Introduction to Electromagnetism
Static Magnetic Fields
(6-4, 6-5, 6-6)

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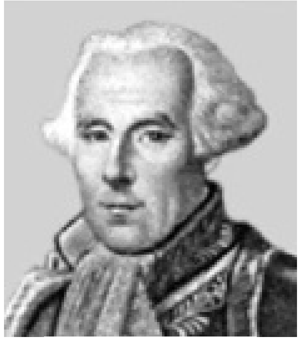
Biot-Savart Law and Applications

Vector magnetic potential:

Jean Baptiste Biot
(1774 - 1862)



Félix Savart
(1791 - 1841)

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}}{R}dv'$$

$$\rightarrow \mathbf{J}dv' = JSd\mathbf{l}' = Id\mathbf{l}'$$ ← For a wire with a cross-section S

$$\rightarrow \boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\mathbf{l}'}{R} \quad \text{(Wb/m)}}$$

$$\rightarrow \mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left[\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\mathbf{l}'}{R}\right] = \frac{\mu_0 I}{4\pi}\oint_{C'}\nabla\times\left(\frac{d\mathbf{l}'}{R}\right)$$

$$= \frac{\mu_0 I}{4\pi}\oint_{C'}\left[\frac{1}{R}\nabla\times d\mathbf{l}' + \left(\nabla\frac{1}{R}\right)\times d\mathbf{l}'\right]$$

$$= \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\mathbf{l}'\times\mathbf{a}_R}{R^2} \quad \text{(T)}$$ ← Biot-Savart law

$$\mathbf{B} = \oint_{C'} d\mathbf{B} \rightarrow \boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\mathbf{l}'\times\mathbf{a}_R}{R^2}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{d\mathbf{l}'\times\mathbf{R}}{R^3}\right)}$$

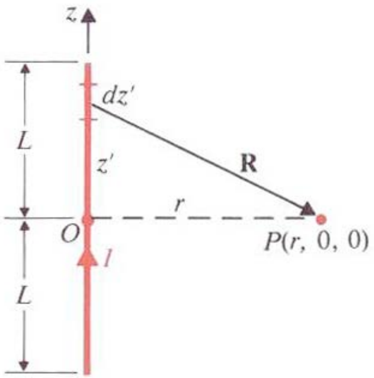
Example 6-4

For a current-carrying straight wire ($L >> r$):

a) By finding **B** from **A**:

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} = \mathbf{a}_z \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz'}{\sqrt{z'^2 + r^2}}$$

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

b) By applying Biot-Savart law:

$$\rightarrow d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)$$

$$\rightarrow \mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$\rightarrow d\mathbf{l}' \times \mathbf{R} = \mathbf{a}_z dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r dz'$$

$$\rightarrow \mathbf{B} = \int d\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{r dz'}{(z'^2 + r^2)^{3/2}} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \frac{1}{r} \left[\frac{z'}{\sqrt{z'^2 + r^2}} \right]_{-L}^{L}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \frac{L}{\sqrt{L^2 + r^2}} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \quad (L >> r)$$

Magnetic Dipole (1)

For a small circular loop carrying current I:

Vector magnetic potential: $\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\mathbf{l}'}{R_1}$

Pick a distant point ($R >> b$):

$$\rightarrow P(R,\theta,\pi/2), \quad R >> b$$

$$\rightarrow d\mathbf{l}' = \mathbf{a}_{\phi'} b d\phi' = (-\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi')bd\phi'$$

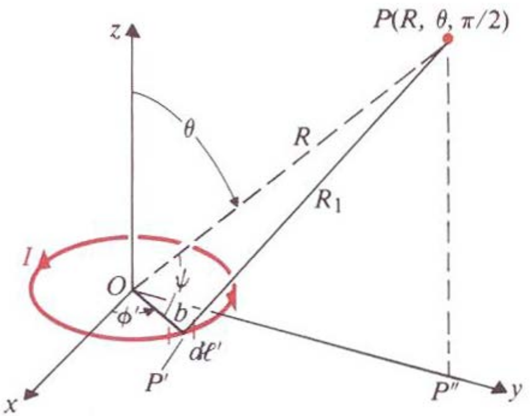
No contribution by symmetry

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}(-\mathbf{a}_x \frac{b\sin\phi'}{R_1})d\phi'$$

