Chapter9. Amplification of light. Lasers
Part 2

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9.2 Stimulated emission and thermal radiation

Einstein in 1917 first introduced the concept of stimulated emission by atomic systems

Consider a molecule with two energy levels 1 and 2.

$$W_{1\to 2} = B_{12} I(\omega_{fi})$$

The rate of transition B_{12} induced by light of frequency ν

$$B_{12} = \frac{2\pi}{3\hbar^2}\left|<\psi_2\left|\vec{d}\right|\psi_1>\right|^2$$

Number of stimulated upward transitions (absorptions): $B_{12}\, u_\nu\, N_1$

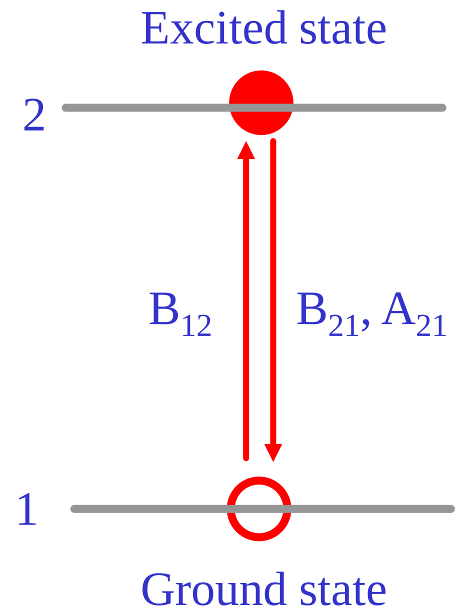
Number of stimulated downward transitions (emissions): $B_{21}\, u_\nu\, N_2$

N_1 (N_2)= population of the level 1 (level 2).

A_{21} = spontaneous emission from the level 1 to the level 2.

At equilibrium the rate of transition $1 \leftrightarrow 2$ must be the same: $N_2 A_{21} + N_2 B_{21} u_\nu = N_1 B_{12} u_\nu$

$$\frac{N_1}{N_2} = \frac{B_{21} I(\nu) + A_{21}}{B_{12} I(\nu)} = 1 + \frac{A_{21}}{B_{12} I(\nu)} \qquad (1)$$



But at equilibrium, N_1 and N_2 in each level is given by the Boltzmann equation

$$\frac{N_1}{N_2} = \exp\left(\frac{E_2 - E_1}{kT}\right) = \exp\left(\frac{h\nu}{kT}\right) \qquad (2)$$

Equation 1 and 2 must be equal. Therefore, we can obtain $I(\nu)$.

$$I(\nu) = \frac{A_{21}/B_{21}}{(B_{12}N_1/B_{21}N_2) - 1} = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})e^{h\nu/kT} - 1} \qquad (3)$$

9.2 Stimulated emission and thermal radiation

At equilibrium $I(\nu)$ is the radiation density of a black body at temperature T: Planck's radiation law.

$$I(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \qquad (4)$$

By comparing the like terms in equations (3) and (4), the Einstein coefficients can be obtained to give

$$\boxed{B_{12} = B_{21}} \qquad (5)$$

The stimulated emission process occurs at the same rate as the absorption process. And we can get

$$\boxed{\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} = \frac{8\pi h}{\lambda^3}} \qquad (6)$$

The rate of spontaneous to stimulated emission increases as ν^3. B_{12} can be measured from the absorption spectrum.

The radiative lifetime:

$$\boxed{\tau_r = \frac{1}{A_{21}} = \frac{c^3}{8\pi h \nu^3 B_{12}} = \frac{\lambda^3}{8\pi h B_{12}}}$$

The stronger the absorption, the shorter the radiative lifetime. And the radiative lifetime decreases with increasing frequency (or decreasing wavelength).

$$\frac{\text{stimulated emission}}{\text{spontaneous emission}} = \frac{B_{21} u(\nu)}{A_{21}} = \frac{1}{e^{h\nu/kT} - 1}$$

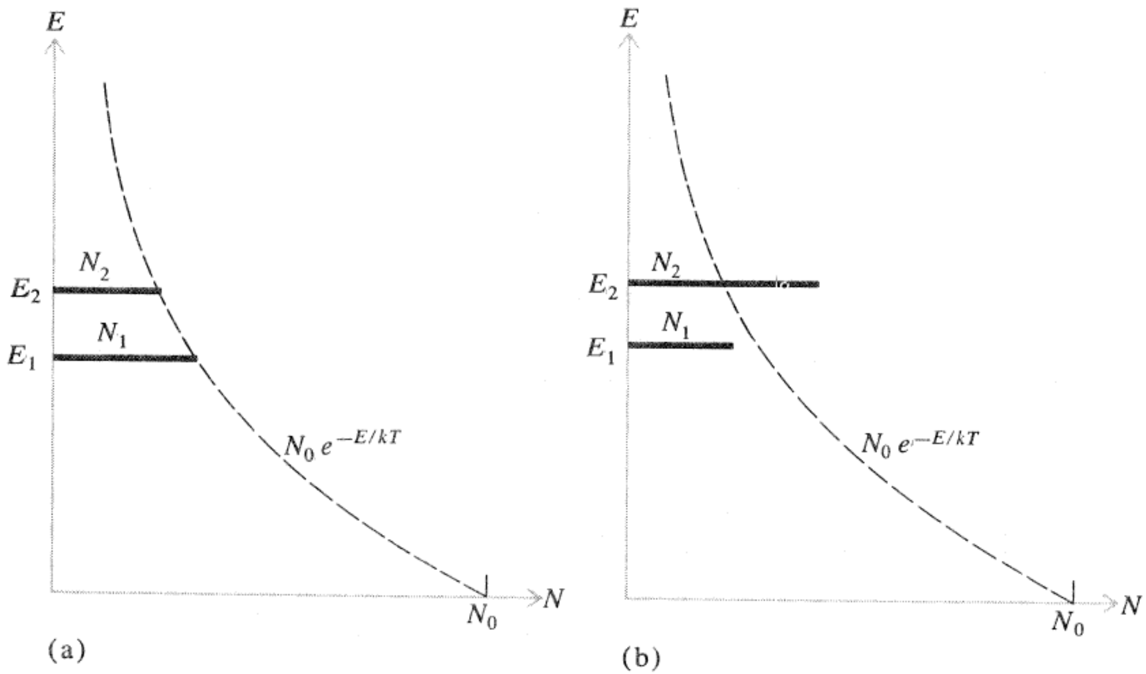
9.3 Amplification in a medium

Stimulated absorption rate ~ $B_{12}\ N_1$

Stimulated emission rate ~ $B_{21}\ N_2$

Since $B_{12} = B_{21}$, the rate of stimulated downward transitions will exceed that of upward transition if $N_2 > N_1$ (**population inversion**).

- If a population inversion exists, a light beam will be ***amplified*** because the gain due to the induced emission exceeds the loss due to absorption.
- The induced radiation is ***coherent*** with the primary beam.
- The act of stimulated emission of a single atom results in the addition of a photon to the particular mode that causes the stimulated emission.



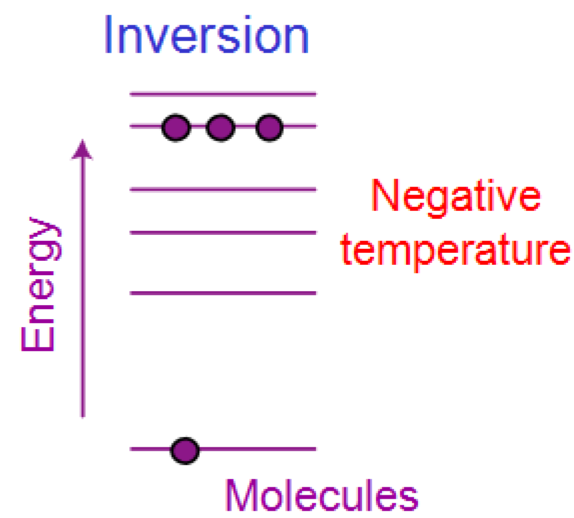


Figure 9.1. Graphs of the population densities of two levels of a system. (a) Normal or Boltzmann distribution; (b) inverted distribution.

Gain constant

Suppose a parallel beam of light propagates through a medium in which there is a population inversion.
For a collimated beam, the spectral energy density u_ν is related to the spectral irradiance I_ν in the frequency interval (ν, $\nu+\Delta\nu$).

$$u_\nu \Delta\nu = \frac{I_\nu \Delta\nu}{c}$$

The rate of upward transitions: $B_{12} u_\nu \Delta N_1 = B_{12} (\frac{I_\nu}{c}) \Delta N_1$

The rate of induced downward transitions : $B_{21} u_\nu \Delta N_2 = B_{21} (\frac{I_\nu}{c}) \Delta N_2$

The net time rate of change of the spectral energy density in the interval $\Delta\nu$ is given by

$$\frac{d}{dt}(u_\nu \Delta\nu) = h\nu (B_{21}\Delta N_2 - B_{12}\Delta N_1) u_\nu$$

$$dx = c dt$$

$$\frac{dI_\nu}{dx} = \frac{h\nu}{c}(\frac{\Delta N_2}{\Delta\nu} - \frac{\Delta N_1}{\Delta\nu}) B_{21} I_\nu$$

The rate of growth of the beam in the direction of propagation: $I_\nu = I_{0\nu} e^{\alpha_\nu x}$

$$\alpha_\nu = \frac{h\nu}{c}(\frac{\Delta N_2}{\Delta\nu} - \frac{\Delta N_1}{\Delta\nu}) B_{21}$$ gain constant

Gain constant and gain curve

Approximate gain constant at the center of a spectral line:

$$\alpha_{\max} \approx \frac{h\nu}{c\Delta\nu}(N_2 - N_1)B_{21} = \frac{\lambda^2}{8\pi\Delta\nu}(N_2 - N_1)A_{21}$$

If $N_2 > N_1$, $\alpha > 0$ (condition for amplification)

The fraction of atoms whose *x* component of velocity lies between u_x and $u_x + \Delta u_x$:

$$Ce^{-au_x^2}\Delta u_x, \quad C = \sqrt{m/2\pi kT} \text{ and } \quad a = m/2kT$$

