

Introduction to Electromagnetism

Solution of Electrostatic Problems

(4-4)

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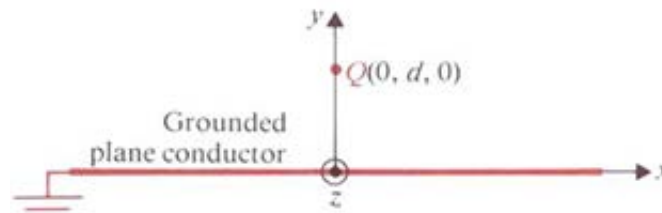
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Method of Images

Boundary conditions for electrostatic problems:

Replaced by appropriate image (equivalent) charges

Example: Point charge and grounded plane conductor



(a) Physical arrangement.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow V(x, y, z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + (y-d)^2 + z^2}} + \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s}{R_1} ds$$

How to solve?

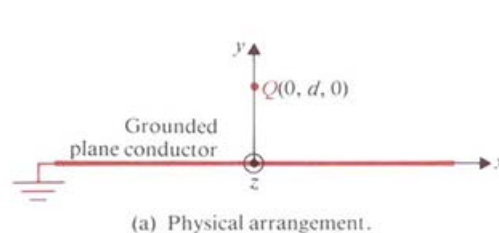
Point Charge and Grounded Plane Conductor

Laplace's equation:

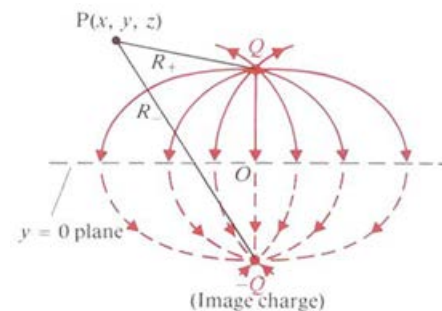
$$\rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Boundary conditions:

1. $V(x, 0, z) = 0$
2. $V \rightarrow \frac{Q}{4\pi\epsilon_0 R}$, as $R \rightarrow 0$
3. $V \rightarrow 0$, as $R \rightarrow \pm\infty$
4. $V(x, y, z) = V(-x, y, z)$,
 $V(x, y, z) = V(x, y, -z)$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.



(b) Image charge and field lines.

Recall for a point charge:

$$\rightarrow \nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

Method of images:

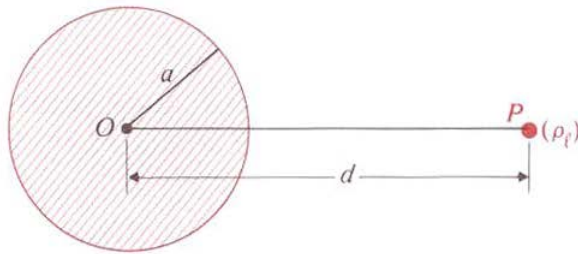
$$\rightarrow V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_+} - \frac{1}{R_-} \right)$$

← Superposition of the solutions of Laplace's eq. satisfying the BCs

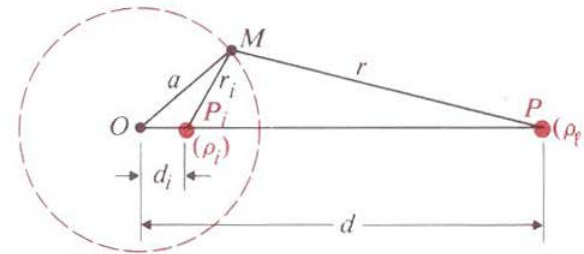
$$R_+ = [x^2 + (y - d)^2 + z^2]^{1/2}, R_- = [x^2 + (y + d)^2 + z^2]^{1/2}$$

Valid solution for $y < 0$?

Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder.



