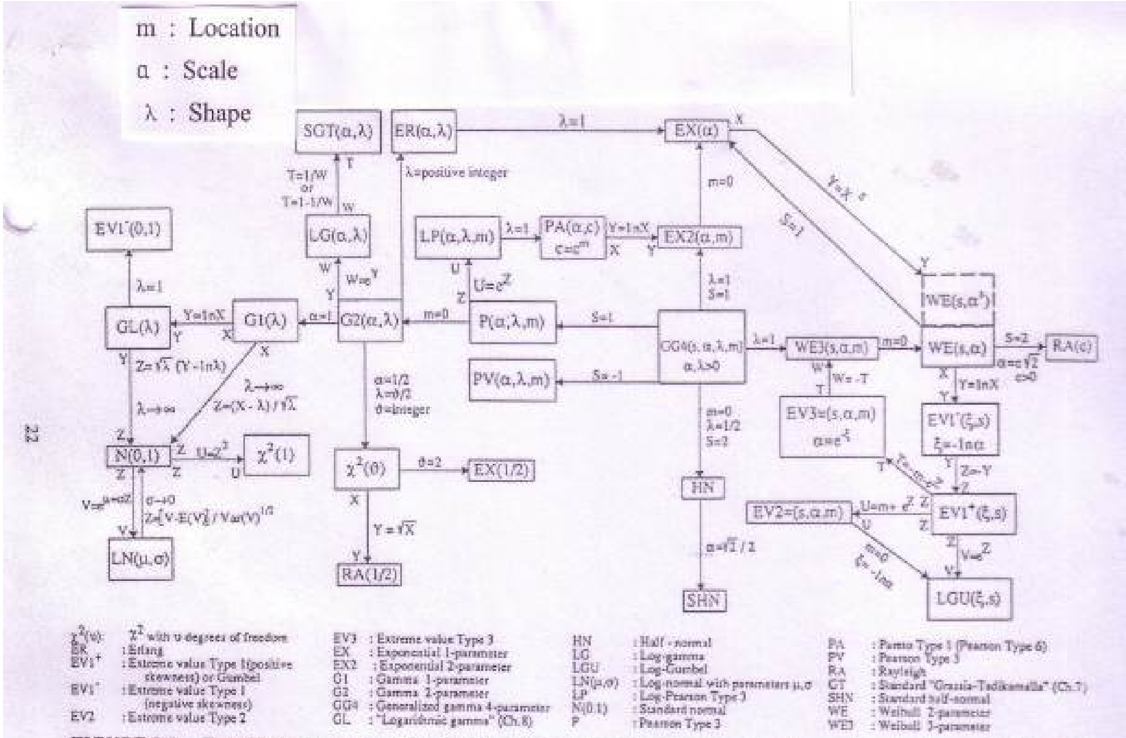

FIGURE 2.2 - Relations between a group of distributions commonly used in water sciences (from Ashkar and Bobée, 1989).

TABLE 5.1: APPLICATION OF THE GAMMA DISTRIBUTION G2(α,λ) IN HYDROLOGY

AUTHORS	DOMAIN OF APPLICATION	COMMENTS
ALEXANDER (1962)	controlled streamflow	Comparison with other methods of calculation
BARGER and THOM (1949), BARGER et al. (1959)	precipitation	Application of the gamma distribution in the USA (Thom's method)
BRIDGES and HAHN (1971) BRIDGES and HAHN (1972)	precipitation	Proposition of a method for determining the reliability of the estimation
CICIONI et al. (1973)	flood flow	Comparison with the Pearson, log-normal and Fisher-Tippet distributions
CRUFF and RANTZ (1965)	flood flow	Use of the distribution in comparison with other distributions
FRIEDMAN and JAMES (1957)	precipitation	Gamma distribution applied in search for the best distribution for modeling precipitation
FRIND (1969)	annual flow	Gamma distribution fitted to simulated data
KOTZ and NEUMANN (1963)	precipitation	Adequacy of the gamma distribution for different rainfall durations
LANDWEHR (1979)	water quality index	Application of the gamma and other distributions derived from it
MARKOVIC (1965)	precipitation and streamflow	Comparison of several distributions for 2506 stations in the U.S.A.
MOOLEY and CRUTCHER (1968) MOOLEY (1973)	precipitation	Application of the gamma distibution for representing monthly monsoon precipitation in India
RICHARDSON (1982)	precipitation	Comparison with the exponential and mixed exponential distributions for generating daily rainfall series
SANTOS (1970)	flood flow	Comparison with the log-normal distribution
THOM (1957)	precipitation	
THOM (1966)	precipitation	Describes Thom's method

TABLE 6.1: APPLICATION OF THE PEARSON TYPE 3 P(α,λ,m) DISTRIBUTION IN HYDROLOGY.

AUTHORS	DOMAIN OF APPLICATION	COMMENTS
BOBÉE (1976)	maximum annual flood flow	The Pearson Type 3 distribution is compared to the log-Pearson Type 3
CHONG and MOORE (1983)	flood flow	Comparison with the log-normal, log-Pearson Type 3 and Gumbel for developing a regional flood frequency curve
CICIONI et al., (1973)	maximum annual flood flow	Comparison with the log-normal, gamma and Fisher-Tippet distributions
CONDIE and NIX (1975)	low flow	Comparisonwith the Pearson Type 5, log-normal and Gumbel distributions
CRUFF and RANTZ (1965)	flood flow	Comparison of different methods of flood frequency estimation
KOPITTKE et al., (1976)	maximum annual flood flow	Comparison with the LP, Gumbel and Weibull distributions for flood flows in Australia
KOTTEGODA (1972)	5-day mean flow	Comparison with Pearson Type 6 and Johnson distributions
MATALAS (1963)	low flow	Comparison with the log-normal and Pearson Type 5 distributions
RAO (1981)	flood flow	Comparison with the Gumbel, log-normal and log-Pearson Type 3 distributions
SHALIGRAM and LELE (1978)	flood flow	Comparison with the Gumbel, Pearson Type 5 and log-normal distributions

TABLE 7.1: COMPARISON OF DIFFERENT DISTRIBUTIONS WITH THE LP DISTRIBUTION (WRC METHOD).