Due to the Doppler effect, these atoms will emit or absorb radiation, propagating in the *x* direction, of slightly different frequency ν than the resonance frequency ν_o if the atom when it is at rest.

$$\frac{\nu - \nu_o}{\nu_o} = \frac{u_x}{c}$$

The number of atoms in a given level that can absorb or emit in the frequency interval (ν, $\nu+\Delta\nu$) is given by

$$\Delta N_i = N_i C e^{-\beta(\nu-\nu_o)^2}\frac{c}{\nu_o}\Delta\nu, \quad \beta = mc^2/(2kT\nu_o^2)$$

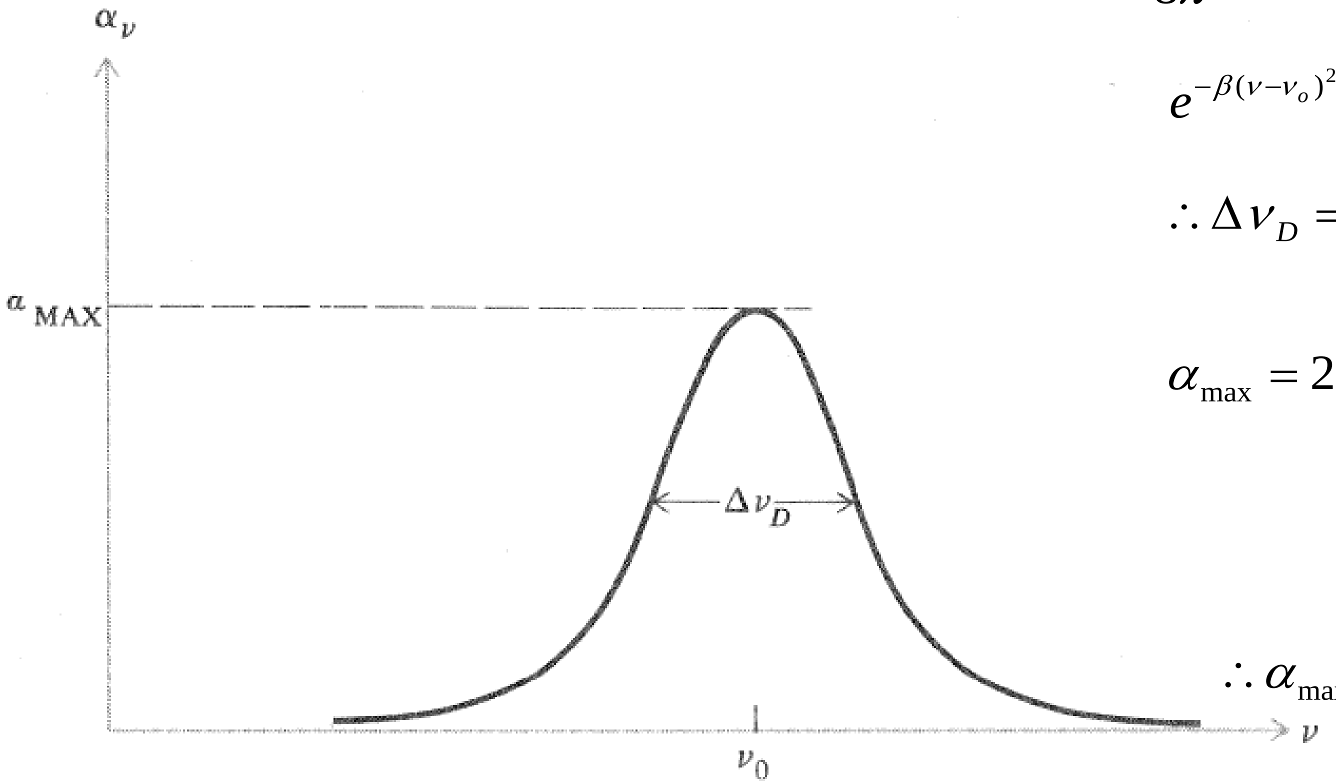
$$\alpha_\nu = Ce^{-\beta(\nu-\nu_o)^2}(N_2 - N_1)hB_{21}$$

Gain curve

The gain for a Doppler-broadened laser transition varies with frequency according to a Gaussian function.

$$\alpha_\nu = Ce^{-\beta(\nu-\nu_o)^2}(N_2 - N_1)hB_{21}$$

$$\alpha_{\max} = C(N_2 - N_1)hB_{21} = C(N_2 - N_1)\frac{\lambda_o^3}{8\pi}A_{21}$$



$$e^{-\beta(\nu-\nu_o)^2} = \frac{1}{2} \rightarrow \nu - \nu_o = \sqrt{\ln 2/\beta}$$

$$\therefore \Delta\nu_D = 2\sqrt{\ln 2/\beta}$$

$$\alpha_{\max} = 2\sqrt{\frac{\ln 2}{\pi}}\frac{\lambda_o^2}{8\pi\Delta\nu_D}(N_2 - N_1)A_{21}$$

$$2\sqrt{\frac{\ln 2}{\pi}} = 0.939$$

$$\therefore \alpha_{\max} \approx \frac{\lambda_o^2}{8\pi\Delta\nu_D}(N_2 - N_1)A_{21}$$

Figure 9.2. Amplification coefficient for a Doppler broadened spectral line.

9.4 Methods of producing a population inversion

(1) Optical pumping or photon excitation, e.g. solid-state lasers such as ruby laser
(2) Electron excitation, e.g. gaseous ion laser such as Ar laser
(3) Inelastic atom-atom collisions, e.g. He-Ne laser
(4) Chemical reactions, e.g. chemical lasers, $H_2 + F_2 \rightarrow 2HF$

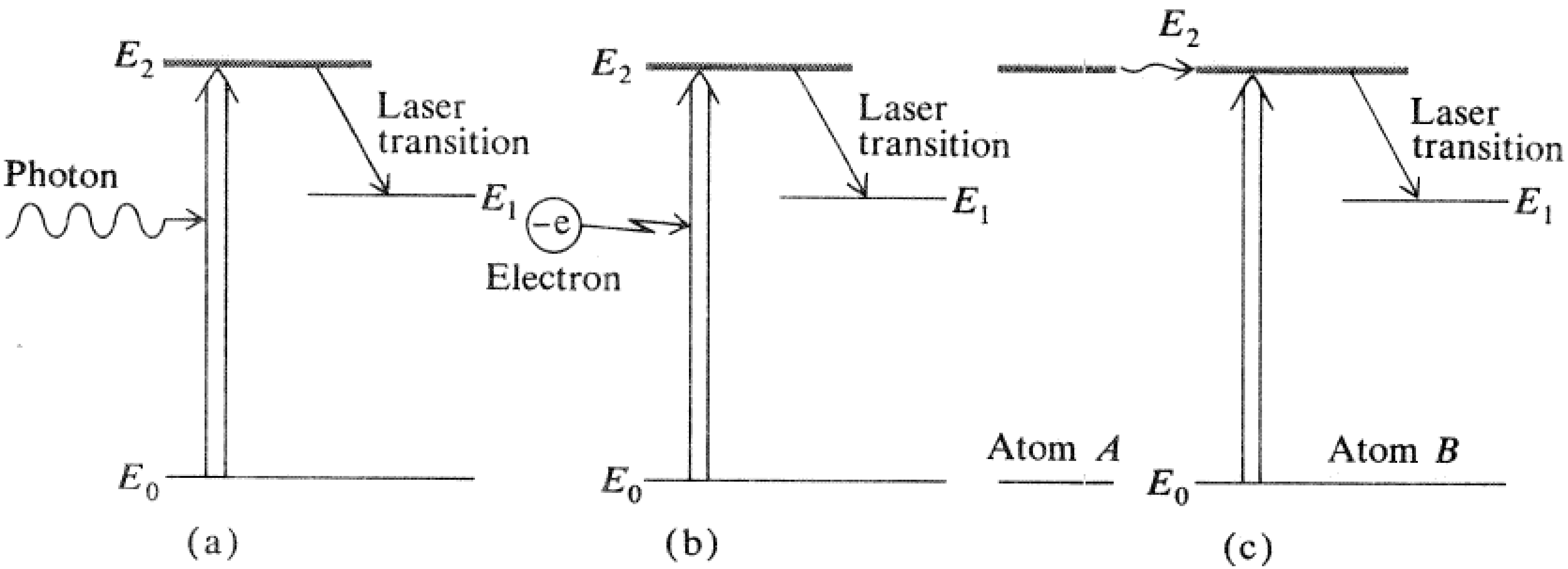
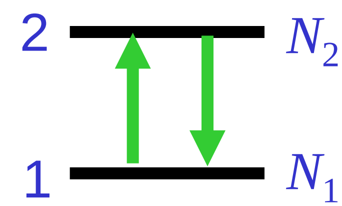


Figure 9.3. Diagrams showing three processes for producing a population inversion. (a) Optical pumping; (b) direct electron excitation; (c) inelastic atom–atom collisions.

Population inversion is impossible in a two-level system

Rate equations for the densities of the two states.



