Introduction to Electromagnetism

Static Magnetic Fields

(6-13)

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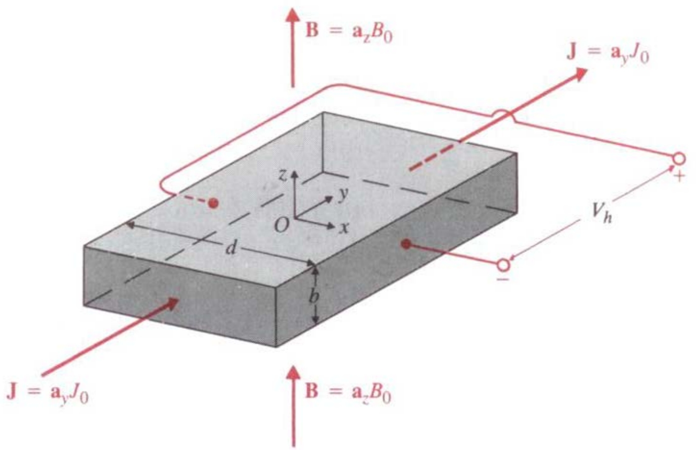
Magnetic Forces (1)

Magnetic force: $\mathbf{F}_m = q\mathbf{u}\times\mathbf{B} \quad (\text{N})$

Hall effect:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

For an n-type semiconductor:

$$\mathbf{B} = \mathbf{a}_z B_0$$

$$\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u} \quad \leftarrow \quad q = -e$$

$$\rightarrow \mathbf{F}_{net,\,steady} = q\mathbf{u}\times\mathbf{B} + q\mathbf{E}_h = 0$$

$$\rightarrow \mathbf{E}_h = -\mathbf{u}\times\mathbf{B}$$ *→ Hall effect*

$$= -(-\mathbf{a}_y u_0)\times\mathbf{a}_z B_0$$

$$= \mathbf{a}_x u_0 B_0$$ *← Hall field*

$$\rightarrow V_h = \int_0^d E_h dx = u_0 B_0 d$$ *← Hall voltage*

$$\rightarrow E_x / J_y B_z = 1/Nq$$ *← Hall coefficient → To determine N*

What if for a p-type semiconductor?

→ V_h will be reversed.

Magnetic Forces (2)

Magnetic force on the differential length element $d\mathbf{l}$:

$$d\mathbf{F}_m = NqS|dl|\mathbf{u}\times\mathbf{B}$$

$$\rightarrow\ d\mathbf{F}_m = Id\mathbf{l}\times\mathbf{B} \quad (\mathrm{N})$$ → *d**l** along the direction of I*

$$\rightarrow\ \boxed{\mathbf{F}_m = I\oint_C d\mathbf{l}\times\mathbf{B} \quad (\mathrm{N})}$$

For two circuits carrying current I_1 and I_2: "Biot-Savart law"

$$\mathbf{F}_{21} = I_1\oint_{C_1} d\mathbf{l}_1\times\mathbf{B}_{21} \quad \leftarrow \quad \mathbf{B}_{21} = \frac{\mu_0 I_2}{4\pi}\oint_{C_2}\frac{d\mathbf{l}_2\times\mathbf{a}_{R_{21}}}{R_{21}^2}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2}\frac{d\mathbf{l}_1\times(d\mathbf{l}_2\times\mathbf{a}_{R_{21}})}{R_{21}^2} \quad (\mathrm{N})$$

$$\mathbf{F}_{21} \overset{?}{=} -\mathbf{F}_{12}$$

Newton's third law of action and reaction

→ *Ampère's law of force*

$$\mathbf{F}_{12} = I_2\oint_{C_2} d\mathbf{l}_2\times\mathbf{B}_{12}$$

$$= \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2}\oint_{C_1}\frac{d\mathbf{l}_2\times(d\mathbf{l}_1\times\mathbf{a}_{R_{12}})}{R_{12}^2} \quad (\mathrm{N})$$

Magnetic Forces (3)

For two circuits carrying current I_1 and I_2:

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2} \quad \text{(N)}$$

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)}$$

Recall: "back-cab" rule

$$\rightarrow \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{21}^2}$$

$$\rightarrow \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} = \oint_{C_2} d\mathbf{l}_2 \oint_{C_1} \frac{(d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2}$$

$$= \oint_{C_2} d\mathbf{l}_2 \oint_{C_1} d\mathbf{l}_1 \cdot [-\nabla_1 \left(\frac{1}{R_{21}} \right)] = 0$$

Newton's third law of action and reaction verified!

$$\rightarrow \mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\mathbf{a}_{R_{21}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{21}^2}$$

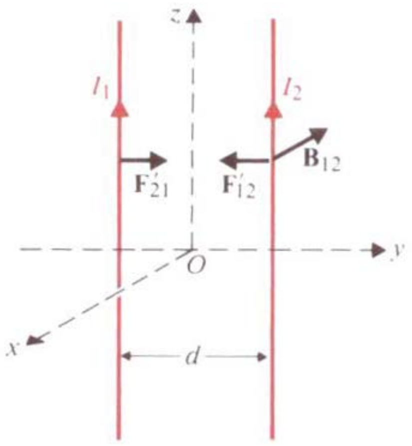
Recall:

$$\leftarrow \nabla \times (\nabla V) = 0$$

$$\rightarrow \mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\mathbf{a}_{R_{12}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{12}^2} = -\mathbf{F}_{21}$$

Example 6-21

For two parallel current-carrying wires:



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$$\frac{\mathbf{F}_{12}}{l_2} = \mathbf{F}'_{12} = I_2(\mathbf{a}_z \times \mathbf{B}_{12})$$

$$\rightarrow \mathbf{B}_{12} = -\mathbf{a}_x \frac{\mu_0 I_1}{2\pi d}$$ *← Ampère's circuital law*

$$\rightarrow \mathbf{F}'_{12} = -\mathbf{a}_y \frac{\mu_0 I_1 I_2}{2\pi d} \quad \text{(N/m)}$$

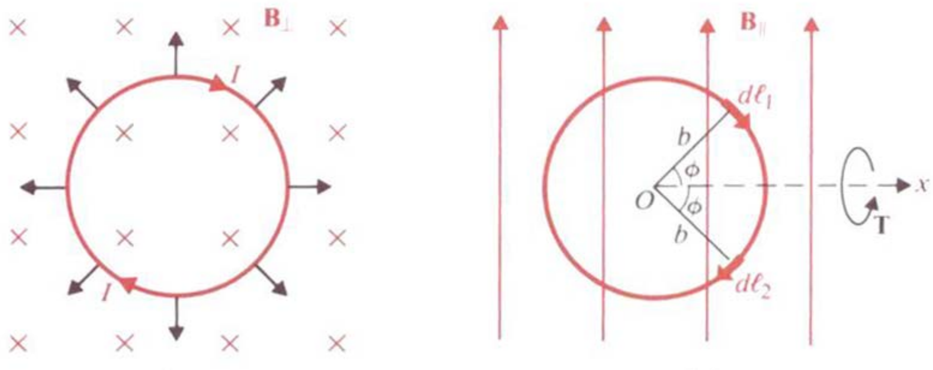
→ Attraction for currents in the same direction

→ Repulsion for currents in opposite directions

Magnetic Forces and Torques

Magnetic flux density:

$$\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_{//}$$
$$(\mathbf{B}_\perp = \mathbf{a}_z B_\perp)$$



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Forces on the loop:

$\rightarrow \mathbf{F}_{\mathbf{B}_\perp} \propto \mathbf{a}_r$ → *Radial symmetry* → *No net contribution*

$\rightarrow d\mathbf{F}_{1,\mathbf{B}_{//}} \propto -\mathbf{a}_z \ (upward)$

$\rightarrow d\mathbf{F}_{2,\mathbf{B}_{//}} \propto +\mathbf{a}_z \ (downward)$

→ *Rotational motion*

→ *Torque*

Torque on the loop:

$$\begin{aligned} d\mathbf{T} &= \mathbf{a}_x 2(dF)b\sin\phi \\ &= \mathbf{a}_x 2(IdlB_{//}\sin\phi)b\sin\phi \\ &= \mathbf{a}_x 2Ib^2 B_{//}\sin^2\phi\, d\phi \end{aligned}$$

$$\rightarrow \mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2Ib^2 B_{//}\int_0^{\pi}\sin^2\phi\, d\phi = \mathbf{a}_x I(\pi b^2)B_{//}$$
$$= \mathbf{m}\times\mathbf{B} \quad (\mathrm{N}\cdot\mathrm{m})$$

