

Introduction to Electromagnetism

Gradient and Divergence

(2-6, 2-7, 2-8)

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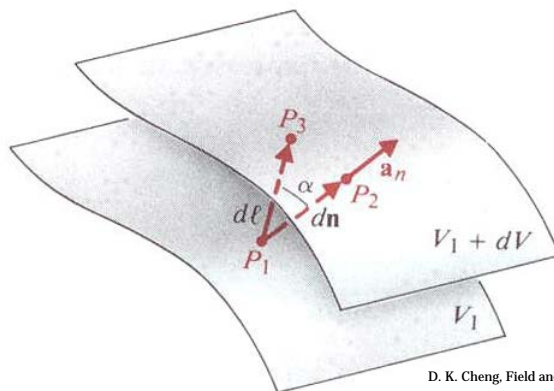
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Gradient of a Scalar Field (1)

Let us consider a scalar function of space coordinates: $V(u_1, u_2, u_3)$

Suppose that there are two surfaces on which V is constant:

$$\rightarrow V = V_1 \text{ \& } V = V_1 + dV$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Space rate of change: $dV / dl \rightarrow$ **Maximum when: $d\mathbf{l} // d\mathbf{n}$**

Gradient of a scalar: The vector of the maximum space rate of increase of the scalar

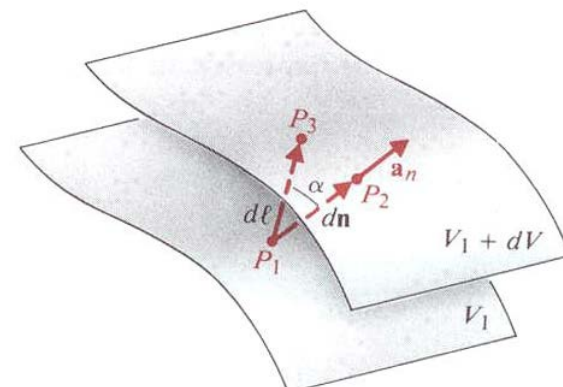
$$\mathbf{grad} V \equiv \mathbf{a}_n \frac{dV}{dn} \quad \leftarrow \mathbf{a}_n : \text{Unit normal vector}$$

$$\rightarrow \nabla V \equiv \mathbf{a}_n \frac{dV}{dn}$$

Gradient of a Scalar Field (2)

Directional derivative along $d\mathbf{l}$:

$$\begin{aligned}\frac{dV}{dl} &= \frac{dV}{dn} \frac{dn}{dl} = \frac{dV}{dn} \cos \alpha \\ &= \frac{dV}{dn} \mathbf{a}_n \cdot \mathbf{a}_l = (\nabla V) \cdot \mathbf{a}_l\end{aligned}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\begin{aligned}\rightarrow dV &= (\nabla V) \cdot d\mathbf{l} & \leftarrow d\mathbf{l} &= \mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3 \\ & & &= \mathbf{a}_{u_1} (h_1 du_1) + \mathbf{a}_{u_2} (h_2 du_2) + \mathbf{a}_{u_3} (h_3 du_3)\end{aligned}$$

Expressed in terms of the differential changes in coordinates:

$$\begin{aligned}\rightarrow dV &= \frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3 \quad \leftarrow \text{Chain rule} \\ &= \left(\mathbf{a}_{u_1} \frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial l_3} \right) \cdot (\mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3) \\ &= \left(\mathbf{a}_{u_1} \frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial l_3} \right) \cdot d\mathbf{l}\end{aligned}$$

$$\therefore \nabla V = \mathbf{a}_{u_1} \frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial l_3} = \mathbf{a}_{u_1} \frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3 \partial u_3}$$

Gradient of a Scalar Field (3)

Recall:

$$\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial l_3} = \mathbf{a}_{u_1} \frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3 \partial u_3}$$

Vector differential operator (del):