Absorption — Stimulated emission — Spontaneous emission

$$\frac{dN_2}{dt} = BI(N_1 - N_2) - AN_2 \qquad \frac{dN_1}{dt} = BI(N_2 - N_1) + AN_2$$

If the total number of molecules is *N*: $N \equiv N_1 + N_2 \qquad \Delta N \equiv N_1 - N_2$

$$\frac{d\Delta N}{dt} = -2BI\Delta N + 2AN_2$$

$$2N_2 = (N_1 + N_2) - (N_1 - N_2) = N - \Delta N$$

$$\frac{d\Delta N}{dt} = -2BI\Delta N + AN - A\Delta N$$

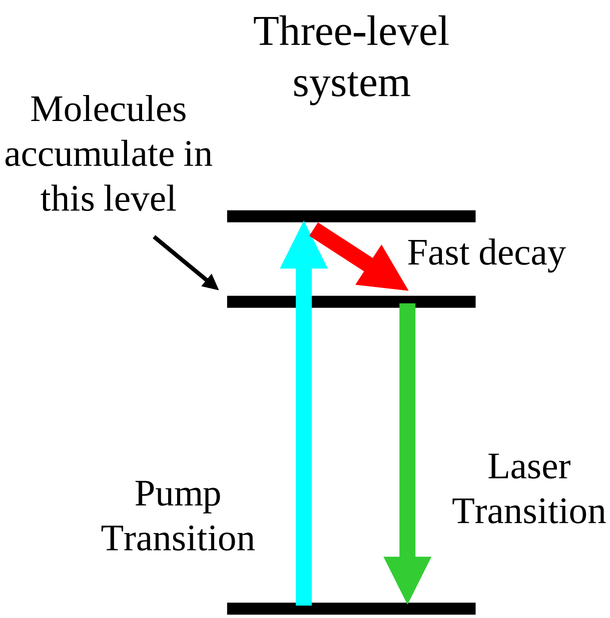
In steady-state: $0 = -2BI\Delta N + AN - A\Delta N \qquad \Rightarrow (A + 2BI)\Delta N = AN$

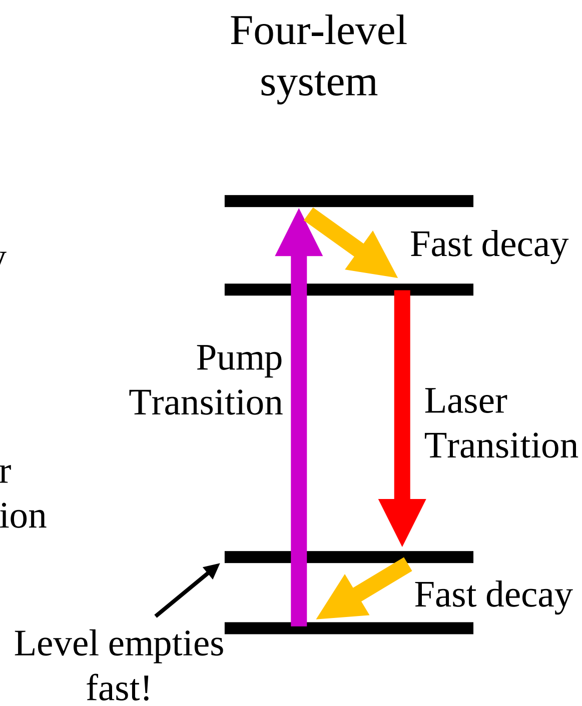
$$\Rightarrow \Delta N = AN/(A + 2BI) \qquad \Rightarrow \Delta N = N/(1 + 2BI/A)$$

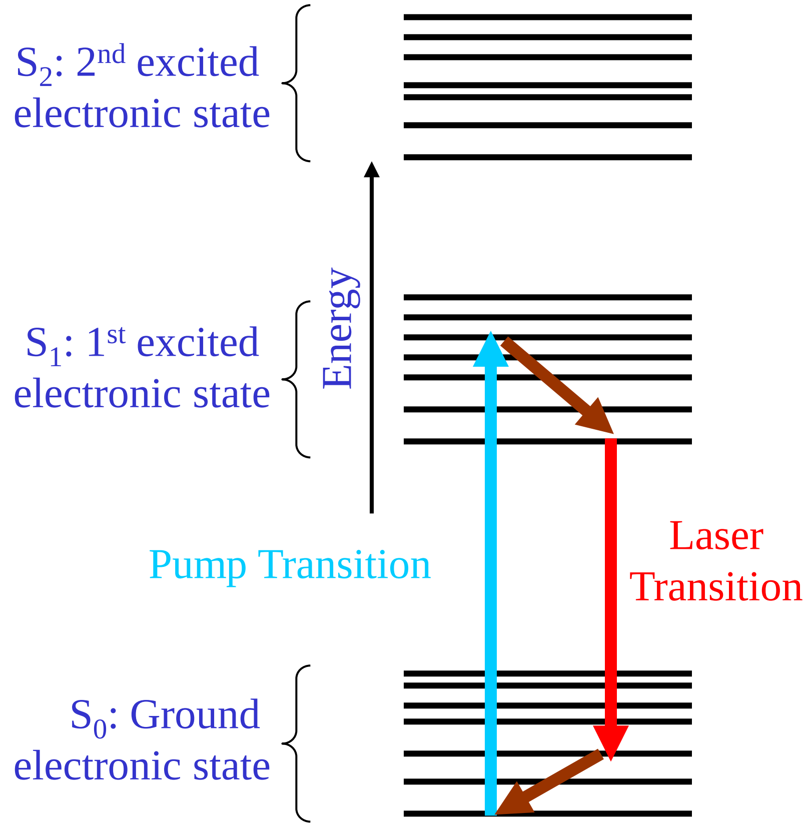
$$\therefore \Delta N = \frac{N}{1 + I/I_{sat}}, \quad I_{sat} = A/2B$$

ΔN is always positive. Thus, it is impossible to achieve an inversion in a two-level system. So we need three-level or four-level systems.

3 and 4-level systems and dye energy levels

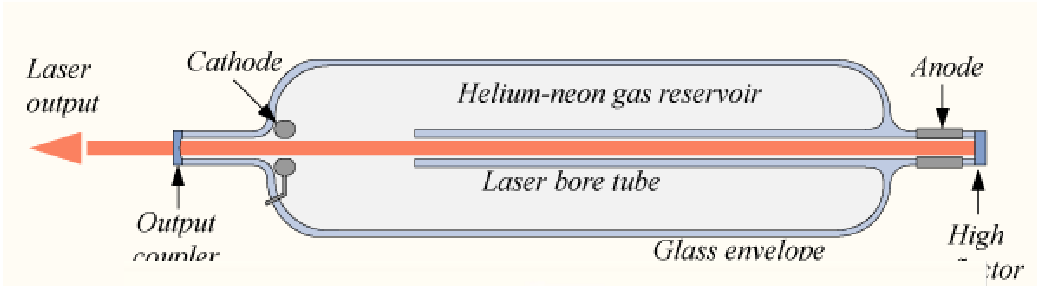


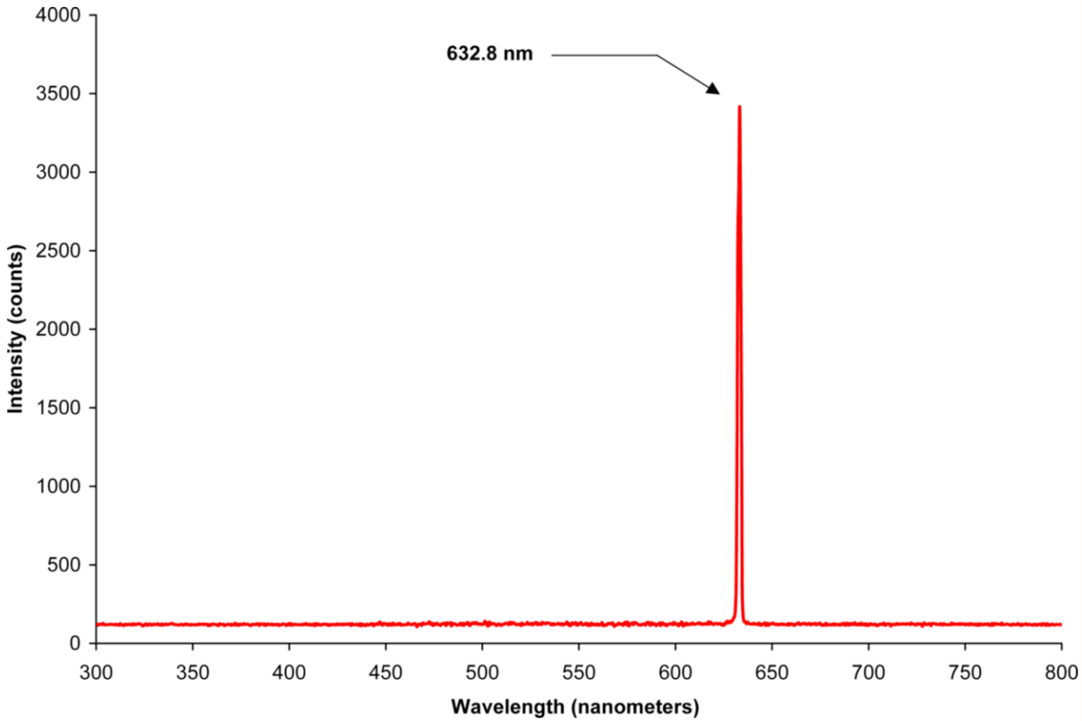


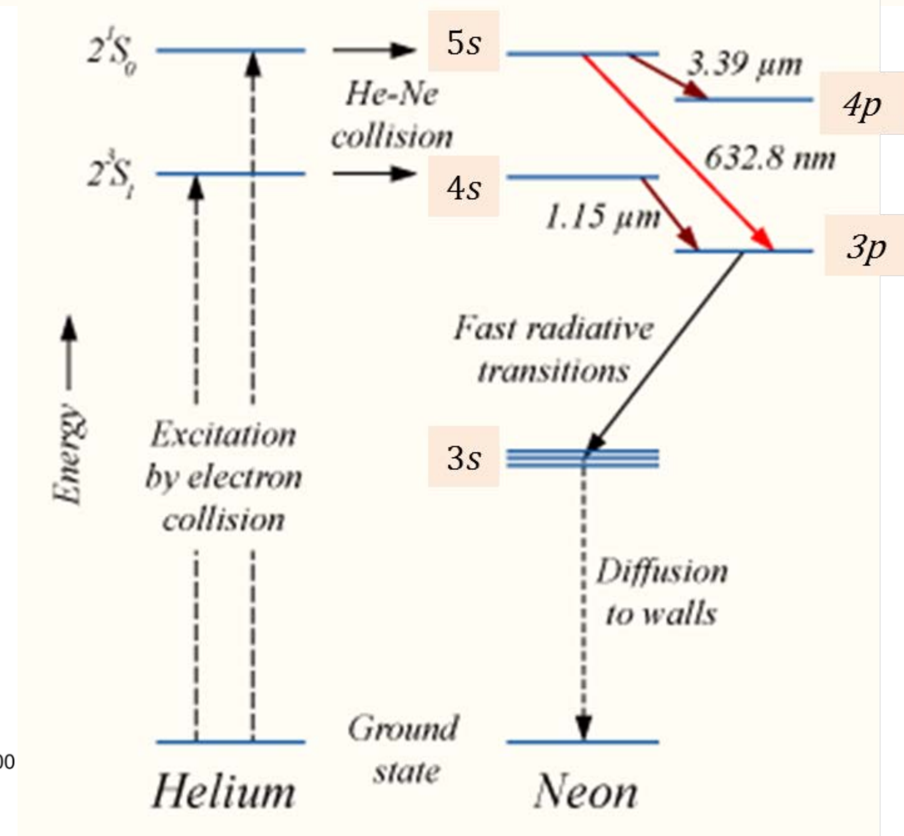


He-Ne laser

A **helium–neon laser** or **HeNe laser**, is a type of gas laser whose gain medium consists of a mixture of helium and neon(10:1) inside of a small bore capillary tube, usually excited by a DC electrical discharge. The best-known and most widely used HeNe laser operates at a wavelength of 632.8 nm