(b) Line charge and its image.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Electric potential from a line charge:

$$V = -\int_{r_0}^r E_r dr = -\frac{\rho_l}{2\pi\epsilon_0} \int_{r_0}^r \frac{1}{r} dr = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

Trial image charge:

$$\rightarrow \rho_i = -\rho_l \quad \text{at } d_i$$

$$\rightarrow V_M = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r} - \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r_i} = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_i}{r}$$

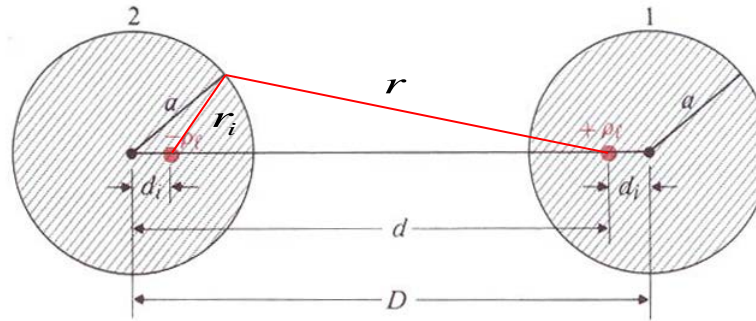
*Constant
at any M
on the surface*

→ Triangles OMP_i and OPM must be similar!

$$\rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} \quad \rightarrow \boxed{d_i = \frac{a^2}{d}}$$

Two-Wire Line of a Radius

Example 4-4: Capacitance per unit length of a two-wire transmission line



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Recall:

$$V_M = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_i}{r}$$

$$\rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d}$$

By image charges:

$$V_2 = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{a}{d}, \quad V_1 = -\frac{\rho_l}{2\pi\epsilon_0} \ln \frac{a}{d}$$

Capacitance per unit length:

$$\rightarrow C = \frac{\rho_l}{V_1 - V_2} = \frac{\pi\epsilon_0}{\ln(d/a)}$$

$$= \frac{\pi\epsilon_0}{\ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]} = \frac{\pi\epsilon_0}{\cosh^{-1}(D/2a)}$$

$$\ln[x + \sqrt{x^2 - 1}] = \cosh^{-1} x$$

$$\rightarrow d = D - d_i = D - \frac{a^2}{d}$$

$$\rightarrow d = \frac{1}{2} (D + \sqrt{D^2 - 4a^2})$$

Two-Wire Line of Different Radii (1)

Potential at any point $P(x,y)$:

$$\rightarrow V_P = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$$

Equipotential line:

$$\rightarrow \frac{r_2}{r_1} = k \text{ (constant)}$$

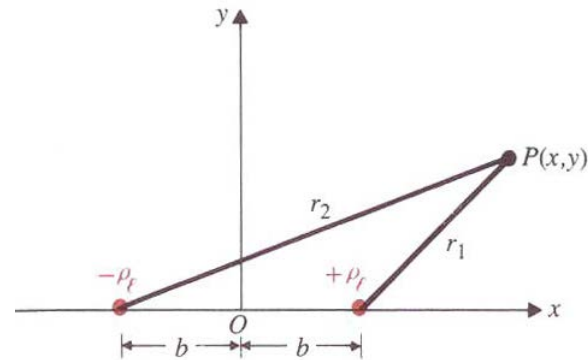
$$\rightarrow \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k$$

$$\rightarrow \left(x - \frac{k^2 + 1}{k^2 - 1}b\right)^2 + y^2 = \left(\frac{2k}{k^2 - 1}b\right)^2$$

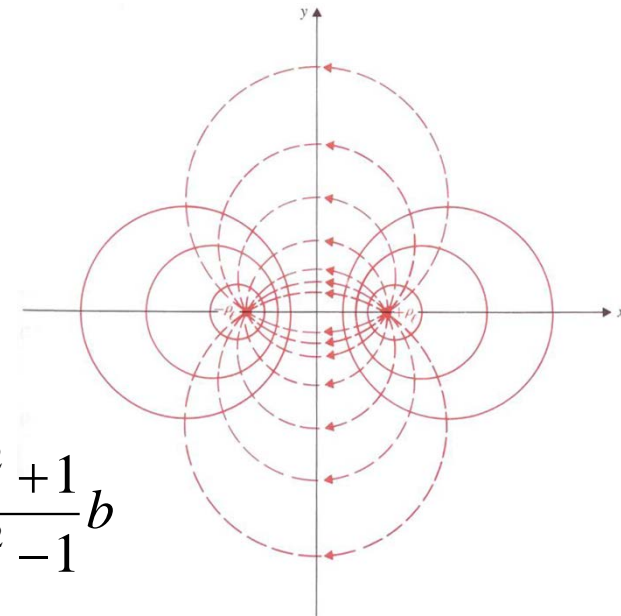
Radius: $\rightarrow a = \left| \frac{2k}{k^2 - 1}b \right|$ Center: $\rightarrow c = \frac{k^2 + 1}{k^2 - 1}b$

$$\rightarrow c^2 = a^2 + b^2$$

$$\rightarrow +\rho_l \text{ for } k > 1, \quad -\rho_l \text{ for } k < 1$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Two-Wire Line of Different Radii (2)