Forces and Torques - Stored Magnetic Energy (1)

Principle of virtual displacement:

Work done by the source

$$dW_s = dW + dW_m$$

Mechanical work done by the system — *Increase in the stored magnetic energy*

System of circuits with constant flux linkage:

Virtual displacement $d\mathbf{l}$ *with no changes in flux linkages*

$\rightarrow$ *No emf* $\rightarrow$ *No energy from to the source to the system*

$$\rightarrow \; dW_s = dW + dW_m = 0$$

Magnetic forces:

$$dW = \mathbf{F}_\Phi \cdot d\mathbf{l} = -dW_m = -(\nabla W_m) \cdot d\mathbf{l}$$

$$\rightarrow \boxed{\mathbf{F}_\Phi = -\nabla W_m \quad (\mathrm{N})}$$

Magnetic torques:

$$\rightarrow \boxed{(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \quad (\mathrm{N \cdot m})}$$

Forces and Torques - Stored Magnetic Energy (2)

System of circuits with constant currents:

Virtual displacement $d\mathbf{l}$ *with constant currents* $\rightarrow$ *Flux linkage changed*

$\rightarrow$ *Emf induced* $\rightarrow$ *Energy from the source to the system*

$$\rightarrow\ dW_s = dW + dW_m$$

Magnetic forces:

$$\left.\begin{aligned} &\rightarrow\ dW_s = \sum_k I_k V_k dt = \sum_k I_k d\Phi_k \\ &\rightarrow\ dW_m = \frac{1}{2}\sum_k I_k d\Phi_k \end{aligned}\right\} \rightarrow\ dW = dW_m$$

$$\rightarrow\ dW = \mathbf{F}_I \cdot d\mathbf{l} = dW_m = \nabla\left(W_m\right)\cdot d\mathbf{l}$$

$$\rightarrow\ \boxed{\mathbf{F}_I = \nabla W_m \quad (\mathrm{N})}$$

Magnetic torques:

$$\rightarrow\ \boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \quad (\mathrm{N\cdot m})}$$

Force and Torque in Terms of Mutual Inductance

For two circuits with currents I_1 and I_2:

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k$$

$$\rightarrow W_m = \frac{1}{2} L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

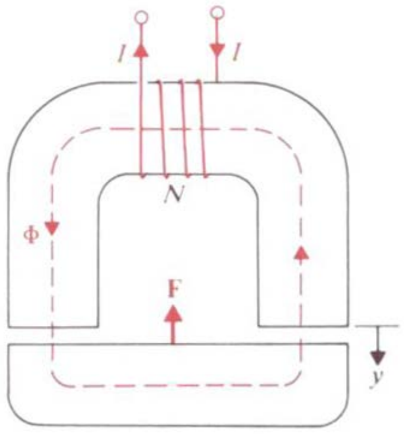
Magnetic force between two circuits:

$$\rightarrow \boxed{\mathbf{F}_I = I_1 I_2 (\nabla L_{12})}$$

Magnetic torque between two circuits:

$$\rightarrow \boxed{(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \quad (\mathrm{N \cdot m})}$$

Example 6-23



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System of circuits with constant flux linkage:

$$\rightarrow dW_m = d(W_m)_{air\ gap}$$

$$= 2\left(\frac{B^2}{2\mu_0} S dy\right)$$

Magnetic flux

$$\rightarrow \mathbf{F}_\Phi = -\mathbf{a}_y \frac{\partial W_m}{\partial y} = -\mathbf{a}_y \frac{B^2 S}{\mu_0} = -\mathbf{a}_y \frac{\Phi^2}{\mu_0 S} \quad \text{(N)}$$

System of circuits with constant currents:

$$\rightarrow \Phi = \frac{\mathcal{V}_m}{\mathcal{R}_c + 2\mathcal{R}_g} = \frac{NI}{\mathcal{R}_c + 2y/\mu_0 S}$$