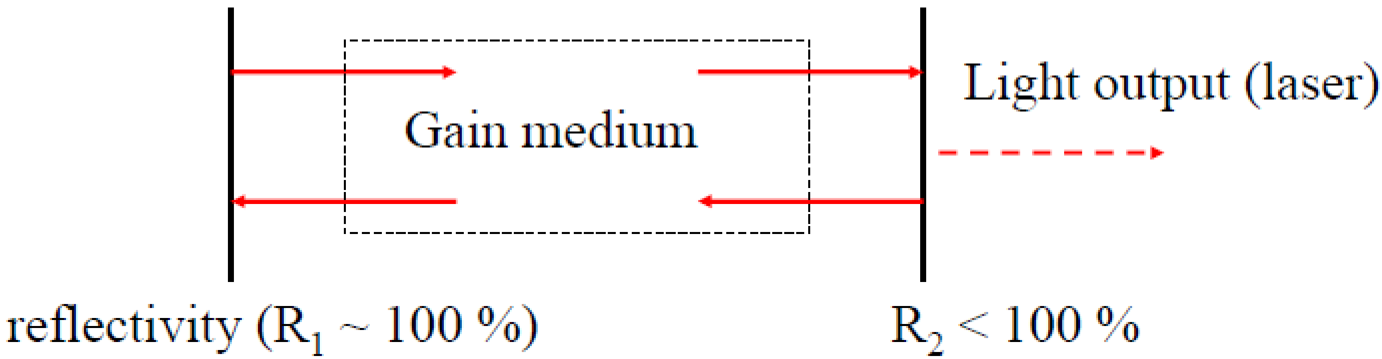




https://en.wikipedia.org/wiki/Helium%E2%80%93neon_laser

9.5 laser oscillation

- In a practical laser device, it is generally necessary to have certain positive optical *feedback in addition to optical* amplification provided by a *gain medium.*
- This requirement can be met by placing the gain medium inan *optical resonator. The optical resonator provides selective* feedback to the amplified optical field.
- In many lasers the optical feedback is provided by placing the *gain medium inside a* ***Fabry-Perot cavity****, formed by using two mirrors or highly reflecting surfaces*



Source: Prof. Andrew W. Poon (HKUST)

9.5 laser oscillation

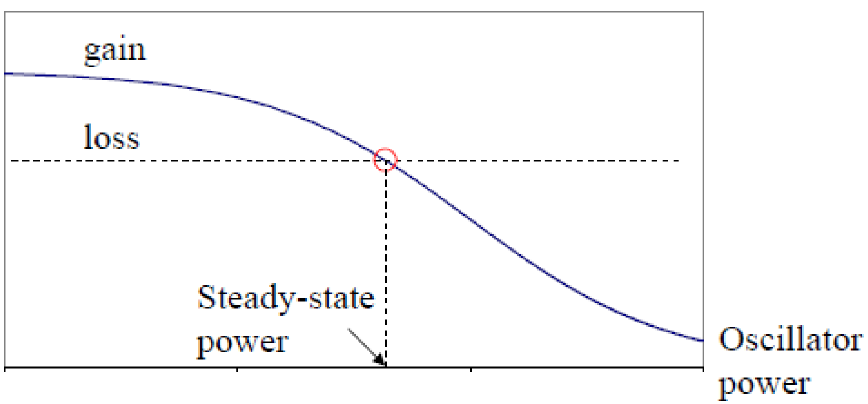
Two conditions must be satisfied for oscillation to occur:

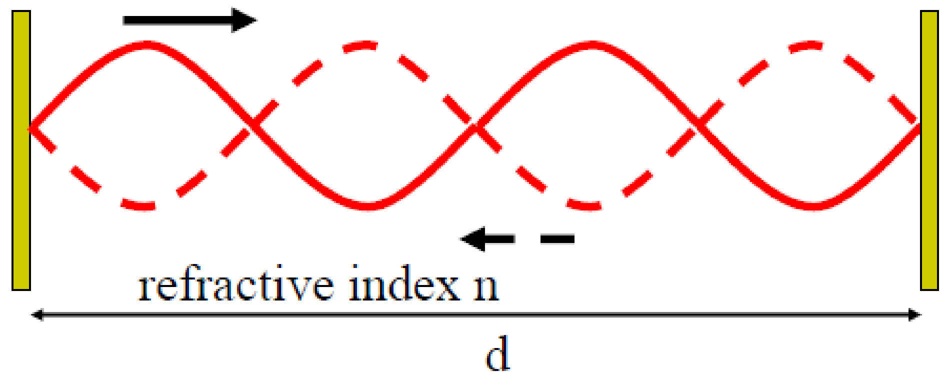
- The amplifier gain must be greater than the loss in the feedback system so that net gain is incurred in a round trip through the feedback loop.

- The total phase shift in a single round trip must be a multiple of 2π so that the feedback input phase matches the phase of the original input.

$$k \cdot 2d = \frac{2\pi n}{\lambda} \cdot 2d = \frac{2\pi n \nu}{c} \cdot 2d = 2\pi q \quad (q = 1,\ 2,\ 3, \ldots)$$

If these conditions are satisfied, the system becomes unstable and oscillation begins.

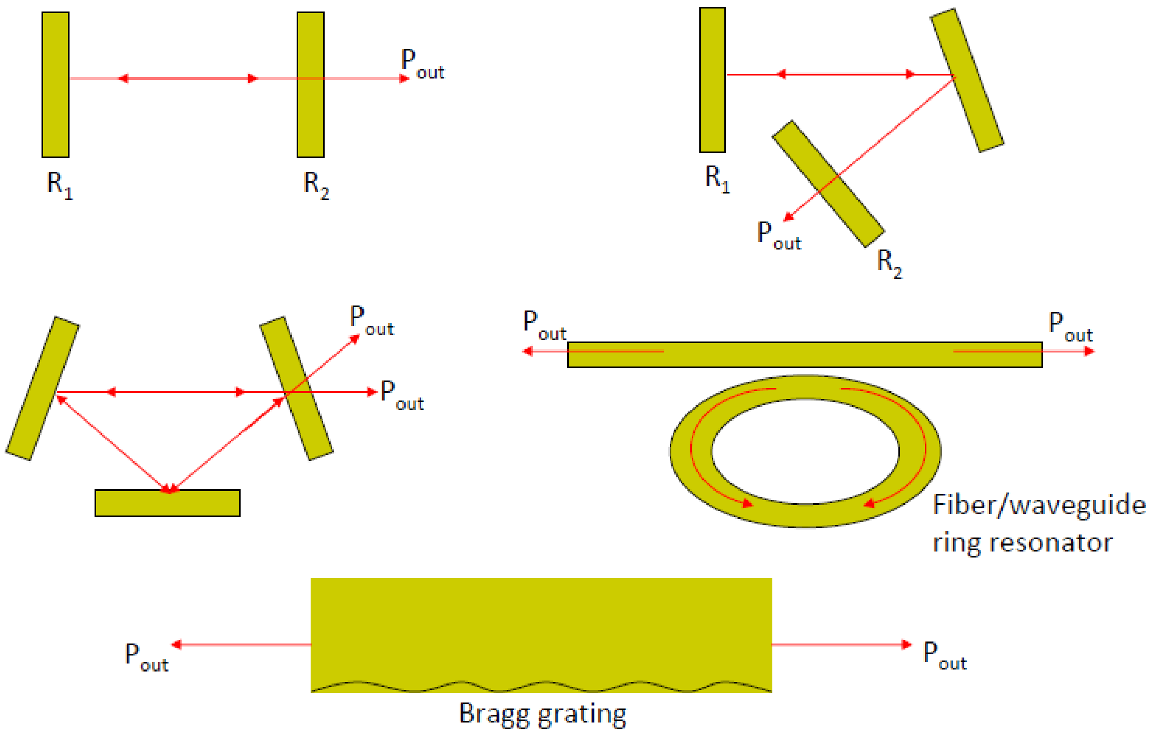




Source: Prof. Andrew W. Poon (HKUST)

Resonant optical cavities

- A linear cavity with two end mirrors is known as a Fabry-Perot cavity because it takes the form of a Fabry-Perot interferometer. In the case of semiconductor diodes, the diode end facets form the two end mirrors.
- A folded cavity can simply be a folded Fabry-Perot cavity with a standing oscillating field.
- The optical cavity can also comprise a distributed Bragg grating with distributed feedback. Distributed Feedback (DFB) diode lasers are the most common single-mode laser diodes for optical communications.



Source: Prof. Andrew W. Poon (HKUST)

9.5 laser oscillation

- The modes along the cavity axis is referred to as ***longitudinal modes***. Many wavelengths may satisfy the resonance condition → ***multimode cavity***

$$\nu_{n+1} - \nu_n = \frac{c}{2nd}$$

- If extremely high spectral purity is needed, it is possible to obtain oscillation on one mode by suitable selection of laser parameters.

- The inherent linewidth in this case is determined mainly by the quality factor Q of the laser resonator.