AUTHOR(S)	MAIN DISTRIBUTIONS COMPARED	CRITERIA OF COMPARISON	COMMENTS AND CONCLUSIONS
BENSON (1968)	G2, GU, LGU, LN, LP	Relative deviation between estimated and theoretical values	Resulted in the recommendation of the LP distribution for the US (WRC, 1967)
CHONG and MOORE (1983)	LN, P, LP, GU	Deviation between estimated and observed values	Application to 22 small watersheds in Illinois. The Gumbel distribution gives better results than the LP
CONWAY (1970)	N, P, GU, LN, LP, LGU	Relative deviation between estimated and theoretical values	Application to 46 stations in Australia. Results in the recommendation of the LP distribution
HADGRAFT (1982)	P, LP, N, PTN, GU, LGU	Goodness of fit test	Application to 41 stations in Queensland (Australia)
HOANG (1978)	N, LN, P, LP	Graphical method.	LP gives best results.
KOPITTKE et al., (1976)	N, LN, P, LP, BO, GU, LGU, PO, LPO, WE	Goodness of fit test.	The LP and BO distributions as a whole give best results for the 40 stations considered.
MCMAHON and SRIKANTHAN (1981)	N, LN, G2, GU, WE, LP	Comparison in the Diagram C_S- C_k	The LP distribution as a whole, gives best results for the 172 rivers considered in Australia.
REICH (1972)	GU, LGU, LP	Survey.	Application to 83 rivers in Pennsylvania. LP distribution is preferred.
SINGH and SINCLAIR (1972)	DN, LP	Deviation between estimated and observed values	Application to 33 basins in Illinois. LP distribution does not give good results.
WALLIS and WOOD (1985)	GEV, WA, LP	Bias and mean square error	Simulation study: GEV and WA distributions fitted by the method of probability weighted moments perform better than the LP.

DISTRIBUTIONS:
BO:BOUGHTON - DN:DOUBLE NORMAL - G2:GAMMA 2-PARAMETER - GEV:GENERALIZED EXTREME VALUE - GU:GUMBEL - LGU:LOG-GUMBEL - LN:LOG-NORMAL - LP:LOG- PEARSON TYPE 3 - LPO:LOG-POTTER - N:NORMAL - P:PEARSON TYPE 3-PO:POTTER - PTN:POWER TRANSFORMATION TO NORMALITY - WA:WAKEBY - WE:WEIBULL

Table 1. summary of studies for at-site variance and cross-site correlation for scale and shape parameters of LP3, GEV, GLO, Gumble, Gamma, and Weibull distributions

Distributions	*Parameters*	*relationships*	*Studies*
LP3	γ of logs	At-site variance	- Analytical: Bobée [1973] - Empirical: Griffis et al. [2004]
		Cross-site correlation	- Empirical: Martins and Stedinger [2002]
	σ of logs	At-site variance	- Analytical: Bobée [1973], Kite [1988]
		Cross-site correlation	- Analytical: Griffis [2006]
	γ & σ of logs	At-site correlation between γ & σ	- Analytical: Bobée [1973], Chowdhury and Stedinger [1991]
		Cross-site correlation with σ	- Analytical: Griffis [2006]
GEV	τ_3	At-site variance	- Analytical: Hosking et al. [1985] - Empirical: Chowdhury et al. [1991]
		Cross-site correlation	- Empirical: Martins and Stedinger [2002]
		Cross-site correlation with τ_2	- Empirical: This study
	τ_2	At-site variance	- Analytical: Hosking et al. [1985] - Empirical: Chowdhury et al. [1991]
		Cross-site correlation	- Empirical: This study
		Variance of the regional averaged $\bar{\tau}_2$ - Kjeldsen and Rosbjerg [2002]	
	τ_2 & τ_3	Cross-site correlation	- Empirical: This study

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GLO	τ_3	At-site variance	- Analytical: Kjeldsen and Jones [2004]
		Cross-site correlation	- Empirical: This study - Bootstrap demonstrate ρ_{xy}^3 adequate: Kjeldsen and Jones [2006]
	τ_2	At-site variance	- Analytical: Kjeldsen and Jones [2004]
		Cross-site correlation	- Empirical: This study - Bootstrap demonstrate ρ_{xy}^2 adequate: Kjeldsen and Jones [2006]
	τ_2 & τ_3	At-site correlation between τ_2 & τ_3	- Analytical: Kjeldsen and Jones [2004]
		Cross-site correlation	- Empirical: This study
Gumbel & Gamma (2 parameter)	σ	Cross-site correlation	- Analytical: Bayazit and Önöz [2004]
		Variance of the regional averaged $\hat{\sigma}$ - Analytical: Bayazit and Önöz [2004]	
Weibull (2 parameter)	σ	Cross-site correlation	- Analytical: Bayazit and Önöz [2004]
		Variance of the regional averaged $\hat{\sigma}$ - Analytical: Bayazit and Önöz [2004]	
	κ	Variance of the regional averaged $\hat{\kappa}$ - Analytical: Hoe et al. [2001]	

TABLE 18.2.1 Commonly Used Frequency Distributions in Hydrology (see also Table 18.1.2)