Recall:

Two cylindrical conductors separated by a distance D : $\rightarrow c^2 = a^2 + b^2$

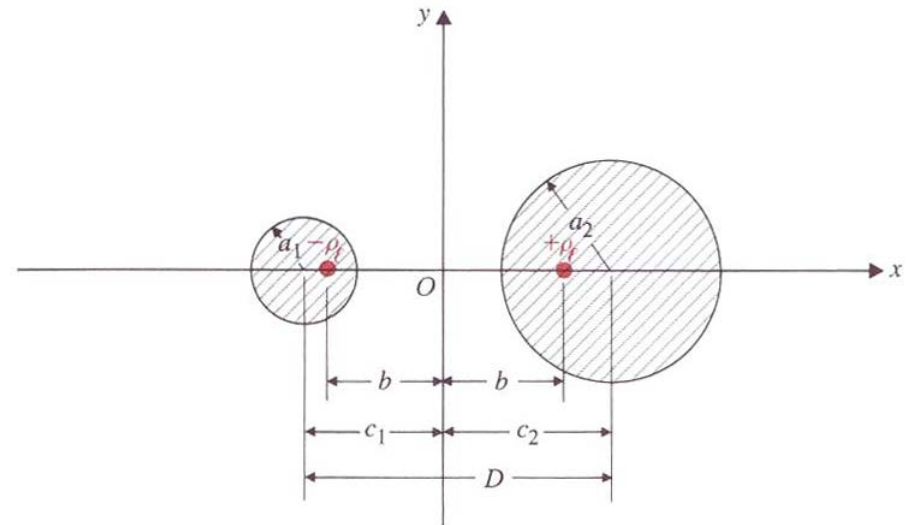
$$\rightarrow b^2 = c_1^2 - a_1^2$$

$$\rightarrow b^2 = c_2^2 - a_2^2$$

$$\rightarrow c_1 + c_2 = D$$

$$\rightarrow c_1 = \frac{1}{2D} (D^2 + a_1^2 - a_2^2)$$

$$\rightarrow c_2 = \frac{1}{2D} (D^2 + a_2^2 - a_1^2)$$



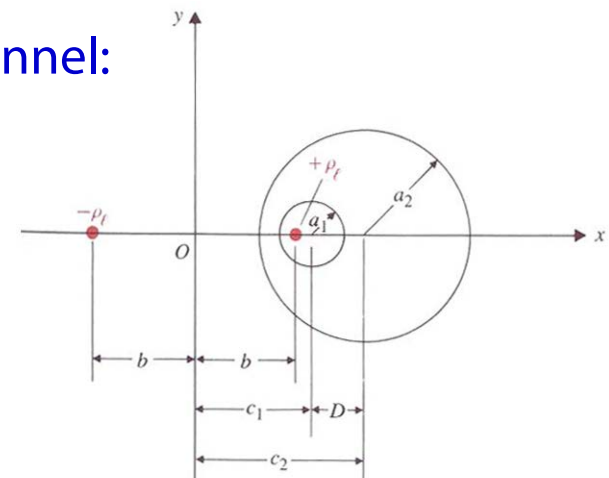
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

An off-center conductor inside a cylindrical tunnel:

$$\rightarrow c_2 - c_1 = D$$

$$\rightarrow c_1 = \frac{1}{2D} (a_2^2 - a_1^2 - D^2)$$

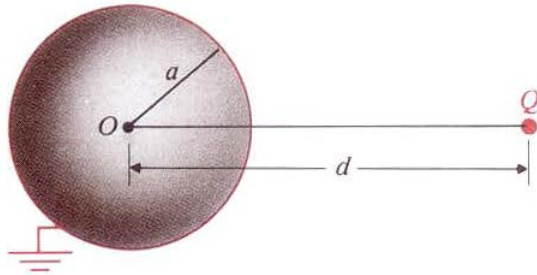
$$\rightarrow c_2 = \frac{1}{2D} (a_2^2 - a_1^2 + D^2)$$



