

# Introduction to Electromagnetism

## Vector Analysis

(2-1, 2-2, 2-3)

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# Introduction

Quantities in electromagnetics (from a mathematical viewpoint)?

- **Scalar:** Completely specified by its **magnitude** (positive or negative, together with its unit)
- **Vector:** Required both a **magnitude** and a **direction** to specify

How do we specify the direction of a vector?

- In a three-dimensional space, three numbers are needed.
- These numbers depend on the choice of a coordinate system:  
e.g.) Cartesian coordinates, cylindrical coordinates, spherical coordinates, etc.

However, physical laws and theorems certainly must hold irrespective of the coordinate system:

- The general expressions of the laws of electromagnetism do not require the specification of a coordinate system
- A particular coordinate system is chosen only when a problem of a given geometry is analyzed.

# What we are going to learn on vector analysis

## 1. Vector algebra:

Addition, subtraction and multiplication of vectors

## 2. Orthogonal coordinate systems:

Cartesian, cylindrical, and spherical coordinates

## 3. Vector calculus:

Differentiation and integration of vectors;

Line, surface, and volume integrals;

“del” operator;

Gradient, divergence, and curl operations

A deficiency in vector analysis in the study of electromagnetics is similar to a deficiency in algebra and calculus in the study of physics.

“It is obvious that these deficiencies cannot yield fruitful outcomes!”

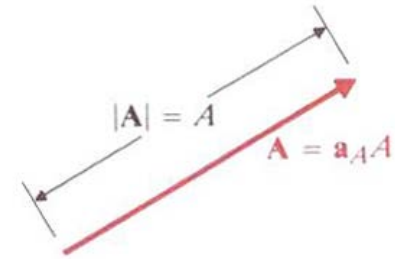
# Vector: Magnitude and Direction

Magnitude and direction of a vector  $\mathbf{A}$ :

$$\mathbf{A} = \mathbf{a}_A A$$

Magnitude:  $A = |\mathbf{A}|$

Dimensionless unit vector:  $\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$



Graphical representation of vector  $\mathbf{A}$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

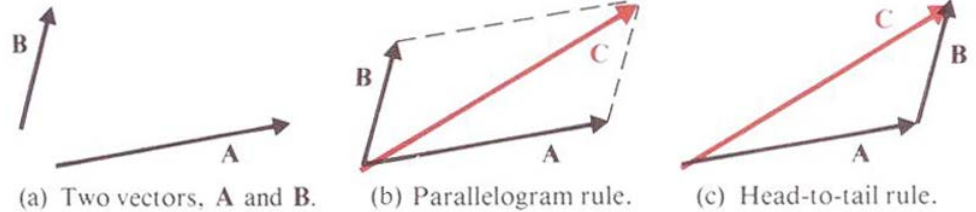
Equality:

Two vectors are equal if they have the same **magnitude** and the same **direction**.

# Vector Addition and Subtraction

Vector addition:  $\mathbf{C} = \mathbf{A} + \mathbf{B}$

1. By the parallelogram rule
2. By the head-to-tail rule



Vector addition

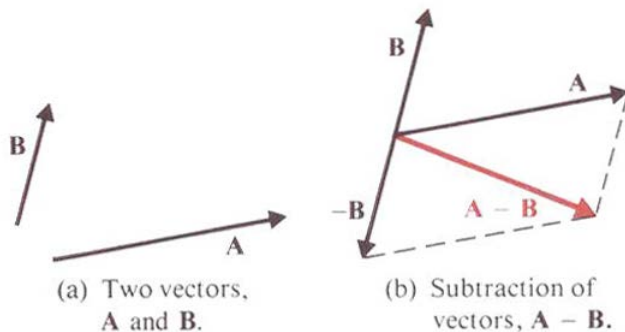
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Rules for vector addition:

**Commutative law:**  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

**Associative law:**  $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Vector subtraction:  $\mathbf{C} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$



Vector subtraction

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$-\mathbf{B}$ : Negative of vector  $\mathbf{B}$

Having the same magnitude but with an opposite direction w.r.t.  $\mathbf{B}$

$$-\mathbf{B} = (-a_B)\mathbf{B}$$

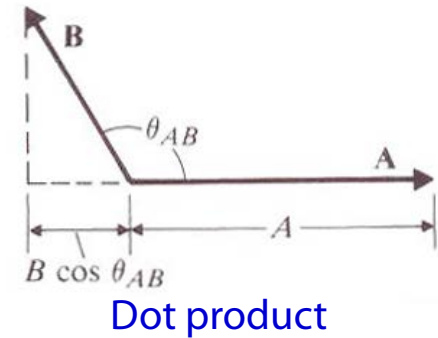
**Scalar multiplication:**  $k\mathbf{A} = (kA)\mathbf{a}_A$

# Scalar or Dot Product

Scalar or dot product of two vectors **A** and **B**:

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta_{AB}$$

where  $\theta_{AB}$  is the smaller angle between **A** and **B**



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

- Note:**
1. Magnitude: Less than or equal to the product of their magnitudes
  2. Either a positive or a negative quantity, depending on  $\theta_{AB}$
  3. Equal to the product of the magnitude of one vector and the **projection** of the other upon the first one
  4. **Zero** when the vectors are perpendicular to each other