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{4\pi}\oint_{C'}\frac{\sin\phi'}{R_1}d\phi'$$

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{2\pi R}\int_{-\pi/2}^{\pi/2}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)\sin\phi' d\phi'$$

$$\rightarrow \boxed{\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2}\sin\theta}$$



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$$\rightarrow R_1^2 = R^2 + b^2 - 2bR\cos\psi$$
$$= R^2 + b^2 - 2bR\sin\theta\sin\phi'$$

$$\rightarrow \frac{1}{R_1} = \frac{1}{R}\left(1+\frac{b^2}{R^2} - \frac{2b}{R}\sin\theta\sin\phi'\right)^{-1/2}$$

$$\cong \frac{1}{R}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)$$

Magnetic Dipole (2)

For a small circular loop carrying current I:

Vector magnetic potential:

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta$$

Magnetic flux density:

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta)$$

Magnetic dipole moment:

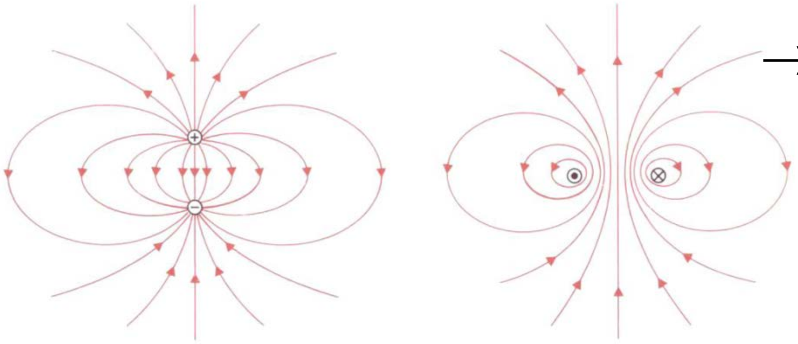
cf. Eq. (3-53b)

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 (I\pi b^2)}{4\pi R^2} \sin\theta \qquad \rightarrow V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\varepsilon_0 R^2} \quad \text{(V)}$$

$$\rightarrow \mathbf{m} = \mathbf{a}_z I\pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \quad (\text{A} \cdot \text{m}^2)$$

$$\rightarrow \boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)}}$$

$$\rightarrow \boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3} (\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(T)}}$$



(a) Electric dipole. (b) Magnetic dipole.

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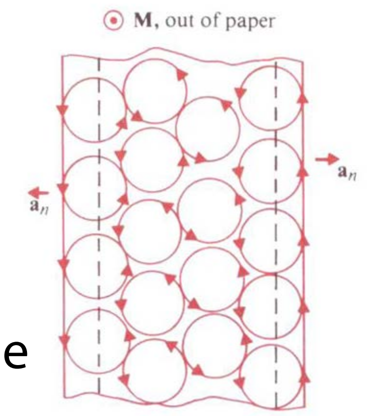
Magnetization and Equivalent Current Densities



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Magnetization vector (macroscopic):

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v} \quad \text{(A/m)}$$

$\leftarrow$ n: the number of atoms per unit volume

Differential dipole moment for an elemental volume dv':

$$\to \ d\mathbf{m} = \mathbf{M}dv' \ \to \ d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2} dv' = \frac{\mu_0}{4\pi} \mathbf{M} \times \nabla' \left(\frac{1}{R} \right) dv'$$

Vector magnetic potential:

$$\to \ \mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla' \left(\frac{1}{R} \right) dv' = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R} \right) dv'$$

$$= \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds' \quad \leftarrow \boxed{\int_{V'} \nabla' \times \mathbf{F} dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{s}'}$$

$$\leftarrow \ \nabla \cdot (\mathbf{F} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{C})$$

Magnetization current densities:

$$\to \ \boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \quad (\text{A/m}^2)} \ \leftarrow \text{Volume current density}$$

$$\to \ \boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \quad (\text{A/m})} \ \leftarrow \text{Surface current density}$$