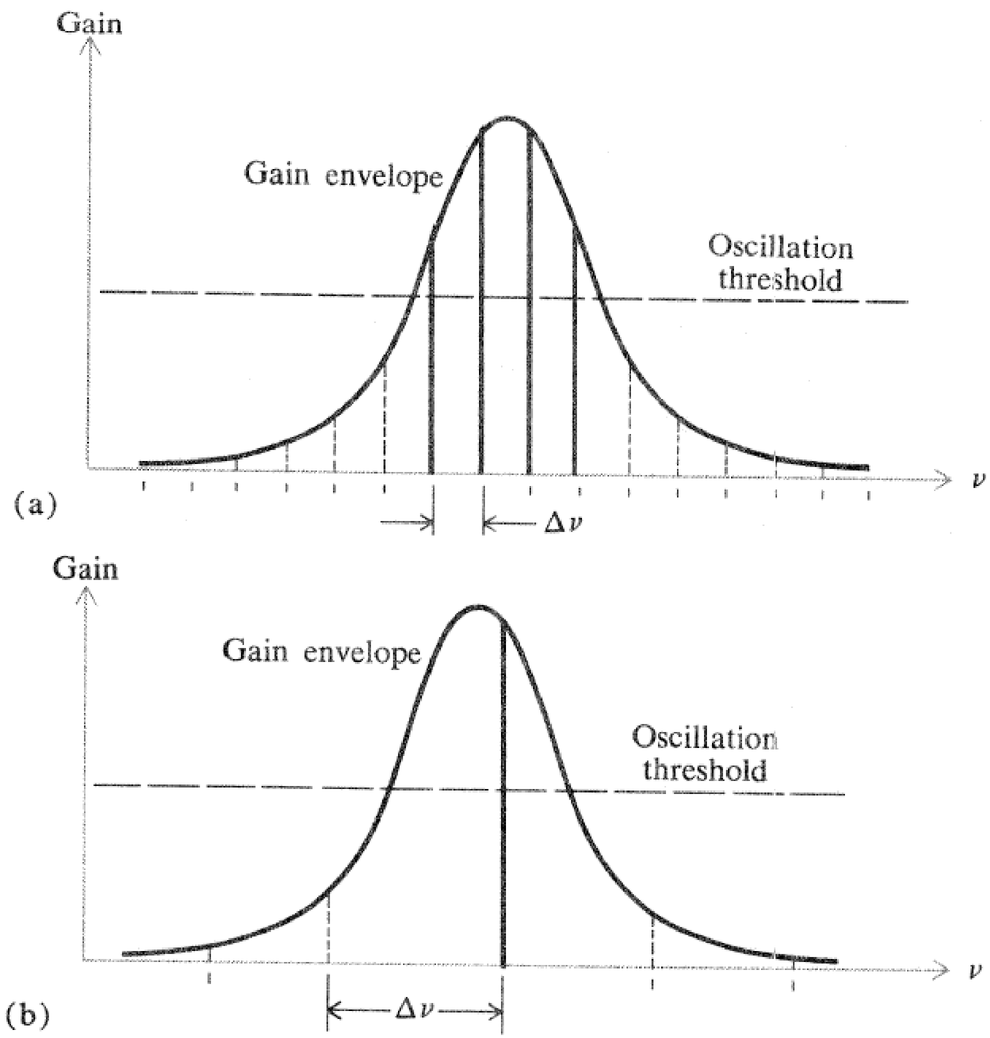
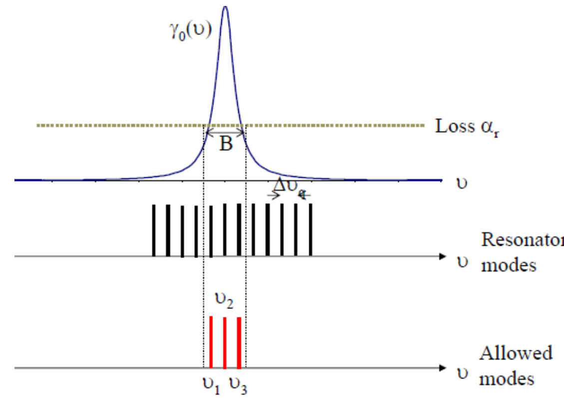


Figure 9.5. Oscillation frequencies in a laser. (a) Four longitudinal modes; (b) one mode.

Threshold condition for laser oscillation

In order for the laser to oscillate, the gain must equal or exceed the loss.

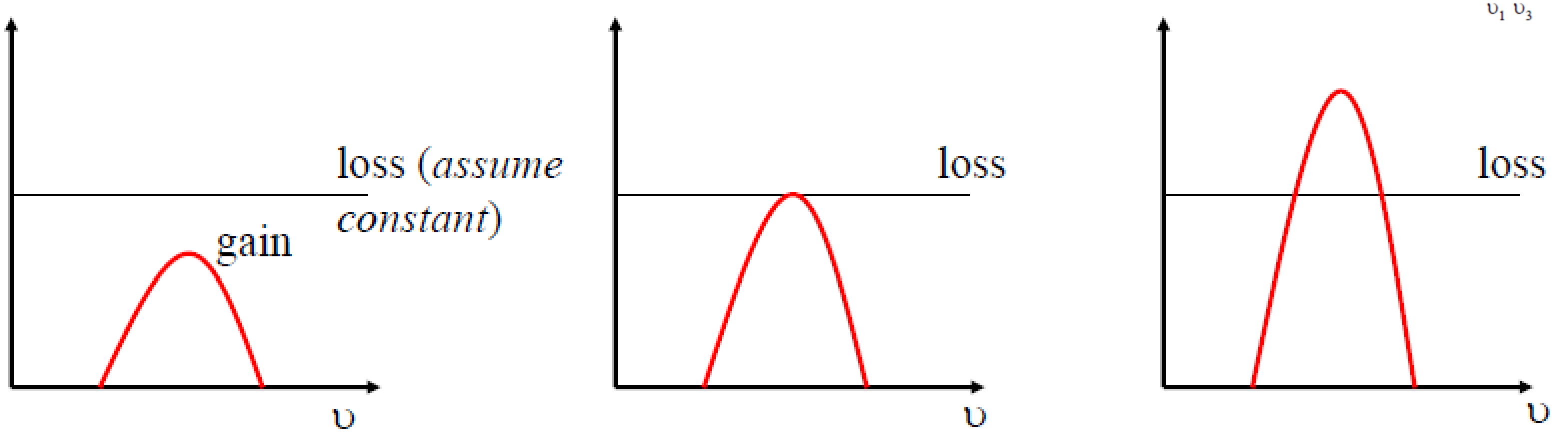


$$I_\nu = I_{0\nu} e^{\alpha_\nu x}$$

$$I_\nu - I_{0\nu} \geq \delta I_\nu$$

$$e^{\alpha_\nu 2l} - 1 \geq \delta \quad (\mathrm{l} = \text{active length of the amplifying medium})$$

$$\text{If } \alpha_\nu 2l << 1, \qquad \alpha_\nu 2l \geq \delta$$



gain < loss — **Sub-threshold (incoherent emission)**

gain = loss — **Threshold (oscillation begins, start to emit coherent light)**

gain > loss — **Above-threshold (increase in coherent output power)**

Source: Prof. Andrew W. Poon (HKUST)

Hole burning

Spectral hole burning is the frequency selective bleaching of the absorption spectrum of a material, which leads to an increased transmission (a "spectral hole") at the selected frequency.

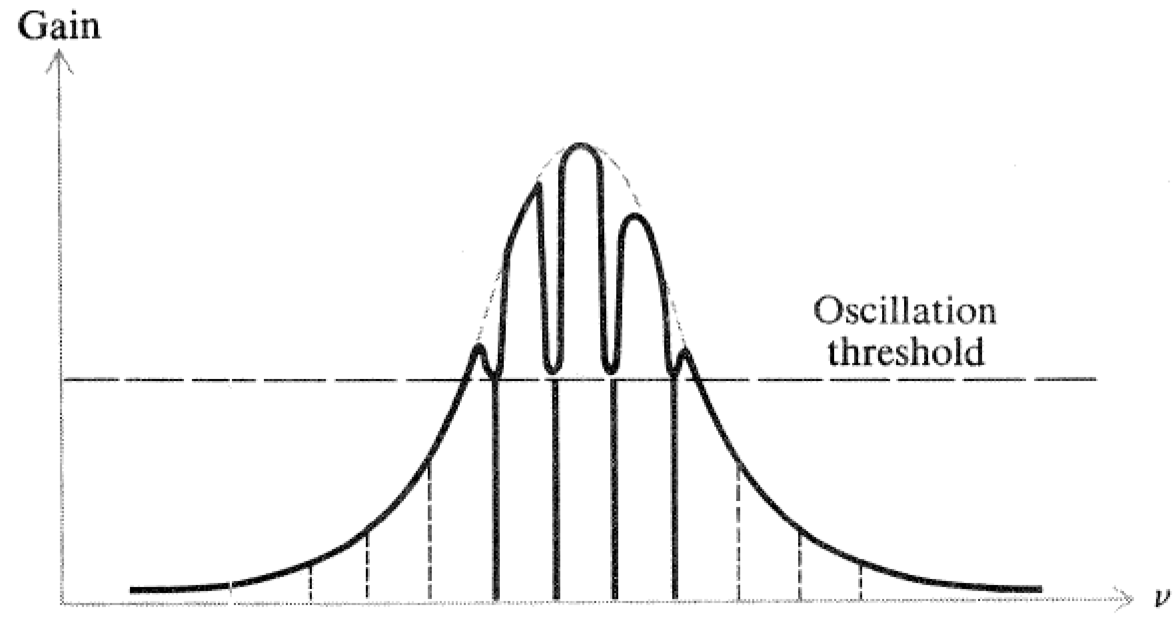


Figure 9.6. Hole burning of the gain envelope in a laser.