Magnetic flux linkgage

Magnetic flux

$$\rightarrow W_m = \frac{1}{2}\sum_k I_k \Phi_k = \frac{1}{2} IN\Phi$$

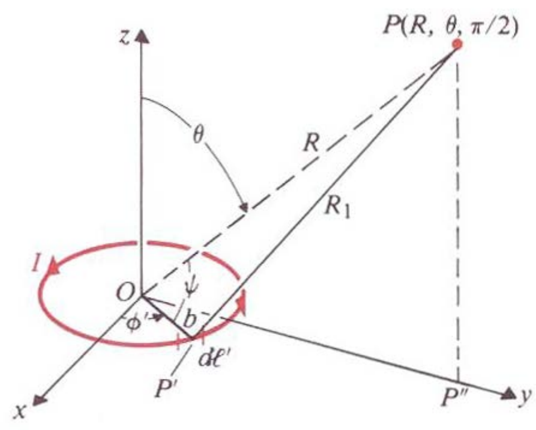
$$\rightarrow \mathbf{F}_I = \mathbf{a}_y \frac{\partial W_m}{\partial y} = \mathbf{a}_y \frac{1}{2}\left(-\Phi^2 \frac{2}{\mu_0 S}\right) = -\mathbf{a}_y \frac{\Phi^2}{\mu_0 S} \quad \text{(N)}$$

Example 6-24

Force between two coaxial circular coils:

Recall:

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta \quad \text{(6-43)}$$



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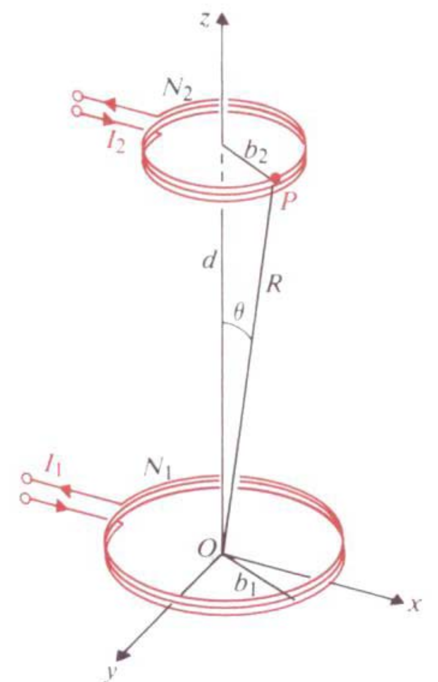
$$\rightarrow \mathbf{A}_{12} = \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2} \sin\theta$$

$$= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2}\left(\frac{b_2}{R}\right) = \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2 b_2}{4(z^2 + b_2^2)^{3/2}}$$

$$\rightarrow \Phi_{12}^L = \oint_{C_2} \mathbf{A}_{12} \cdot d\mathbf{l}_2 = \frac{\mu_0 N_1 I_1 b_1^2 b_2}{4(z^2 + b_2^2)^{3/2}} \oint_{C_2} b_2 \, d\phi$$

$$= \frac{\mu_0 N_1 N_2 \pi I_1 b_1^2 b_2^2}{2(z^2 + b_2^2)^{3/2}}$$

$$\rightarrow L_{12} = \frac{\Phi_{12}^L}{I_1} = \frac{\mu_0 N_1 N_2 \pi b_1^2 b_2^2}{2(z^2 + b_2^2)^{3/2}}$$



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$$\rightarrow \mathbf{F}_{12} = \mathbf{a}_z I_1 I_2 \left.\frac{dL_{12}}{dz}\right|_{z=d} = -\mathbf{a}_z I_1 I_2 \frac{3\mu_0 N_1 N_2 \pi b_1^2 b_2^2 d}{2(d^2 + b_2^2)^{5/2}}$$

$$\cong -\mathbf{a}_z \frac{3\mu_0 m_1 m_2}{2\pi d^4} \quad \text{(N)} \qquad \leftarrow m_1 = N_1 I_1 \pi b_1^2, \quad m_2 = N_2 I_2 \pi b_2^2$$