Distribution	pdf and/or cdf	Range	Moments
Normal	$f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]$	$-\infty<x<\infty$	μ_X and σ_X^2; $\gamma_X=0$
Lognormal*	$f_X(x)=\frac{1}{x\sqrt{2\pi\sigma_Y^2}}\exp\left[-\frac{1}{2}\left(\frac{\ln(x)-\mu_Y}{\sigma_Y}\right)^2\right]$	$x>0$	$\mu_X=\exp\left(\mu_Y+\frac{\sigma_Y^2}{2}\right)$ $\sigma_X^2=\mu_X^2[\exp(\sigma_Y^2)-1]$ $\gamma_x=3\mathrm{CV}_x+\mathrm{CV}_x^3$
Pearson type 3	$f_X(x)=\lvert\beta\rvert[\beta(x-\xi)]^{\alpha-1}\frac{\exp[-\beta(x-\xi)]}{\Gamma(\alpha)}$ $\Gamma(\alpha)$ is the gamma function (for $\beta>0$ and $\xi=0$: $\gamma_x=2\,\mathrm{CV}_x$)	$\alpha>0$ for $\beta>0$: $x>\xi$ for $\beta<0$: $x<\xi$	$\mu_x=\xi+\frac{\alpha}{\beta}$; $\sigma_X^2=\frac{\alpha}{\beta^2}$ and $\gamma_x=\frac{2}{\sqrt{\alpha}}$ and $\gamma_x=\frac{-2}{\sqrt{\alpha}}$
log-Pearson type 3	$f_X(x)=\lvert\beta\rvert\beta[\ln(x)-\xi]^{\alpha-1}\frac{\exp\{-\beta[\ln(x)-\xi]\}}{x\Gamma(\alpha)}$	for $\beta<0$, $0<x<\exp(\xi)$; for $\beta>0$, $\exp(\xi)<x<\infty$	See Eq. (18.2.34)
Exponential	$f_X(x)=\beta\exp[-\beta(x-\xi)]$ $F_X(x)=1-\exp[-\beta(x-\xi)]$	$x>\xi$ for $\beta>0$	$\mu_X=\xi+\frac{1}{\beta}$; $\sigma_X^2=\frac{1}{\beta^2}$ $\gamma_X=2$
Gumbel	$f_X(x)=\frac{1}{\alpha}\exp\left[-\frac{x-\xi}{\alpha}-\exp\left(-\frac{x-\xi}{\alpha}\right)\right]$ $F_X(x)=\exp\left[-\exp\left(-\frac{x-\xi}{\alpha}\right)\right]$	$-\infty<x<\infty$	$\mu_x=\xi+0.5772\alpha$ $\sigma_X^2=\frac{\pi^2\alpha^2}{6}=1.645\alpha^2$; $\gamma_x=1.1396$
GEV	$F_X(x)=\exp\left\{-\left[1-\frac{\kappa(x-\xi)}{\alpha}\right]^{1/\kappa}\right\}$	(σ_x^2 exists for $\kappa>-0.5$) when $\kappa>0$, $x<\left(\xi+\frac{\alpha}{\kappa}\right)$; $\kappa<0$, $x>\left(\xi+\frac{\alpha}{\kappa}\right)$	$\mu_X=\xi+\left(\frac{\alpha}{\kappa}\right)[1-\Gamma(1+\kappa)]$ $\sigma_X^2=\left(\frac{\alpha}{\kappa}\right)^2(\Gamma(1+2\kappa)-[\Gamma(1+\kappa)$
Weibull	$f_X(x)=\left(\frac{k}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{k-1}\exp\left[-\left(\frac{x}{\alpha}\right)^k\right]$ $F_X(x)=1-\exp[-(x/\alpha)^k]$	$x>0$; $\alpha,k>0$	$\mu_x=\alpha\Gamma\left(1+\frac{1}{k}\right)$ $\sigma_X^2=\alpha^2\left\{\Gamma\left(1+\frac{2}{k}\right)-\left[\Gamma\left(1+\frac{1}{k}\right)\right.\right.$
Generalized Pareto	$f_X(x)=\left(\frac{1}{\alpha}\right)\left[1-\kappa\frac{(x-\xi)}{\alpha}\right]^{1/\kappa-1}$ $F_X(x)=1-\left[1-\kappa\frac{(x-\xi)}{\alpha}\right]^{1/\kappa}$	for $\kappa<0$, $\xi\le x<\infty$ for $\kappa>0$, $\xi\le x\le\xi+\frac{\alpha}{\kappa}$ (γ_X exists for $\kappa>-0.33$)	$\mu_X=\xi+\frac{\alpha}{(1+\kappa)}$ $\sigma_X^2=\alpha^2/[(1+\kappa)^2(1+2\kappa)]$ $\gamma_X=\frac{2(1-\kappa)(1+2\kappa)^{1/2}}{(1+3\kappa)}$

* Here $Y=\ln(X)$. Text gives formulas for three-parameter lognormal distribution, and for two- and three-parameter lognormal with common base 10 logarithm

Chapter 7 Frequency Analysis

- Graphical Construction of Probability Paper

- Mathematical Construction of Probability Paper
  - transformation of the CDF to Y = aZ+b
    where Y = function of parameters and F_X
    Z = function of parameters and x
  - plot Y vs. Z --> straight line

((example 1)) exponential distribution

((example 2)) Gumbel distribution for max

- Probability Plotting
  - Gumbel's criteria (p153, Text)
  - plotting position equations (table 18.3.1, Handbook of Hydrology)

- Frequency Factors

$$x_T = u + \triangle x_T = u + K_T\sigma$$

where x_T = magnitude of X with a T−year return period
$\triangle x_T$ = departure of the variate from the mean
K_T = frequency factor with a T−year return period

  - normal distribution

$$K_T = \frac{x_T - u}{\sigma} = Z_T$$

  - lognormal distribution
    Eq (7.12)

  - P3
    (i) Wilson−Hilferty transformation

$$K_T = \frac{2}{\gamma}\left(1 + \frac{\gamma z_T}{6} - \frac{\gamma^2}{36}\right)^3 - \frac{2}{\gamma}$$

    (ii) table 7.7, Text

  - Gumbel distribution
    (i) $K_T = -0.45 - 0.7797 \ln[\ln\frac{T}{1-T}]$
    (ii) table 7.8, Text

- Treatment of Zeros

mal distribution

$$G^{-1}(1 - q_i) = \mu + \sigma\, \Phi^{-1}(1 - q_i) \tag{18.3.7}$$

Thus, except for intercept and slope, a plot of the observations $x_{(i)}$ versus $G^{-1}[1 - q_i]$ is visually identical to a plot of $x_{(i)}$ versus $\Phi^{-1}(1 - q_i)$. The values of q_i are often printed along the abscissa or horizontal axis. Lognormal paper is obtained by using a log scale to plot the ordered logarithms log $(x_{(i)})$ versus a normal-probability scale, which is equivalent to plotting log $(x_{(i)})$ versus $\Phi^{-1}(1 - q_i)$. Figure 18.3.1 illustrates use of lognormal paper with Blom's plotting positions.