9.6 Optical-resonator theory

In a laser resonator, part of energy spills around the reflecting mirrors and is lost (***diffusion loss*** of the resonator). If $U(x,y)$ and $U'(x',y')$ represent the complex amplitudes of the radiation over the mirror surfaces, then by applying the Fresnel-Kirchhoff diffraction theory we can write

$$U'(x',y') = -\frac{ik}{4\pi}\iint U(x,y)\frac{e^{ikr}}{r}(1+\cos\theta)dxdy$$

$$r = \left[d^2 + (x'-x)^2 + (y'-y)^2\right]^{1/2}$$

$$\cos = \frac{d}{r}$$

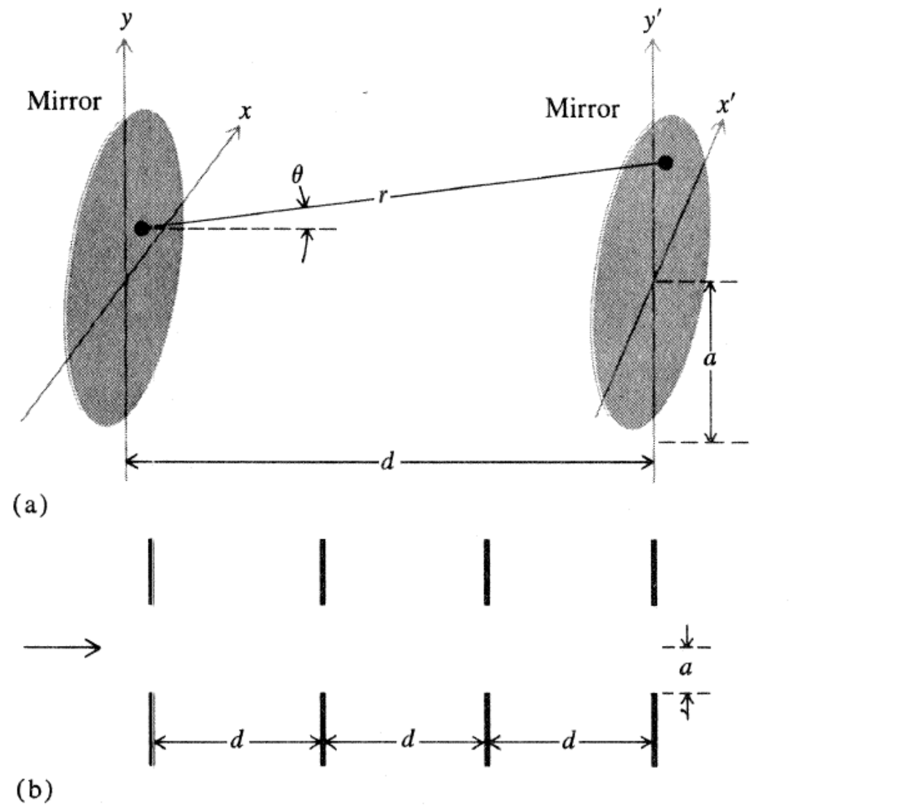


Figure 9.7. (a) Geometry of a Fabry-Perot laser cavity; (b) equivalent multiple diffraction problem.

If the mirrors are identical, the two functions U and U' will become identical except for a constant factor γ.

eigenvalue

kernel

$$\gamma U(x',y') = \iint U(x,y)K(x,y,x',y')dxdy$$

$$K(x,y,x',y') = -\frac{ik}{4\pi}(1+\cos\theta)\frac{e^{ikr}}{r}$$

9.6 Optical-resonator theory

There are an infinite number of solutions U_n, n=1, 2, …, each with an associated eigenvalue γ_n.

$$\gamma_n = |\gamma_n| e^{i\phi_n}$$

$$1 - |\gamma_n|^2 = \text{relative energy loss per transit due to diffraction}$$

Accurate solutions of the Fabry-Perot resonator problem are quite involved. Let's find a simple approximation by employing the same procedure as that of the Fraunhoffer diffraction case.

$$K(x, y, x', y') = Ce^{ik_1(xx' + yy')}$$

$$\gamma U(x', y') = C \iint U(x, y) e^{ik_1(xx' + yy')} dxdy$$

Thus the functions $U(x,y)$ is its own Fourier transform. The simplest of such functions is the Gaussian.

$$U(x, y) = e^{-\rho^2/w^2} = e^{-(x^2+y^2)/w^2}$$

More general functions that are their own Fourier transforms are products of *Hermite polynomials* and the Gaussians.

$$U_{pq}(x, y) = H_p\left(\frac{\sqrt{2}x}{w}\right) H_q\left(\frac{\sqrt{2}y}{w}\right) e^{-(x^2+y^2)/w^2}$$

9.6 Optical-resonator theory

The integers *p* and *q* are the order of the Hermite polynomials and each set (*p*,*q*) corresponds to a particular transverse mode of the resonator.
The lowest-order Hermite polynomial Ho is a constant, hence the simplest Gaussian mode corresponds to the set (0,0) and is called the $TEM_{0,0}$ mode. *TEM* refers to the transverse electromagnetic waves in the cavity

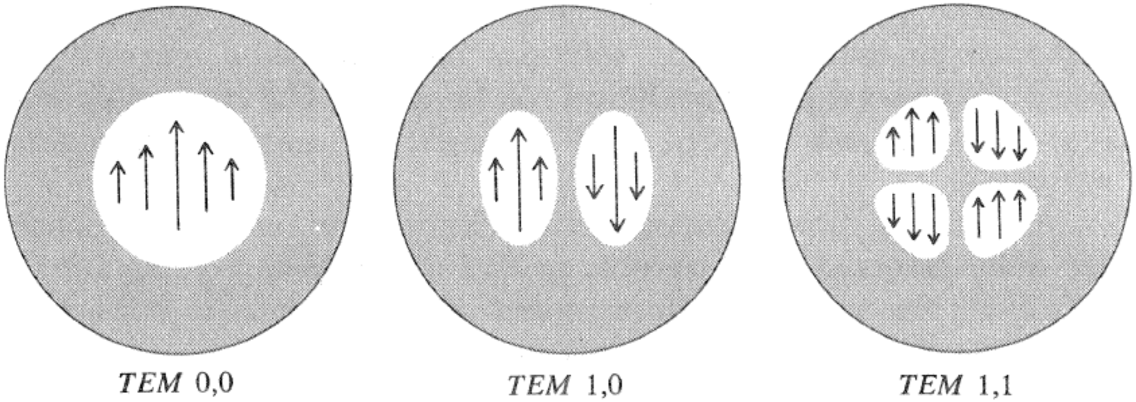


Figure 9.8. Field distributions at the mirrors for some low-order modes.

- Lasers are often designed to operate on a single transverse mode. This is usually the $TEM_{0,0}$ Gaussian mode because it has the smallest beam diameter and can be focused to the smallest spot size.
- Higher-order modes occupy a larger volume and therefore can have larger gain.

$H_n(u)$: Hermite polynomials

$$H_n(u) = e^{u^2/2}\left(u - \frac{d}{du}\right)^n e^{-u^2/2} = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2}$$

$$\frac{dH_n(u)}{du} = 2nH_{n-1}(u)$$

$$H_o(u) = 1$$

$$H_1(u) = 2u$$

$$H_2(u) = -2 + 4u^2$$

$$H_3(u) = -12u + 8u^3$$

$$\vdots$$

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2}$$

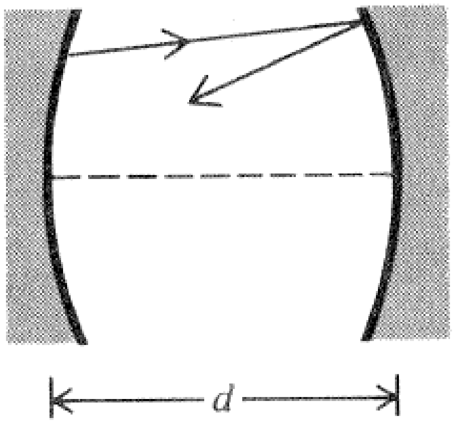
Resonator configurations, Stability

One of the most commonly used cavity configurations is known as the *confocal resonator* consisting of two identical concave spherical mirrors separated by a distance equal to the radius of curvature.

A *stable* resonator is one in which a ray inside the cavity will remain clos to the optic axis upon multiple reflections between the end mirrors.

stability criterion $0 < d < 4f$ or $0 < d < 2r$

In the confocal resonator $d = 2f = r$



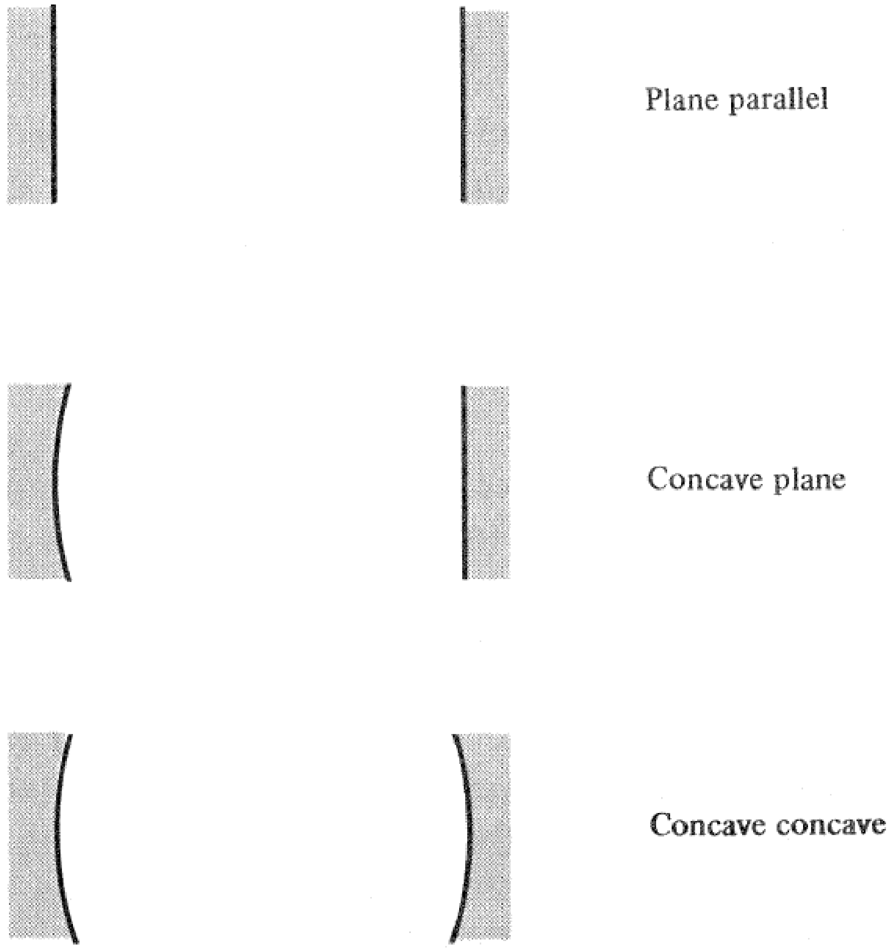


Figure 9.9. Some common laser cavities.

