

Introduction to Electromagnetism

Orthogonal Coordinate Systems

(2-4, 2-5)

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Orthogonal Coordinate systems (1)

How many surfaces are required for determining the location of a point in a three-dimensional space?

A point can be defined as the intersection of three surfaces.

Orthogonal coordinate system:

If these three surfaces are mutually perpendicular to one another

Base vectors:

The unit vectors perpendicular to the three surfaces

Right-handed system \rightarrow

$$\begin{cases} \mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \\ \mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1} \\ \mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \end{cases} \quad \begin{aligned} &\rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0 \\ &\rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1 \end{aligned}$$

Vector representation:

$$\rightarrow \mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}$$

$$\rightarrow A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}$$

Orthogonal Coordinate systems (2)

Example 2-4: $\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}$

$$\mathbf{B} = \mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3}$$

Recall:

$$\rightarrow \begin{cases} \mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3} & \rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0 \\ \mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1} & \rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1 \\ \mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2} & \end{cases}$$

Scalar product:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (\mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}) \cdot (\mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3}) \\ &= A_{u_1} B_{u_1} + A_{u_2} B_{u_2} + A_{u_3} B_{u_3} \end{aligned}$$

Vector product:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (\mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}) \times (\mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3}) \\ &= \mathbf{a}_{u_1} (A_{u_2} B_{u_3} - A_{u_3} B_{u_2}) + \mathbf{a}_{u_2} (A_{u_3} B_{u_1} - A_{u_1} B_{u_3}) + \mathbf{a}_{u_3} (A_{u_1} B_{u_2} - A_{u_2} B_{u_1}) \\ &= \begin{vmatrix} \mathbf{a}_{u_1} & \mathbf{a}_{u_2} & \mathbf{a}_{u_3} \\ A_{u_1} & A_{u_2} & A_{u_3} \\ B_{u_1} & B_{u_2} & B_{u_3} \end{vmatrix} \end{aligned}$$

Conversion factors

Some of the coordinates, say u_i ($i = 1, 2, \text{ or } 3$), may not be a length:

A conversion factor is needed to convert a differential change du_i into a change in length dl_i :

$$dl_i = h_i du_i \quad \leftarrow h_i \text{ is called a "metric coefficient"}$$

Directed differential length change: $d\mathbf{l} = \mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3$

$$\rightarrow d\mathbf{l} = \mathbf{a}_{u_1} (h_1 du_1) + \mathbf{a}_{u_2} (h_2 du_2) + \mathbf{a}_{u_3} (h_3 du_3)$$

Differential volume change: $dv = dl_1 dl_2 dl_3$

$$\rightarrow dv = h_1 h_2 h_3 du_1 du_2 du_3$$

Differential area change: $ds = \mathbf{a}_n ds \rightarrow ds_1 = dl_2 dl_3$

$$\rightarrow ds_1 = h_2 h_3 du_2 du_3$$

$$\rightarrow ds_2 = h_1 h_3 du_1 du_3$$

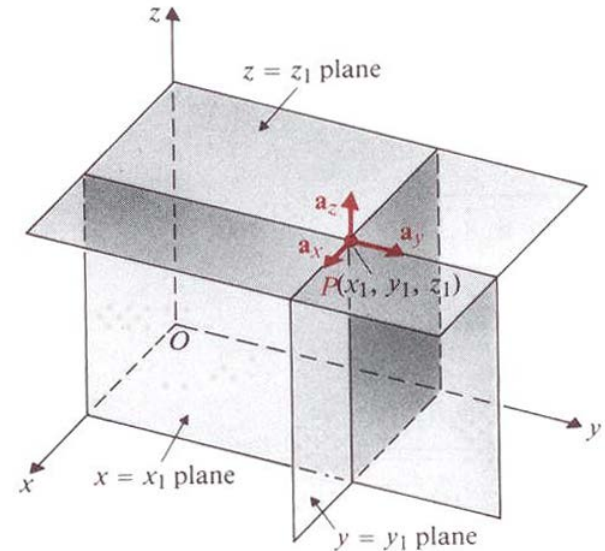
$$\rightarrow ds_3 = h_1 h_2 du_1 du_2$$

Cartesian (or Rectangular) Coordinates (1)

Notation: $(u_1, u_2, u_3) = (x, y, z)$

$$\rightarrow \begin{cases} \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \\ \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \\ \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \end{cases}$$

$$\rightarrow \overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector representation:

$$\rightarrow \mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

Dot product:

$$\rightarrow \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Cross product:

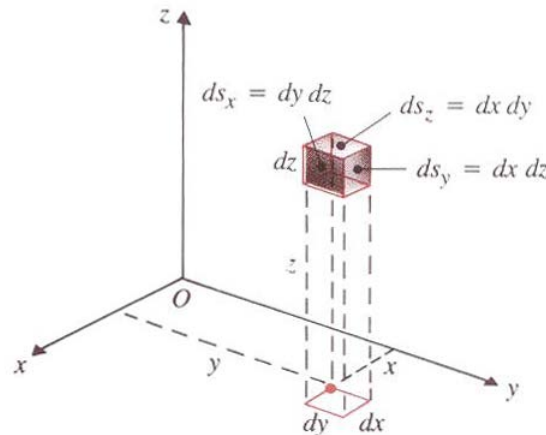
$$\rightarrow \mathbf{A} \times \mathbf{B} = \mathbf{a}_x (A_y B_z - A_z B_y) + \mathbf{a}_y (A_z B_x - A_x B_z) + \mathbf{a}_z (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cartesian (or Rectangular) Coordinates (2)

Metric coefficients:

$$\rightarrow h_1 = h_2 = h_3 = 1$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Directed differential length change: $d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$

Differential area change:

$$ds_x = dy dz$$

$$ds_y = dx dz$$

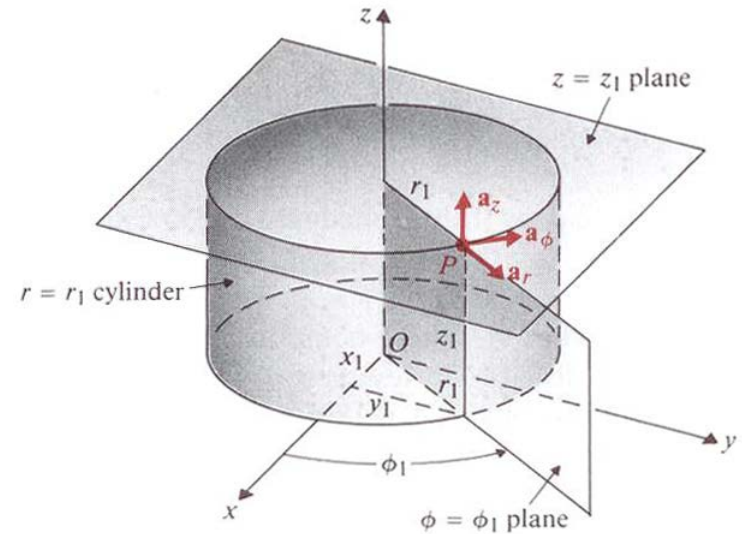
$$ds_z = dx dy$$

Differential volume change: $dv = dx dy dz$

Cylindrical Coordinates (1)

Notation: $(u_1, u_2, u_3) = (r, \phi, z)$

$$\rightarrow \begin{cases} \mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z \\ \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r \\ \mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi \end{cases}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector representation:

$$\rightarrow \mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

Dot product:

$$\rightarrow \mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

Cross product:

$$\rightarrow \mathbf{A} \times \mathbf{B} = \mathbf{a}_r (A_\phi B_z - A_z B_\phi) + \mathbf{a}_\phi (A_z B_r - A_r B_z) + \mathbf{a}_z (A_r B_\phi - A_\phi B_r)$$

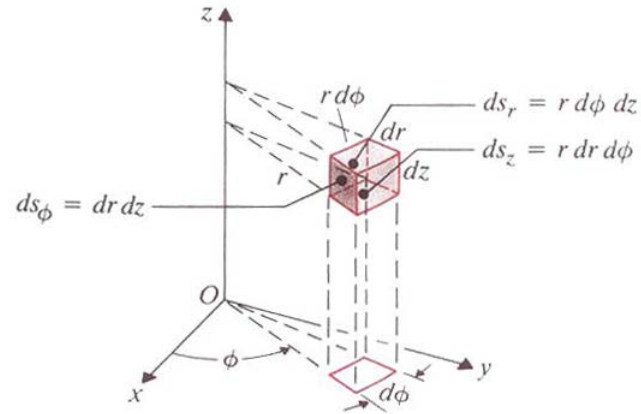
$$= \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi & \mathbf{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

Cylindrical Coordinates (2)

Metric coefficients:

$$\rightarrow h_1 = h_3 = 1$$

$$\rightarrow h_2 = r$$



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Directed differential length change: $d\mathbf{l} = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_z dz$

Differential area change:

$$ds_r = r d\phi dz$$

$$ds_\phi = dr dz$$

$$ds_z = r dr d\phi$$

Differential volume change: $dv = r dr d\phi dz$

Cylindrical Coordinates (3)

