

Introduction to Electromagnetism

Solution of Electrostatic Problems

(4-1, 4-2, 4-3)

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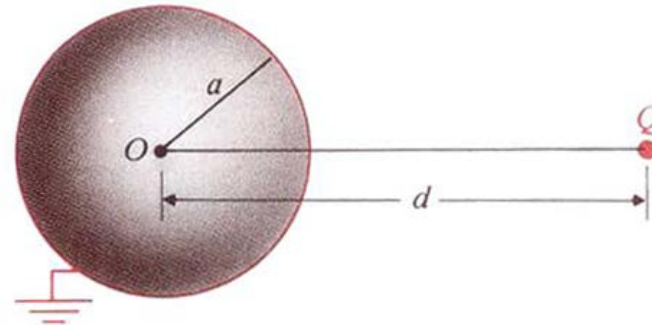
Electrostatic problems

How to deal with the electrostatics problems?

- Electric potential
- Electric field intensity
- Charge distribution

Boundary-value problems:

- Conductors
- Method of images



(a) Point charge and grounded conducting sphere.

D. K. Cheng, *Field and Wave Electromagnetics*, 2nd ed., Addison-Wesley, 1989.

Poisson's and Laplace's Equations (1)

Master equations for electrostatics:

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{E} = 0$$

$$\rightarrow \mathbf{E} = -\nabla V$$

For a linear and isotropic medium:

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\rightarrow \nabla \cdot (\epsilon \mathbf{E}) = \rho$$

$$\rightarrow \nabla \cdot (\epsilon \nabla V) = -\rho$$

For a simple medium (**linear/isotropic/homogeneous**):

$$\rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon}} \quad \rightarrow \text{Poisson's equation}$$

∇^2 : \rightarrow *Laplacian operator:*
"the divergence of the gradient of"

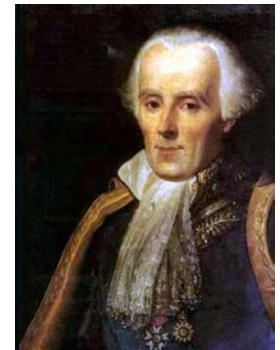


Simeon D. Poisson
(1781–1842)

Poisson's and Laplace's Equations (2)

In Cartesian coordinates:

$$\begin{aligned}\nabla^2 V &= \nabla \cdot \nabla V \\ &= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z} \right) \\ &\rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon}} \quad (\text{V/m}^2)\end{aligned}$$



Pierre-Simon Laplace
(1749–1827)

In cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical coordinates:

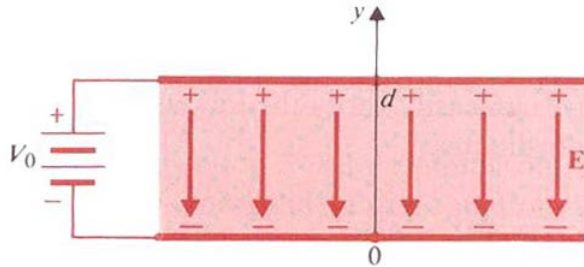
$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

Laplace's equation:

$$\nabla^2 V = 0 \quad \leftarrow \text{No free charge } (\rho = 0)$$

Example 4-1

A parallel-plate capacitor:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Between the plates:

$$\nabla^2 V = 0$$

$$\rightarrow \frac{d^2 V}{dy^2} = 0$$

$$\rightarrow V = C_1 y + C_2$$

Boundary conditions:

$$V(y=0) = 0, \quad V(y=d) = V_0$$

$$\rightarrow V = \frac{V_0}{d} y$$

Electric field:

$$\rightarrow \mathbf{E} = -\nabla V = -\mathbf{a}_y \frac{dV}{dy} = -\mathbf{a}_y \frac{V_0}{d}$$

Surface charge density:

$$\rightarrow E_n = \mathbf{a}_n \cdot \mathbf{E} = \frac{\rho_s}{\epsilon} \quad \rightarrow \rho_{su} = \frac{\epsilon V_0}{d}, \quad \rho_{sl} = -\frac{\epsilon V_0}{d}$$

Uniqueness of Electrostatic Solutions

Uniqueness theorem:

A solution of Poisson's equation (or Laplace's equation) that satisfies the given boundary conditions is a unique solution.

Proof:

Let us suppose that there are two solutions:

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon}, \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon}$$

$$V_d = V_1 - V_2 \quad \rightarrow \quad \nabla^2 V_d = 0$$

← Boundary conditions: $V_d = 0$

$$\rightarrow \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2$$

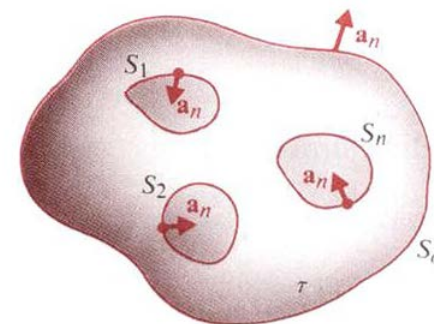
$$\rightarrow \oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n ds = \int_{\tau} |\nabla V_d|^2 dv \quad \rightarrow \quad \int_{\tau} |\nabla V_d|^2 dv = 0$$

$$S = \{S_0, S_1, \dots, S_n\}$$

$$\rightarrow V_d = \text{const.}$$

$$= 0$$

∴ V_1 and V_2 must be identical.



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Bare Copper Wires