Rules for dot product:

$$\mathbf{A} \cdot \mathbf{A} = A^2 \quad \text{or} \quad A = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

**Commutative law:**  $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

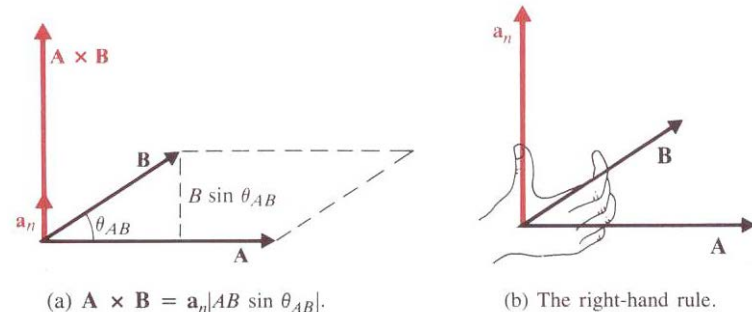
**Distributive law:**  $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$  **Proof? HW**

# Vector or Cross Product

Vector or cross product of two vectors  $\mathbf{A}$  and  $\mathbf{B}$ :

$$\mathbf{A} \times \mathbf{B} \equiv |AB \sin \theta_{AB}| \mathbf{a}_n$$

where  $\mathbf{a}_n$  is perpendicular to the plane containing  $\mathbf{A}$  and  $\mathbf{B}$ ; its direction follows that of the thumb of the right hand when the fingers rotate from  $\mathbf{A}$  to  $\mathbf{B}$  through the angle  $\theta_{AB}$  (the right-hand rule)



Cross product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Rules for cross product:

**Not commutative:**  $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$

**Distributive law:**  $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$  Proof? HW

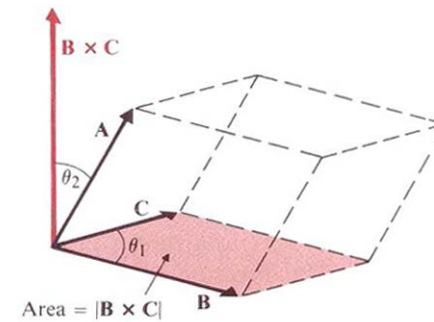
**Not associative:**  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

# Product of Three Vectors

Scalar triple product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Cyclic permutation



Scalar triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

“back-cab” rule



# Vector Triple Product

Proof:

Let  $\mathbf{A} = \mathbf{A}_{//} + \mathbf{A}_{\perp}$  (w.r.t.  $\mathbf{B} - \mathbf{C}$  plane)

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A}_{//} + \mathbf{A}_{\perp}) \times (\mathbf{B} \times \mathbf{C}) \\ &= \mathbf{A}_{//} \times (\mathbf{B} \times \mathbf{C}) \equiv \mathbf{D}\end{aligned}$$

$$\begin{aligned}D &= \mathbf{D} \cdot \mathbf{a}_D = A_{//} BC \sin(\theta_1 - \theta_2) \\ &= A_{//} BC (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2) \\ &= (B \sin \theta_1)(A_{//} C \cos \theta_2) - (C \sin \theta_2)(A_{//} B \cos \theta_1) \\ &= [\mathbf{B}(A_{//} \cdot \mathbf{C}) - \mathbf{C}(A_{//} \cdot \mathbf{B})] \cdot \mathbf{a}_D \quad ?\end{aligned}$$

Let  $\mathbf{B}(A_{//} \cdot \mathbf{C}) - \mathbf{C}(A_{//} \cdot \mathbf{B}) = \mathbf{D} + k\mathbf{A}_{//}$

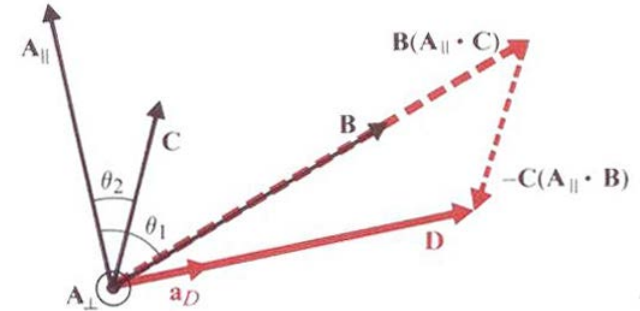
$$\rightarrow \mathbf{A}_{//} \cdot [\mathbf{B}(A_{//} \cdot \mathbf{C}) - \mathbf{C}(A_{//} \cdot \mathbf{B})] = \mathbf{A}_{//} \cdot (\mathbf{D} + k\mathbf{A}_{//})$$

$$\rightarrow (\mathbf{A}_{//} \cdot \mathbf{B})(A_{//} \cdot \mathbf{C}) - (\mathbf{A}_{//} \cdot \mathbf{C})(A_{//} \cdot \mathbf{B}) = 0 = \mathbf{A}_{//} \cdot \mathbf{D} + kA_{//}^2$$

$$\leftarrow \mathbf{A}_{//} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}, \quad \mathbf{A}_{//} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} \quad \rightarrow k = 0$$

$$\therefore \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$



Vector triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Note that  $\mathbf{A}_{//}$ ,  $\mathbf{B}$  &  $\mathbf{C}$  are on the same plane.