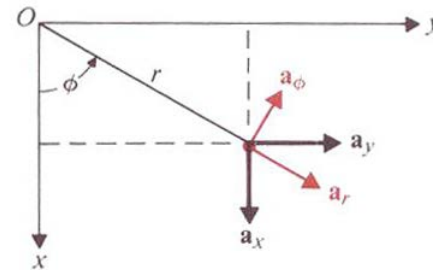
Transformation from cylindrical coordinates to Cartesian coordinates:

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

$$\begin{aligned} \rightarrow A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x \\ &= A_r \cos \phi - A_\phi \sin \phi \end{aligned}$$

$$\begin{aligned} \rightarrow A_y &= \mathbf{A} \cdot \mathbf{a}_y \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_y + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_y \\ &= A_r \sin \phi + A_\phi \cos \phi \end{aligned}$$

$$\rightarrow A_z = A_z$$



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$$\rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Conversion formulas:

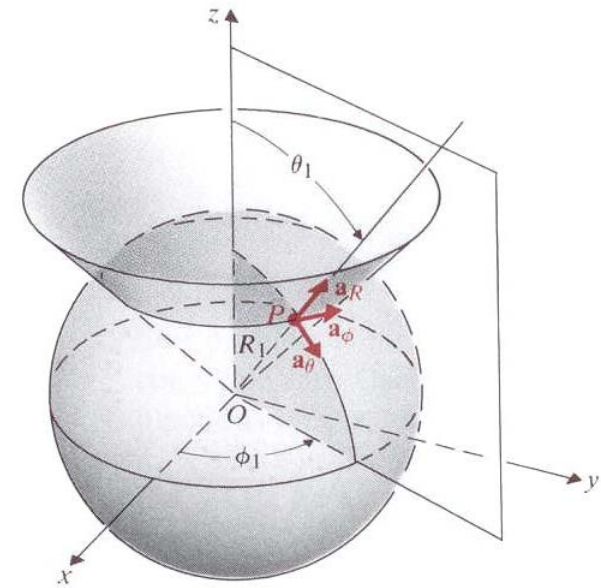
$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \\ z &= z \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1} \frac{y}{x} \\ z &= z \end{aligned}$$

Spherical Coordinates (1)

Notation: $(u_1, u_2, u_3) = (R, \theta, \phi)$

$$\rightarrow \begin{cases} \mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi \\ \mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R \\ \mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta \end{cases}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector representation:

$$\rightarrow \mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$

Dot product:

$$\rightarrow \mathbf{A} \cdot \mathbf{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

Cross product:

$$\begin{aligned} \rightarrow \mathbf{A} \times \mathbf{B} &= \mathbf{a}_R (A_\theta B_\phi - A_\phi B_\theta) + \mathbf{a}_\theta (A_\phi B_R - A_R B_\phi) + \mathbf{a}_\phi (A_R B_\theta - A_\theta B_R) \\ &= \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta & \mathbf{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix} \end{aligned}$$

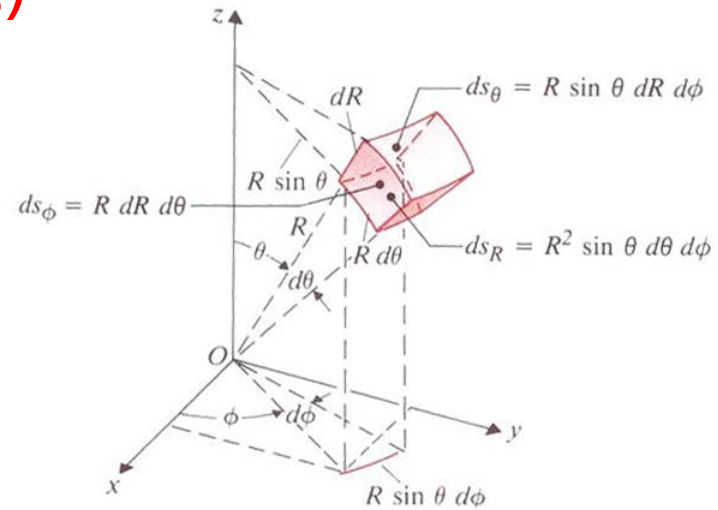
Spherical Coordinates (2)

Metric coefficients:

$$\rightarrow h_1 = 1$$

$$\rightarrow h_2 = R$$

$$\rightarrow h_3 = R \sin \theta$$



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Directed differential length change: $d\mathbf{l} = \mathbf{a}_R dR + \mathbf{a}_\theta R d\theta + \mathbf{a}_\phi R \sin \theta d\phi$