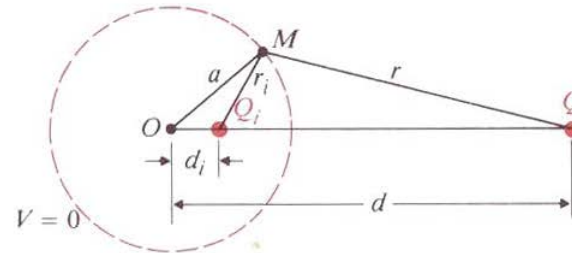
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Point Charge and Conducting Sphere

Potential on the spherical surface grounded:



(a) Point charge and grounded conducting sphere.



(b) Point charge and its image.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow V_M = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{Q_i}{r_i} \right) = 0 \quad \leftarrow Q_i = -Q?$$

$$\rightarrow \frac{r_i}{r} = -\frac{Q_i}{Q} = \frac{a}{d}$$

Recall:

$$\rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d}$$

$$\rightarrow \boxed{Q_i = -\frac{a}{d}Q} \quad \rightarrow \boxed{d_i = \frac{a^2}{d}}$$

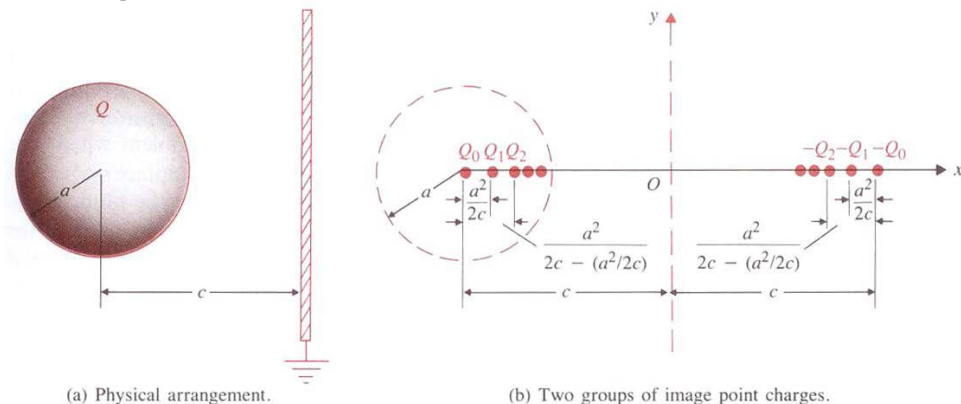
For a conducting sphere that is electrically neutral and is not grounded:

Additional image charge to the center: $\rightarrow Q' = -Q_i = \frac{a}{d}Q$

Charged Sphere and Grounded Plane

Boundary conditions:

- (1) Equipotential on the conducting sphere
- (2) Zero potential at $x = 0$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Image charge for the B.C. at $x = 0$

$$\rightarrow Q_0 \text{ at } (-c_0, 0) \quad \rightarrow -Q_0 \text{ at } (+c_0, 0)$$

Image charge for the B.C. on the sphere

$$\rightarrow Q_1 \text{ at } (-c_1, 0) \quad \rightarrow -Q_1 \text{ at } (+c_1, 0)$$

$$\rightarrow Q_2 \text{ at } (-c_2, 0) \quad \rightarrow -Q_2 \text{ at } (+c_2, 0)$$

$$\rightarrow Q = \sum_{k=0}^{\infty} Q_k = Q_0 \left(1 + \alpha + \frac{\alpha^2}{1 - \alpha^2} + \dots \right) \quad \leftarrow \alpha = \frac{a}{2c}$$

Capacitance: $\rightarrow V_0 = \frac{Q_0}{4\pi\epsilon_0 a} \quad \rightarrow C = \frac{Q}{V_0} = 4\pi\epsilon_0 a \left(1 + \alpha + \frac{\alpha^2}{1 - \alpha^2} + \dots \right)$