For the Gumbel distribution,

$$G^{-1}(1 - q_i) = \xi - \alpha \ln [-\ln (1 - q_i)] \tag{18.3.8}$$

Thus a plot of $x_{(i)}$ versus $G^{-1}(1 - q_i)$ is identical to a plot of $x_{(i)}$ versus the *reduced Gumbel variate*

$$y_i = -\ln [-\ln (1 - q_i)] \tag{18.3.9}$$

It is easy to construct probability paper for the Gumbel distribution by plotting $x_{(i)}$ as a function of y_i; the horizontal axis can show the actual values of y or, equivalently, the associated q_i, as in Fig. 18.3.1 for the lognormal distribution.

Special probability papers are not available for the Pearson type 3 or log Pearson type 3 distributions because the frequency factors depend on the skew coefficient. However, for a given value for the coefficient of skewness γ one can plot the observation $x_{(i)}$ for a P3 distribution, or log $(x_{(i)})$ for the LP3 distribution, versus the frequency factors $K_{p_i}(\gamma)$ defined in Eq. (18.2.29) with $p_i = 1 - q_i$. This should yield a straight line except for sampling error if the correct skew coefficient is employed. Alternatively for the P3 or LP3 distributions, normal or lognormal probability paper is often used to compare the $x_{(i)}$ and a fitted P3 distribution, which plots as a curved line. Table 18.3.2 summarizes how probability plots may be constructed for these and other distributions.

TABLE 18.3.2 Generation of Probability Plots for Different Distributions

Normal probability paper. Plot $x_{(i)}$ versus z_{p_i} given in Eq. (18.2.3), where $p_i = 1 - q_i$. Blom's formula ($a = 3/8$) provides quantile-unbiased plotting positions.

Lognormal probability paper. Plot ordered logarithms log $[x_{(i)}]$ versus z_{p_i}. Blom's formula ($a = 3/8$) provides quantile-unbiased plotting positions.

Exponential probability paper. Plot ordered observations $x_{(i)}$ versus $\xi - \ln (q_i)/\beta$ or just $-\ln (q_i)$. Gringorten's plotting positions ($a = 0.44$) work well.

Gumbel and Weibull probability paper. For Gumbel distribution plot ordered observations $x_{(i)}$ versus $\xi - \alpha \ln [-\ln (1 - q_i)]$ or just $y_i = -\ln [-\ln (1 - q_i)]$. Gringorten's plotting positions ($a = 0.44$) were developed for this distribution. For Weibull distribution plot ln $[x_{(i)}]$ versus $\ln (\alpha) + \ln [-\ln (q_i)]/k$ or just $\ln [-\ln (q_i)]$. (See Ref. 154.)

GEV distribution. Plot ordered observations $x_{(i)}$ versus $\xi + (\alpha/\kappa)\{1 - [-\ln (1 - q_i)]^\kappa$, or just $(1/\kappa)\{1 - [-\ln (1 - q_i)]^\kappa\}$. Alternatively employ Gumbel probability paper on which GEV will be curved. Cunnane's plotting positions ($a = 0.4$) are reasonable.[32]

Pearson type 3 probability paper. Plot ordered observations $x_{(i)}$ versus $K_{p_i}(\gamma)$, where $p_i = 1 - q_i$. Blom's formula ($a = 3/8$) is quantile-unbiased for normal distribution and makes sense for small γ. Or employ normal probability paper. (See Ref. 158.)

Log Pearson type 3 probability paper. Plot ordered logarithms log $[x_{(i)}]$ versus $K_{p_i}(\gamma)$ where $p_i = 1 - q_i$. Blom's formula ($a = 3/8$) makes sense for small γ. Or employ lognormal probability paper. (See Ref. 158.)

Uniform probability paper. Plot $x_{(i)}$ versus $1 - q_i$, where q_i are the Weibull plotting positions ($a = 0$). (See Ref. 154.)

18.3.3 Goodness-of-Fit Tests and L-Moment Diagrams

Rigorous statistical tests are available and are useful for assessing whether or not a given set of observations might have been drawn from a particular family of distributions, as discussed in Sec. 18.3.1. For example, the *Kolmogorov-Smirnov test* provides bounds within which *every observation* on a probability plot should lie if the sample is actually drawn from the assumed distribution; it is useful for evaluating visually the adequacy of a fitted distribution. Stephens[136] gives critical Kolmogorov-Smirnov values for the normal and exponential distributions (reproduced in Ref. 95, p. 112); Chowdhury et al.[22] provide tables for the GEV distribution.

The *probability plot correlation test* discussed below is a more powerful test of whether a sample has been drawn from a postulated distribution; a test with greater *power* has a greater probability of correctly determining that a sample is not from the postulated distribution. *L-moment tests* are also relatively powerful and can be used to determine if a proposed Gumbel, GEV, or normal distribution is consistent with the data. L-moment diagrams are useful as a guide in selecting an appropriate family of distributions for describing a set of variables, such as flood distributions in a region.

Probability Plot Correlation Coefficient Test. A simple but powerful goodness-of-fit test is the probability plot correlation test developed by Filliben.[41] The test uses the correlation r between the ordered observations $x_{(i)}$ and the corresponding fitted quantiles $w_i = G^{-1}(1 - q_i)$, determined by plotting positions q_i for each $x_{(i)}$. Values of r near 1.0 suggest that the observations could have been drawn from the fitted distribution. Essentially, r measures the linearity of the probability plot, providing a quantitative assessment of fit. If $\bar{x}$ denotes the average value of the observations and $\bar{w}$ denotes the average value of the fitted quantiles, then

$$r = \frac{\Sigma (x_{(i)} - \bar{x})(w_i - \bar{w})}{[\Sigma (x_{(i)} - \bar{x})^2\, \Sigma (w_i - \bar{w})^2]^{0.5}} \tag{18.3.10}$$

Table 18.3.3 gives critical values of r for the normal distribution, or the logarithms of lognormal variates, based on a plotting position with $a = 3/8$. Values for the Gumbel distribution are reproduced in Table 18.3.4 for use with $a = 0.44$; the table also applies to logarithms of Weibull variates (see Table 18.3.2 and Sec. 18.2.2). Other tables are available for the uniform,[154] the GEV,[22] the Pearson type 3,[158] and exponential and other distributions.[30]