Differential area change:

$$ds_R = R^2 \sin \theta d\theta d\phi$$

$$ds_\theta = R \sin \theta dR d\phi$$

$$ds_\phi = R dR d\theta$$

Differential volume change: $dv = R^2 \sin \theta dR d\theta d\phi$

Spherical Coordinates (3)

Transformation from spherical coordinates to Cartesian coordinates:

See Example 2-11 (HW)

Conversion formulas:

$$x = R \sin \theta \cos \phi$$

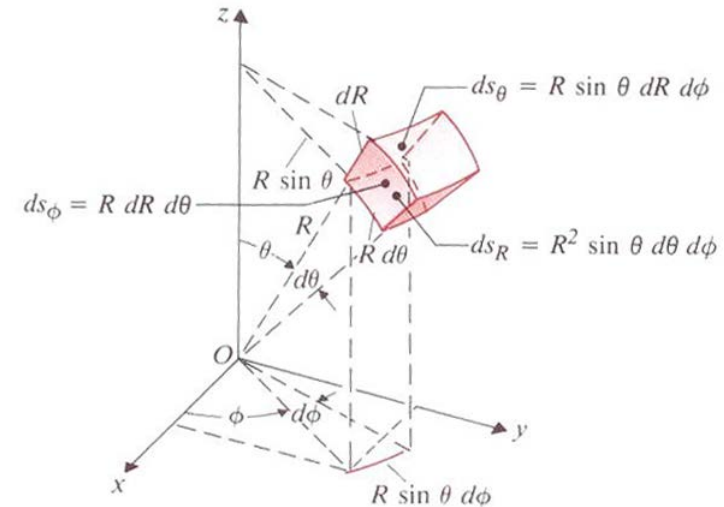
$$y = R \sin \theta \sin \phi$$

$$z = R \cos \theta$$

$$R = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1} \frac{y}{x}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Three Basic Orthogonal Coordinate Systems

| Coordinate System Relations | | Cartesian Coordinates (x, y, z) | Cylindrical Coordinates (r, ϕ, z) | Spherical Coordinates (R, θ, ϕ) |
|-----------------------------|--|--|---|--|
| Base vectors | \mathbf{a}_{u_1} \mathbf{a}_{u_2} \mathbf{a}_{u_3} | \mathbf{a}_x \mathbf{a}_y \mathbf{a}_z | \mathbf{a}_r \mathbf{a}_ϕ \mathbf{a}_z | \mathbf{a}_R \mathbf{a}_θ \mathbf{a}_ϕ |
| Metric coefficients | h_1 h_2 h_3 | 1 1 1 | 1 r 1 | 1 R $R \sin \theta$ |
| Differential volume | dv | $dx dy dz$ | $r dr d\phi dz$ | $R^2 \sin \theta dR d\theta d\phi$ |

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Integrals Containing Vector Functions

Common forms of integrals containing vector functions:

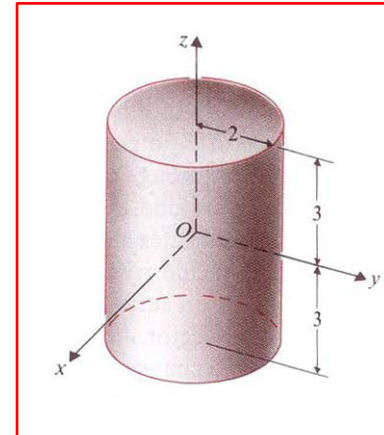
$$\int_V \mathbf{F} dv$$

$$\int_C V d\mathbf{l}$$

$$\int_C \mathbf{F} \cdot d\mathbf{l}$$

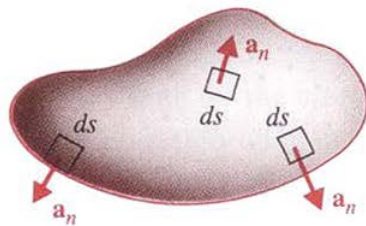
$$\int_S \mathbf{A} \cdot d\mathbf{s}$$

Surface normal vectors?

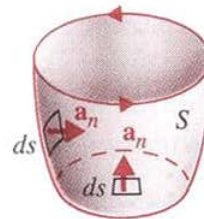


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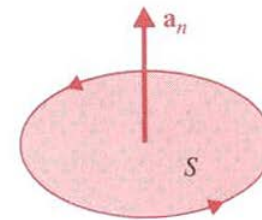
Defining the surface normal vector:



(a) A closed surface.



(b) An open surface.



(c) A disk.

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