Diffraction loss

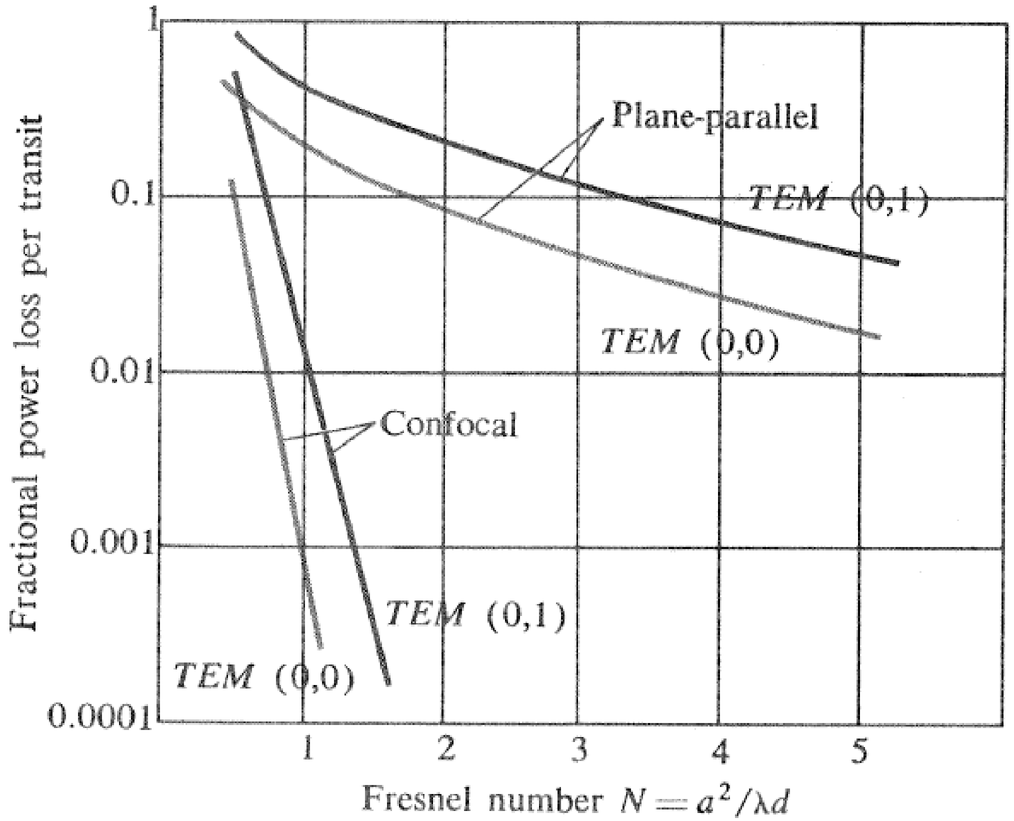


Figure 9.10. Loss curves for the first two modes in plane-parallel and confocal laser cavities.

Fresnel number $N = \dfrac{a^2}{\lambda d}, \; a = \text{mirror radius}, \; d = \text{mirror separation}$

With confocal spherical mirrors, the diffraction losses of low-order modes are negligibly small when N>1..

Spot size

The scale parameter *w* is a measure of the lateral distribution of the energy in the optical beam inside the resonator.

$$U(x,y) = e^{-\rho^2/w^2} = e^{-(x^2+y^2)/w^2}$$

The Gaussian function falls to e^{-1} when $\rho=w$; so the energy will fall to e^{-1} of its maximum value. Hence *w* is called the **spot size** of the dominant (0,0) mode.

$$w^2 = w_o^{\ 2} + \frac{\lambda^2 z^2}{\pi^2 w_o^{\ 2}}$$

For the symmetrical cavity formed by two mirrors each of radius of curvature *R* and separated by a distance *d*, the parameter w_o is given by

$$w_o^{\ 2} = \frac{\lambda}{\pi}\left[\frac{d}{2}\left(R - \frac{d}{2}\right)\right]^{1/2} \qquad r_c = z + \frac{d(2R-d)}{4z}$$

In the case of confocal resonator $R=d$, the spot size at the center is $w_o = \sqrt{\dfrac{\lambda d}{2\pi}}$

and the spot size at either mirror $z=\pm d/2$ is $w = \sqrt{\dfrac{\lambda d}{\pi}}$

Spot size

At the mirrors the wave surfaces match the curvature of the mirror surface. At the center, where the spot size is minimum, the wave surface becomes planar.

Any two wave surfaces will define a cavity if the wave surfaces are replaced by mirrors that match the curvatures of the wave surfaces.

Figure 9.11. Standing wave pattern and lateral distribution of the $TEM_{0,0}$ mode of a confocal laser cavity.

9.11 Q-switching and mode locking

Q-switching is a technique by which a laser can be made to produce a pulsed output beam with extremely high (~GW) peak power, much higher than would be produced by the same laser if it were operating in a continuous wave (constant output) mode.

- Q-switching is achieved by putting some type of variable attenuator (**Q-switch**) inside the laser's optical resonator. When the attenuator is functioning, light which leaves the gain medium does not return, and lasing cannot begin.

- Initially the laser medium is pumped while the Q-switch is set to prevent feedback of light into the gain medium (producing an optical resonator with low Q). This produces a population inversion, but laser operation cannot yet occur since there is no feedback from the resonator. Since the rate of stimulated emission is dependent on the amount of light entering the medium, the amount of energy stored in the gain medium increases as the medium is pumped. Due to losses from spontaneous emission and other processes, after a certain time the stored energy will reach some maximum level; the medium is said to be **gain saturated**. At this point, the Q-switch device is quickly changed from low to high Q, allowing feedback and the process of optical amplification by stimulated emission to begin. Because of the large amount of energy already stored in the gain medium, the intensity of light in the laser resonator builds up very quickly; this also causes the energy stored in the medium to be depleted almost as quickly. The net result is a short pulse of light output from the laser (**giant pulse**) which may have a very high peak intensity.

https://en.wikipedia.org/wiki/Q-switching

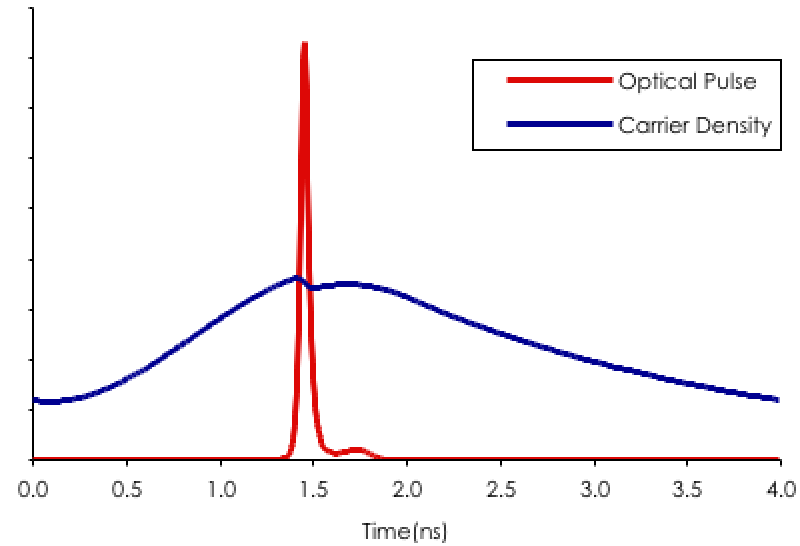
9.11 Q-switching and mode locking

- There are two main types of Q-switching:
  (1) **Active Q-switching**: The Q-switch is an externally controlled variable attenuator. This may be a mechanical device such as a shutter, chopper wheel, or spinning mirror/prism placed inside the cavity, or (more commonly) some form of modulator such as an acousto-optic device, a magneto-optic effect device or an electro-optic device - a **Pockels cell** or **Kerr cell**.

  (2) **Passive Q-switching**: the Q-switch is a saturable absorber, a material whose transmission increases when the intensity of light exceeds some threshold.

https://en.wikipedia.org/wiki/Q-switching

- **Gain-switching**: In a semiconductor laser, the optical pulses are generated by injecting a large number of carriers (electrons) into the active region of the device, bringing the carrier density within that region from below to above the lasing threshold. When the carrier density exceeds that value, the ensuing stimulated emission results in the generation of a large number of photons. However, carriers are depleted as a result of stimulated emission faster than they are injected. So the carrier density eventually falls back to below lasing threshold which results in the termination of the optical output. If carrier injection has not ceased during this period, then the carrier density in the active region can increase once more and the process will repeat itself.



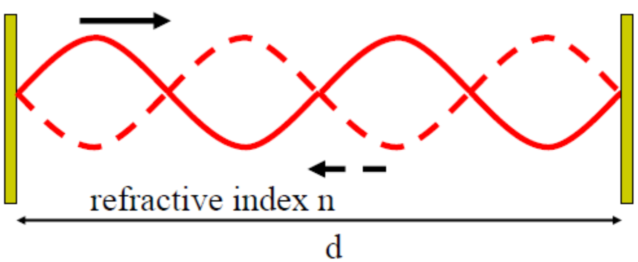
https://en.wikipedia.org/wiki/Gain-switching

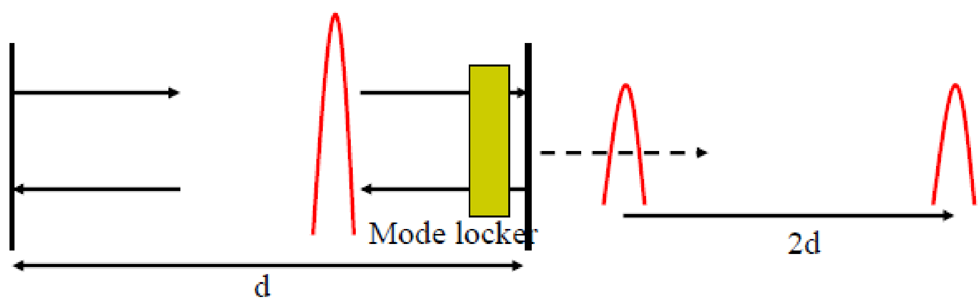
9.11 Q-switching and mode locking

Mode locking is the most important technique for the generation of repetitive, ultrashort laser pulses.