L-Moment Diagrams and Ratio Tests. Figure 18.1.1 provides an example of an L-moment diagram.[72,143] Sample L moments are less biased than traditional product-moment estimators, and thus are better suited for use in constructing moment diagrams. (See Sec. 18.1.4.) Plotting sample statistics on such diagrams allows a choice between alternative families of distributions (Ref. 29). L-moment diagrams include plots of τ_3 versus τ_2 for choosing among two-parameter distributions, or of τ_4 versus τ_3 for choosing among three-parameter distributions. Chowdhury et al.[22] derive the sampling variance of $\hat{\tau}_2$, $\hat{\tau}_3$, and $\hat{\tau}_4$ as a function of κ for the GEV distribution to provide a powerful test of whether a particular data set is consistent with a GEV distribution with a regionally estimated value of κ, or a regional κ and CV. Equation (18.2.24) provides a very powerful test for the Gumbel versus a general

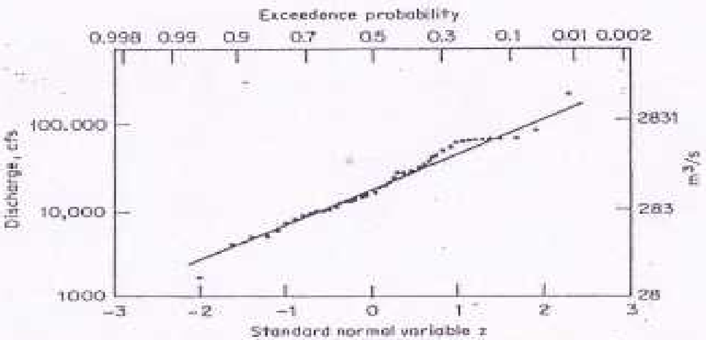


FIGURE 18.3.1 A probability plot using a normal scale of 44 annual maxima for the Guadalupe River near Victoria, Texas. *(Reproduced with permission from Ref. 20, p. 398.)*

ance probability of the ith-largest event is often estimated using the *Weibull plotting position:*

$$q_i = \frac{i}{n+1} \tag{18.3.4}$$

corresponding to the mean of U_i.

Choice of plotting position. Hazen[39] originally developed probability paper and imagined the probability scale divided into n equal intervals with midpoints $q_i = (i - 0.5)/n$, $i = 1, \ldots, n$; these served as his plotting positions. Gumbel[31] rejected this formula in part because it assigned a return period of $2n$ years to the largest observation (see also Harter[38]); Gumbel promoted Eq. (18.3.4).

Cunnane[28] argued that plotting positions q_i should be assigned so that on average $X_{(i)}$ would equal $G^{-1}(1 - q_i)$; that is, q_i would capture the mean of $X_{(i)}$ so that

$$E[X_{(i)}] = G^{-1}(1 - q_i) \tag{18.3.5}$$

Such plotting positions would be almost quantile-unbiased. The Weibull plotting positions $i/(n + 1)$ equal the average exceedance probability of the ranked observations $X_{(i)}$, and hence are probability-unbiased plotting positions. The two criteria are different because of the nonlinear relationship between $X_{(i)}$ and $U_{(i)}$.

Different plotting positions attempt to achieve almost quantile-unbiasedness for different distributions; many can be written

$$q_i = \frac{i - a}{n + 1 - 2a} \tag{18.3.6}$$

which is symmetric so that $q_i = 1 - q_{n+1-i}$. Cunanne recommended $a = 0.40$ for obtaining nearly quantile-unbiased plotting positions for a range of distributions.

Other alternatives are Blom's plotting position ($a = \frac{3}{8}$), which gives nearly unbiased quantiles for the normal distribution, and the Gringorten position ($a = 0.44$) which yields optimized plotting positions for the largest observations from a Gumbel distribution.[49] These are summarized in Table 18.3.1, which also reports the return period, $T_1 = 1/q_1$, assigned to the largest observation. Section 18.6.3 develops plotting positions for records that contain censored values.

The differences between the Hazen formula, Cunanne's recommendation, and the Weibull formula is modest for i of 3 or more. However, differences can be appreciable for $i = 1$, corresponding to the largest observation (and $i = n$ for the smallest observation). Remember that the actual exceedance probability associated with the largest observation is a random variable with mean $1/(n + 1)$ and a standard deviation of nearly $1/(n + 1)$; see Eqs. (18.3.2) and (18.3.3). Thus all plotting positions give crude estimates of the unknown exceedance probabilities associated with the largest (and smallest) events.

A good method for illustrating this uncertainty is to consider quantiles of the beta distribution of the actual exceedance probability associated with the largest observation $X_{(1)}$. The actual exceedance probability for the largest observation $X_{(1)}$ in a sample is between $0.29/n$ and $1.38/(n + 2)$ nearly 50 percent of the time; and between $0.052/n$ and $3/(n + 2)$ nearly 90 percent of the time. Such bounds allow one to assess the consistency of the largest (or, by symmetry, the smallest) observation with a fitted distribution better than does a single plotting position.

Probability Paper. It is now possible to see how probability papers can be constructed for many distributions. A probability plot is a graph of the ranked observations $x_{(i)}$ versus an approximation of their expected value $G^{-1}(1 - q_i)$. For the nor-

TABLE 18.3.1 Alternative Plotting Positions and their Motivation*

Name	Formula	a	T_1	Motivation
Weibull	$\frac{i}{n+1}$	0	$n+1$	Unbiased exceedance probabilities for all distributions
Median†	$\frac{i-0.3175}{n+0.365}$	0.3175	$1.47n + 0.5$	Median exceedance probabilities for all distributions
APL	$\frac{i-0.35}{n}$	−0.35	$1.54n$	Used with PWMs [Eq. (18.1.13)]
Blom	$\frac{i-3/8}{n+1/4}$	0.375	$1.60n + 0.4$	Unbiased normal quantiles
Cunnane	$\frac{i-0.40}{n+0.2}$	0.40	$1.67n + 0.3$	Approximately quantile-unbiased
Gringorten	$\frac{i-0.44}{n+0.12}$	0.44	$1.79n + 0.2$	Optimized for Gumbel distribution
Hazen	$\frac{i-0.5}{n}$	0.50	$2n$	A traditional choice

* Here a is the plotting-position parameter in Eq. (18.3.6) and T_1 is the return period each plotting position assigns to the largest observation in a sample of size n.
† For $i = 1$ and n, the exact value is $q_1 = 1 - q_n = 1 - 0.5^{1/n}$.