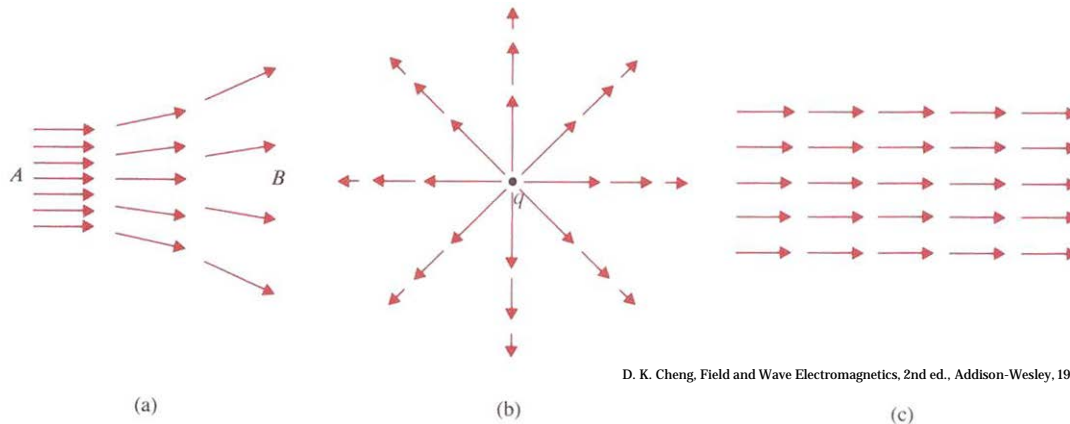
$$\nabla \equiv \mathbf{a}_{u_1} \frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3 \partial u_3}$$

In Cartesian coordinates:

$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Divergence of a Vector Field (1)

Directed field lines for vector fields: Flux lines or streamlines



Consider a volume with an enclosed surface:

1. An excess of outward flow through the surface:
→ A net **positive** divergence → The volume contains a **source**.
2. An excess of inward flow through the surface:
→ A net **negative** divergence → The volume contains a **sink**.
3. No excess of outward or inward flow through the surface:
→ A net **zero** divergence → **Neither** a **source** nor a **sink**

Divergence of a Vector Field (2)

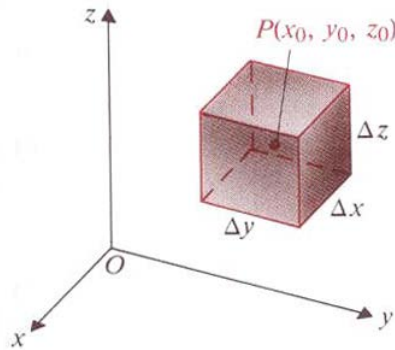
Definition of the divergence a vector field \mathbf{A} at a point:

The net outward flux of \mathbf{A} per unit volume as the volume about the point tends to zero:

$$\text{div } \mathbf{A} \equiv \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

Consider a differential volume of sides Δx , Δy and Δz in Cartesian coordinates:

$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \left[\int_{\text{front face}} + \int_{\text{back face}} + \int_{\text{right face}} + \int_{\text{left face}} + \int_{\text{top face}} + \int_{\text{bottom face}} \right] \mathbf{A} \cdot d\mathbf{s}$$

$$\int_{\text{front face}} \mathbf{A} \cdot d\mathbf{s} = \mathbf{A}_{\text{front face}} \cdot \Delta \mathbf{s}_{\text{front face}} = \mathbf{A}_{\text{front face}} \cdot \mathbf{a}_x (\Delta y \Delta z)$$

$$= A_x \left(x_0 + \frac{\Delta x}{2}, y_0, z_0 \right) \Delta y \Delta z$$

→ Taylor series expansion:

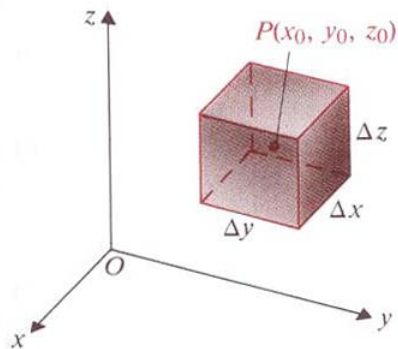
$$A_x \left(x_0 + \frac{\Delta x}{2}, y_0, z_0 \right) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2} \frac{\partial A_x}{\partial x} \Bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}$$

Divergence of a Vector Field (3)

Continued:

$$\int_{\text{back face}} \mathbf{A} \cdot d\mathbf{s} = \mathbf{A}_{\text{back face}} \cdot \Delta\mathbf{s}_{\text{back face}} = \mathbf{A}_{\text{back face}} \cdot (-\mathbf{a}_x \Delta y \Delta z)$$

$$= -A_x \left(x_0 - \frac{\Delta x}{2}, y_0, z_0 \right) \Delta y \Delta z$$



$$A_x \left(x_0 - \frac{\Delta x}{2}, y_0, z_0 \right) = A_x(x_0, y_0, z_0) + \left(-\frac{\Delta x}{2} \right) \frac{\partial A_x}{\partial x} \Bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}$$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \left[\int_{\text{front face}} + \int_{\text{back face}} \right] \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \text{H.O.T.} \right) \Bigg|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

Similarly:

$$\rightarrow \left[\int_{\text{right face}} + \int_{\text{left face}} \right] \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_y}{\partial y} + \text{H.O.T.} \right) \Bigg|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

$$\rightarrow \left[\int_{\text{top face}} + \int_{\text{bottom face}} \right] \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_z}{\partial z} + \text{H.O.T.} \right) \Bigg|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

In result:

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Bigg|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z + \text{H.O.T.}$$

Divergence of a Vector Field (4)

Recall:

$$\operatorname{div} \mathbf{A} \equiv \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Big|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z + \text{H.O.T.}$$

$$\operatorname{div} \mathbf{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

$$\operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\leftarrow \nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{A} \equiv \operatorname{div} \mathbf{A}$$

In general orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

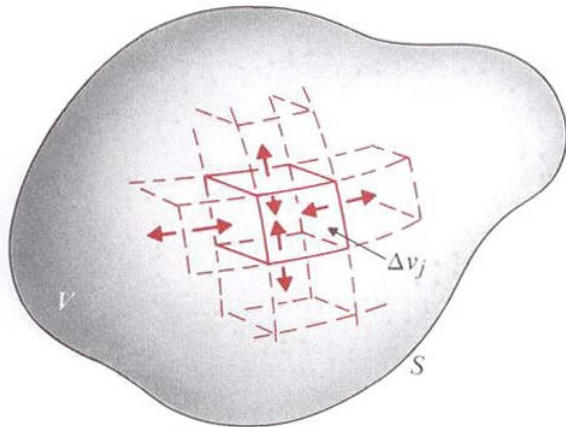
→ HW

Divergence Theorem

The **volume** integral of the divergence of a vector field equals the total outward flux of the vector through the **surface** that bounds the volume:

$$\int_V \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s} \quad \leftarrow \mathbf{A}: \text{Smooth or piecewise smooth!}$$

Divide the volume into small differential volume elements:



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$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \quad \leftarrow \text{By definition}$$

$$\begin{aligned} \int_V \nabla \cdot \mathbf{A} dv &= \lim_{\Delta v_j \rightarrow 0} \left[\sum_{j=1}^N (\nabla \cdot \mathbf{A})_j \Delta v_j \right] \\ &= \lim_{\Delta v_j \rightarrow 0} \left[\sum_{j=1}^N \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \right] \\ &= \oint_S \mathbf{A} \cdot d\mathbf{s} \end{aligned}$$



Carl Friedrich Gauss
(1777 - 1855)

What if not continuously smooth?

A **volume integral of the divergence** of a vector can be converted into a **closed surface integral of the vector**, and vice versa!