- The basis of the technique is to induce a fixed-phase relationship between the longitudinal modes of the laser's resonant cavity. The laser is then said to be '**phase-locked**' or '**mode-locked**'. Interference between these modes causes the laser light to be produced as a train of pulses. Depending on the properties of the laser, these pulses may be of extremely brief duration, as short as a few femtoseconds.
- A laser can oscillate on many longitudinal modes, with frequencies that are equally separated by the Fabry-Perot intermodal spacing $\Delta\nu_q = c/2nd$. Although these modes normally oscillate independently (they are then called *free-running modes*), external means can be used to couple them and lock their phases together. The modes can then be regarded as the components of a Fourier series expansion of a periodic function of time of period $T_F = 1/\Delta\nu_q = 2nd/c$, which constitute a periodic pulse train. The multiple monochromatic waves of equally spaced frequencies with locked phase constructively interfere.
- The mode-locking operation is accomplished by a nonlinear optical element known as the **mode locker** that is placed inside the laser cavity, typically near one end of the cavity if the laser has the configuration of a linear cavity.

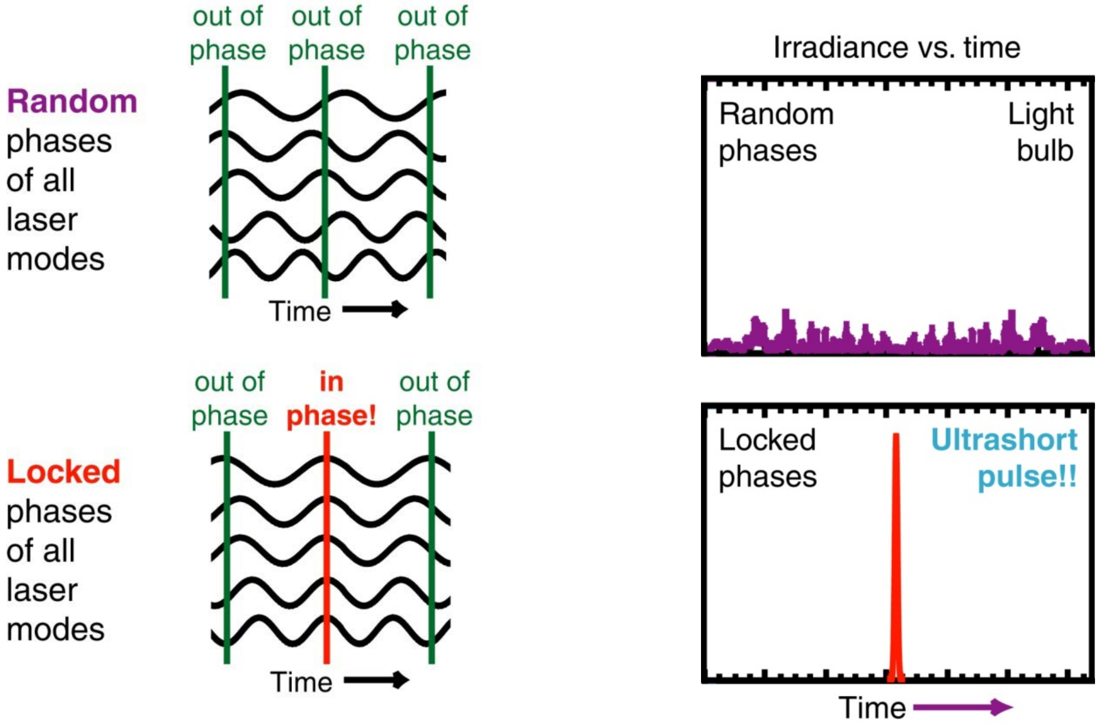
$$k \cdot 2d = \frac{2\pi n}{\lambda} \cdot 2d = \frac{2\pi n \nu}{c} \cdot 2d = 2\pi q \quad (q = 1, 2, 3, \ldots)$$



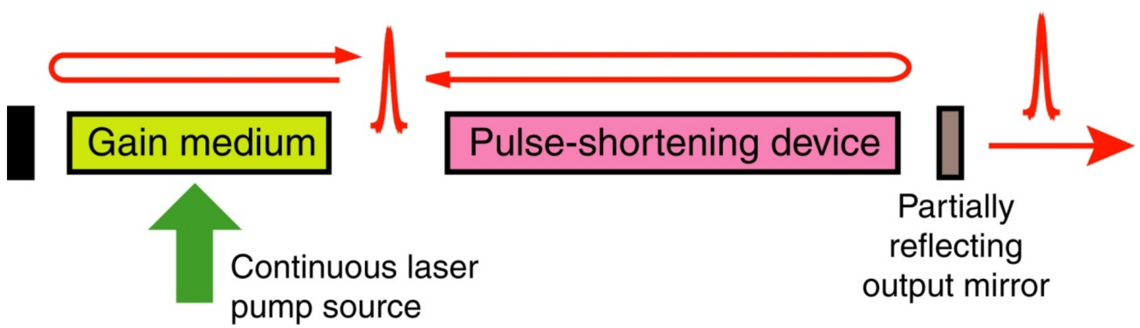


https://en.wikipedia.org/wiki/Mode-locking

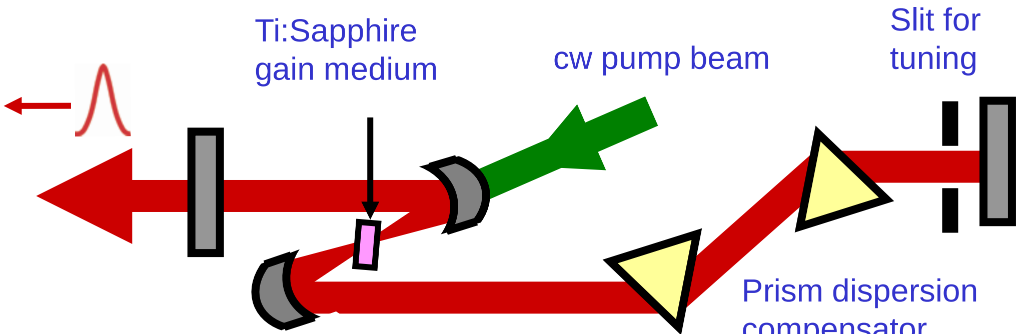
9.11 Q-switching and mode locking



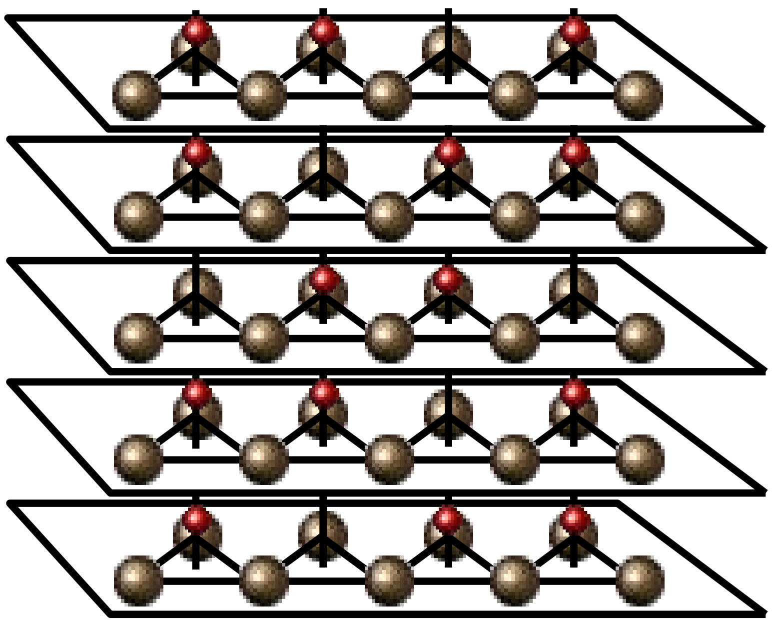
A generic ultrafast laser has a broadband gain medium, a pulse-shortening device, and two or more mirrors.



Titanium Sapphire (Ti:Sapphire)



- Spectra-Physics, Tunami (<35 fs pulse length, 1 W average power)
- Coherent, Mira

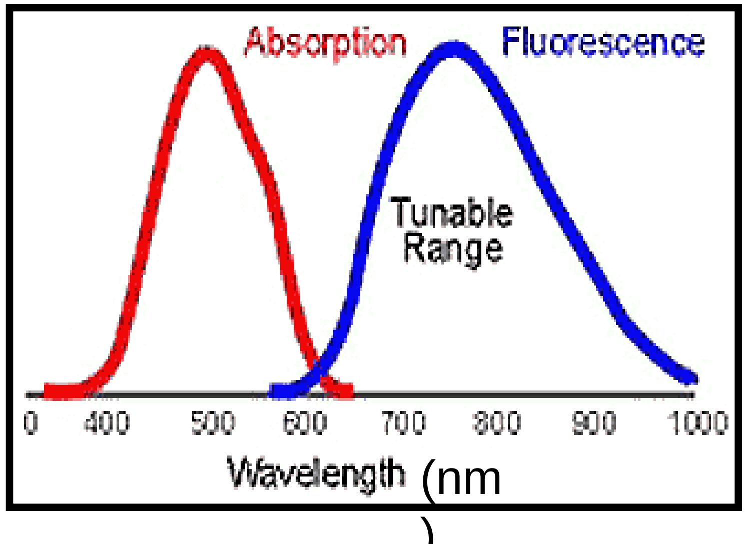


Al_2O_3 lattice

oxygen

aluminum

Absorption and emission spectra of Ti:Sapphire



It can be pumped with a (continuous) Argon laser (~450-515 nm)

Ti:Sapphire lases from ~700 nm to ~1000 nm.