Chapter 8 Confidence Intervals and Hypothesis Testing

•Confidence Intervals (CI)

- 100(1−α)% CI for μ_X

$$\overline{X} - \frac{s_X}{\sqrt{n}} t_{1-\alpha/2, n-1} \leqq \mu_X \leqq \overline{X} + \frac{s_X}{\sqrt{n}} t_{1-\alpha/2, n-1}$$

where $t_{1-\alpha/2,n-1}$ is the upper 100(α/2)% percentile of Student's t distribution with n−1 degree of freedom
((note)) In large samples (n > 40), $t_{1-\alpha/2,n-1} \cong z_{1-\alpha/2}$

- 100(1−α)% CI for x_p

$$\widehat{x_p} - z_{1-\alpha/2}\sqrt{Var(\widehat{x_p})} \leqq x_p \leqq \widehat{x_p} + z_{1-\alpha/2}\sqrt{Var(\widehat{x_p})}$$

where $\widehat{x_p} = \overline{x} + K_p s_x$

- Hypothesis Test for Differences in Means When the Variances are Unknown but same
  - assumptions
    (i) X_1, X_2 ~ normal
    (ii) $\sigma_1 = \sigma_2$
    (iii) σ_1 and σ_2 are unknown
  - hypothesis

$$H_0: \mu_1 = \mu_2$$
$$H_a: \mu_1 \neq \mu_2$$

  - test statistic

- Hypothesis Test for Differences in Means When the Variances are UnKnown but different
  - assumptions
    (i) X_1, X_2 ~ normal
    (ii) $\sigma_1 \neq \sigma_2$
    (iii) σ_1 and σ_2 are known
  - hypothesis

$$H_0: \mu_1 = \mu_2$$
$$H_a: \mu_1 \neq \mu_2$$

  - test statistic

- Hypothesis Test for Differences in Variances
  - assumptions: $X_1 \sim N_1(\mu_1, \sigma_1)$ and $X_2 \sim N_2(\mu_2, \sigma_2)$
  - hypothesis

    H_0: $\sigma_1^2 = \sigma_2^2$

    H_a: $\sigma_1^2 \neq \sigma_2^2$
  - test statistic

- Hypothesis Test for Goodness of Fit
  - Chi-Square Test
  - the Kolmogorov-Smirnov Test
  - the Filliben Test

theoretical frequency distributions is the chi-square test. This test makes a comparison between the actual number of observations and the expected number of observations (expected according to the distribution under test) that fall in the class intervals. The expected numbers are calculated by multiplying the expected relative frequency by the total number of observations. The test statistic is calculated from the relationship

$$\chi_c^2 = \sum_{i=1}^{k} (O_i - E_i)^2 / E_i \quad (8.21)$$

re k is the number of class intervals, O_i is the observed and E_i the expected (according to the distribution being tested) number of observations in the i^{th} class interval. The distribution of χ_c^2 is a chi-square distribution with k-p-1 degrees of freedom where p is the number of parameters estimated from the data. The hypothesis that the data are from the specified distribution is rejected if

$$\chi_c^2 > \chi^2_{1-\alpha,k-p-1} \quad (8.22)$$

Example 8.6. As an example of using the chi-square test, consider the data of table 2.2 and test the hypothesis that the data are from a normal distribution. The observed and expected numbers in each class interval are obtained by multiplying the relative frequency by 66 which is the number of observations. Table 8.2 shows the calculation χ_c^2. The degrees of freedom is k-3 or 7 since two parameters (μ_x and σ_x^2) were estimated for the normal distribution. Comparing χ_c^2 of 10.31 with $\chi^2_{0.90,7}$ = 12.02, it is concluded that the normal distribution adequately describes the data for α = 0.10. If χ_c^2 exceeded $\chi^2_{1-\alpha,k-p-1}$, the hypothesis that the normal distribution describes the data would be rejected.

Comment: By examining table 8.2 and equation 8.21, it is apparent that the chi-square goodness of fit test is quite sensitive in the tails of the assumed distribution. Because of this many statisticians recommend that classes be combined if the expected number in a class is less than 3 (or 5). If the 3 criteria is used, the first two classes and the last two classes must be combined. This makes the calculation of χ_c^2 as shown in table 8.3 and the value is reduced to 3.55. The degrees of freedom are reduced to 5.

Table 8.2. Chi-square test on Kentucky River data.

Class Mark	Observed Number	Expected Number	O-E	$\frac{(O-E)^2}{E}$
000	2	1.65	0.35	0.0742
000	3	3.76	-0.76	0.1536
000	10	7.07	2.93	1.2142
000	9	10.49	-1.49	0.2116
000	11	12.41	-1.41	0.1602
000	10	11.75	-1.75	0.2606
000	12	8.44	3.56	1.5016
000	6	5.35	0.65	0.0789
000	0	2.57	-2.57	2.5700
000	3	0.99	2.01	4.0809
al	66	64.48	+1.52	10.3058

Perhaps a better way of conducting the chi-square goodness of fit test is to define the class intervals so that under the hypothesis being tested the expected number of observations in each class interval is the same. This means that the class intervals will be of unequal width and that the interval widths will be a function of the distribution being tested.

Example 8.7. A chi-square test for normality of Kentucky River data using 10 class intervals each having the same expected frequency can be conducted as follows.

Ten class intervals means that the expected relative frequency or probability in each interval is 0.1. The class boundaries can be determined by using a standard normal table. For instance the boundaries of the 4^{th} class interval on a standard normal distribution are $z_{0.3}$ = -0.52 to $z_{0.4}$ = -0.25. The actual class boundaries are

$$x_{0.3} = s_x z_{0.3} + \bar{x}$$

$$= 21{,}000(-0.52) + 67{,}500 = 56{,}580$$

$$x_{0.4} = 21{,}000(-0.25) + 67{,}500 = 62{,}250$$

Table 8.4 contains the data for conducting the chi-square test based on 10 class intervals having equal expected numbers of observations (66/10 or 6.6) in each interval. In this case χ_c^2 is 4.913 which is less than $\chi^2_{0.90,7}$ of 12.02. The hypothesis is again accepted.

The Kolmogorov-Smirnov Test

An alternative to the chi-square goodness of fit test is the Kolmogorov-Smirnov test. This test is conducted as follows:

1) Let $P_X(x)$ be the completely specified theoretical cumulative distribution function under the null hypothesis.
2) Let $S_n(x)$ be the sample cumulative density function based on n observations. For any observed x, $S_n(x)$ = k/n where k is the number of observations less than or equal to x.

Table 8.3. Chi-square test on Kentucky River data (modified).

Class Mark	Observed Number	Expected Number	O-E	$\frac{(O-E)^2}{E}$
25,000	5	5.41	-0.41	0.0311
35,000				
45,000	10	7.07	2.93	1.2142
55,000	9	10.49	-1.49	0.2116
65,000	11	12.41	-1.41	0.1602
75,000	10	11.75	-1.75	0.2606
85,000	12	8.44	3.56	1.5016
95,000	6	5.35	0.65	0.0789
105,000				
115,000	3	3.56	-0.56	0.0881
Total	66	64.48	+1.52	3.5463

Table 8.4. Chi-square test based on equal expected numbers per class interval.

Class Number	Boundaries lower	Boundaries upper	Observed Number	Expected Number	$\frac{(O-E)^2}{E}$
1	-∞	40,620	6	6.6	0.055
2	40,620	49,860	9	6.6	0.873
3	49,860	56,580	7	6.6	0.024
4	56,580	62,250	4	6.6	1.024
5	62,250	67,500	6	6.6	0.055
6	67,500	72,750	8	6.6	0.300
7	72,750	78,420	4	6.6	1.024
8	78,420	85,140	9	6.6	0.873
9	85,140	94,380	8	6.6	0.300
10	94,380	∞	5	6.6	0.388
		Totals	66	66.0	4.913

3) Determine the maximum deviation, D, defined by

$$D = \max |P_X(x) - S_n(x)|$$

4) If, for the chosen significance level, the observed value of D is greater than or equal to the critical tabulated value of the Kolmogorov-Smirnov statistic, the hypothesis is rejected. The Kolmogorov-Smirnov test statistic is contained in table E.9.

This test can be conducted by calculating the quantities $P_X(x)$ and $S_n(x)$ at each observed point or by plotting the data as in figure 7.5 and selecting the greatest deviation on the probability scale of a point from the theoretical line. If the latter approach is used care must be taken to select the largest deviation on the probability scale which is not necessarily linear. The data should not be grouped for this test.

Note that for the Kolmogorov-Smirnov test, $P_X(x)$ is a completely specified, cumulative probability distribution. That is no parameters for the distribution must be estimated from observed data. Crutcher (1975) points out that when parameters must be estimated to specify $P_X(x)$, the Kolmogorov-Smirnov test is conservative with respect to the Type I error. That is if the critical value is exceeded by the test statistic obtained from the observed values, the hypothesis is rejected with considerable confidence. Crutcher (1975) presents a table of critical values for sample sizes of 25 and 30 as well as infinitely large samples for the exponential, gamma, normal and extreme value distributions when parameters of these distributions must be estimated. In general these critical values are smaller than the values given in table E.9.

Example 8.8. Test the hypothesis that the Kentucky River peak flow data are normally distributed. Use the Kolmogorov-Smirnov test.

Solution: The data are plotted in figure 7.5. The maximum deviation between the best fitting line, $P_X(x)$, and the plotted points, $S_n(x)$, on the probability scale is about 0.06 at X = 47,000 cfs and again at 67,000 cfs. The critical value of the Kolmogorov-Smirnov test statistic from table E.9 for α = 0.10 and n = 66 is 0.150. Therefore we cannot re-

TABLE 17.2.4 Often-Used Critical Values of Four Distributions

The table entry x_p is the value for which Prob $(X < x_p) = p$.

p	0.9	0.95	0.975	0.99	0.995
			t distribution		
df = 1*	3.08	6.31	12.71	31.82	63.66
df = 2	1.89	2.92	4.30	6.96	9.92
df = 3	1.64	2.35	3.18	4.54	5.84
df = 4	1.53	2.13	2.78	3.75	4.60
df = 5	1.48	2.01	2.57	3.36	4.03
df = 7	1.41	1.89	2.36	3.00	3.50
df = 10	1.37	1.81	2.23	2.76	3.17
df = 15	1.34	1.75	2.13	2.60	2.95
df = 20	1.33	1.72	2.09	2.53	2.85
df = 25	1.32	1.71	2.06	2.49	2.79
df = 30	1.31	1.70	2.04	2.46	2.75
			Normal distribution		
	1.28	1.64	1.96	2.33	2.58
			Chi-square distribution†		
df = 1	2.71	3.84	5.02	6.64	7.88
df = 2	4.60	5.99	7.38	9.21	10.60
df = 3	6.35	7.82	9.35	11.34	12.84
df = 4	7.78	9.49	11.14	13.28	14.86
df = 5	9.24	11.07	12.83	15.09	16.75
df = 7	12.02	14.07	16.01	18.48	20.28
df = 10	15.99	18.31	20.48	23.21	25.19
df = 15	22.31	25.00	27.49	30.58	32.80
df = 20	28.41	31.41	34.17	37.57	40.00
df = 30	40.26	43.77	46.98	50.89	53.67
			F distribution‡		
df = 1, *n*					
df = 2, 10	2.92	4.10	5.46	7.56	9.43
df = 2, 20	2.59	3.49	4.46	5.85	6.99
df = 2, 30	2.49	3.32	4.18	5.39	6.35
df = 2, 60	2.39	3.18	3.92	4.98	5.79
df = 2, 120	2.36	3.07	3.80	4.79	5.54
df = 2, ∞	2.30	3.00	3.69	4.60	5.30
df = 3, 10	2.73	3.71	4.83	6.55	8.08
df = 3, 20	2.38	3.10	3.86	4.94	5.82
df = 3, 30	2.28	2.92	3.59	4.51	5.24
df = 3, 60	2.18	2.76	3.34	4.13	4.73
df = 3, 120	2.13	2.68	3.23	3.95	4.50
df = 3, ∞	2.08	2.60	3.12	3.78	4.28
df = 4, 10	2.60	3.48	4.47	5.99	7.34
df = 4, 20	2.25	2.87	3.52	4.43	5.17
df = 4, 30	2.14	2.69	3.25	4.02	4.62
df = 4, 60	2.04	2.52	3.01	3.65	4.14
df = 4, 120	1.99	2.45	2.89	3.48	3.92
df = 4, ∞	1.94	2.37	2.68	3.32	3.72

* df = degrees of freedom.

† For the chi-square distribution a good approximation of the cdf is $x_p = 0.5(z_p + \sqrt{2df - 1})^2$, where z_p is the p percentage point on the standard normal distribution (read from the table above).

‡ For the *F* distribution with 1 and *n* degrees of freedom, the F value for a given *p* level is equal to the *t* value squared for *n* degrees of freedom and the same *p* level.

TABLE 18.3.3 Lower Critical Values of the Probability Plot Correlation Test Statistic for the Normal Distribution Using $p_i = (i - 3/8)/(n + 1/4)$

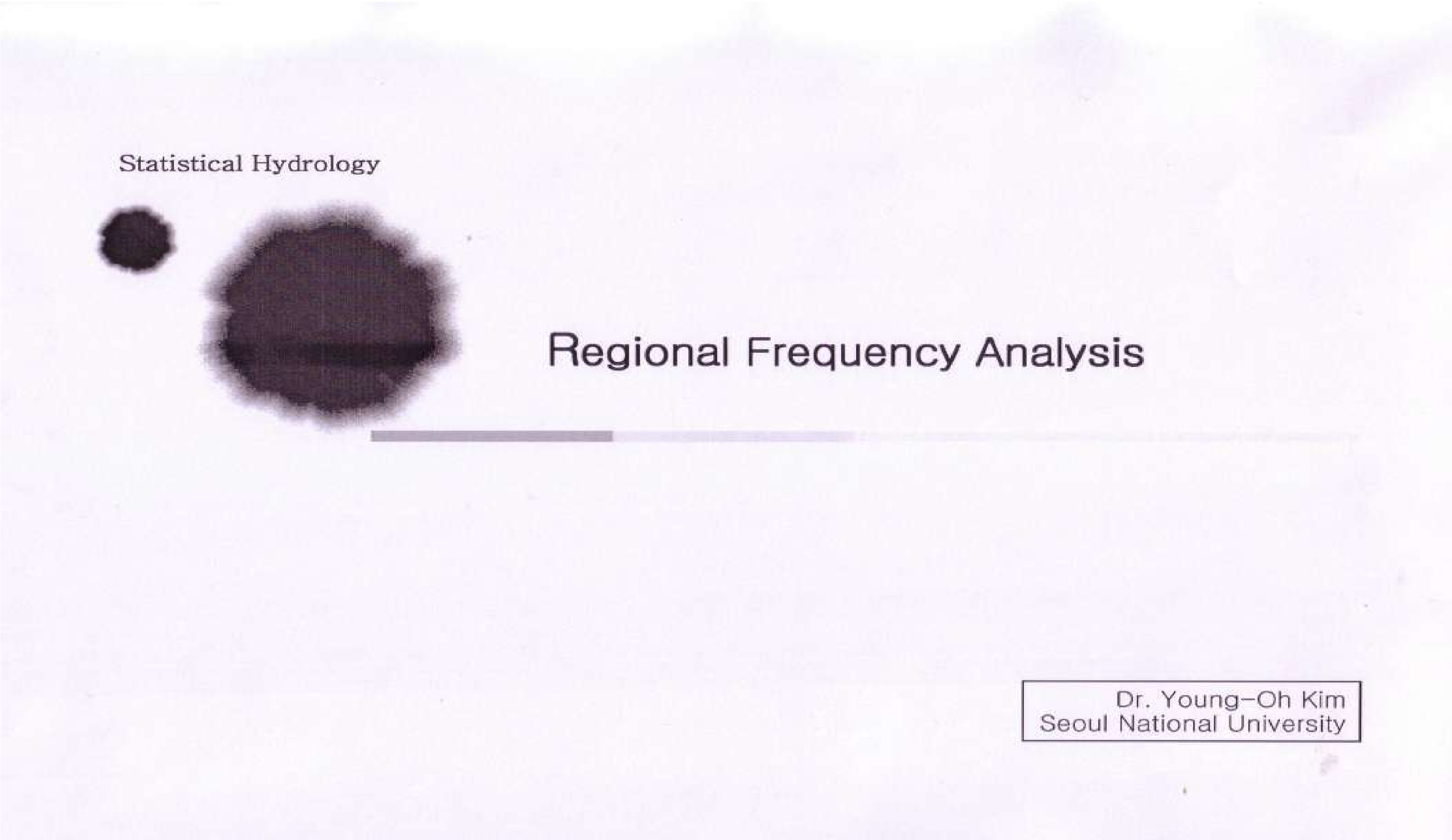
	Significance level		
n	0.10	0.05	0.01
10	0.9347	0.9180	0.8804
15	0.9506	0.9383	0.9110
20	0.9600	0.9503	0.9290
30	0.9707	0.9639	0.9490
40	0.9767	0.9715	0.9597
50	0.9807	0.9764	0.9664
60	0.9835	0.9799	0.9710
75	0.9865	0.9835	0.9757
100	0.9893	0.9870	0.9812
300	0.99602	0.99525	0.99354
1000	0.99854	0.99824	0.99755

Source: Refs. 101, 152, 153. Used with permission.

TABLE 18.3.4 Lower Critical Values of the Probability Plot Correlation Test Statistic for the Gumbel and Two-Parameter Weibull Distributions Using $p_i = (i - 0.44)/(n + 0.12)$

	Significance level		
n	0.10	0.05	0.01
10	0.9260	0.9084	0.8630
20	0.9517	0.9390	0.9060
30	0.9622	0.9526	0.9191
40	0.9689	0.9594	0.9286
50	0.9729	0.9646	0.9389
60	0.9760	0.9685	0.9467
70	0.9787	0.9720	0.9506
80	0.9804	0.9747	0.9525
100	0.9831	0.9779	0.9596
300	0.9925	0.9902	0.9819
1000	0.99708	0.99622	0.99334

Source: Refs. 152, 153. See also Table 18.3.2.



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