

# Introduction to Electromagnetism

## Curl of a Vector Field, etc.

(2-9, 2-10, 2-11, 2-12)

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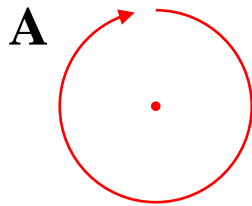
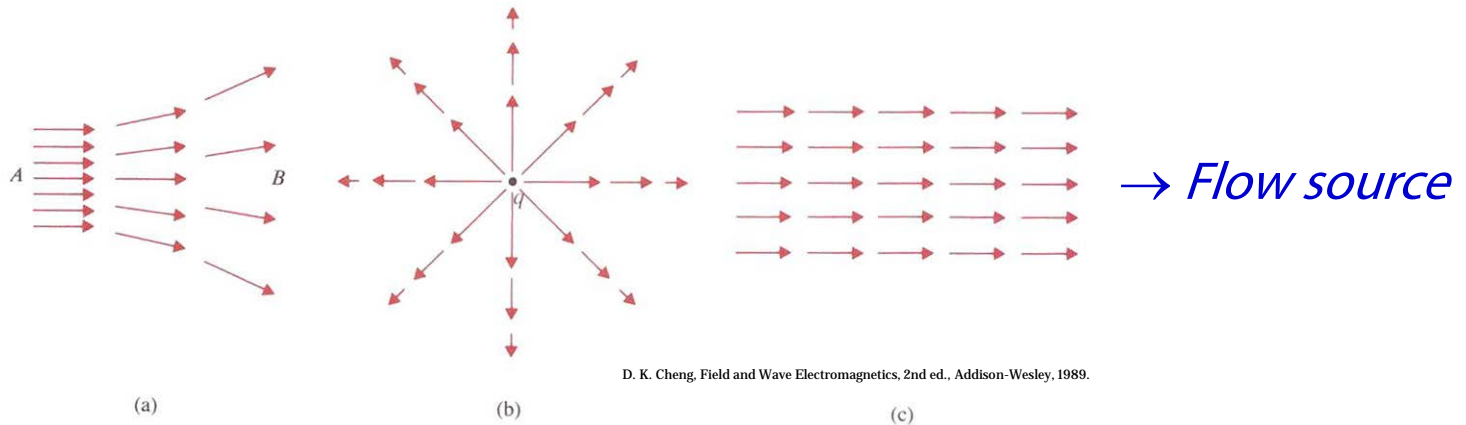
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# Curl of a Vector Field (1)

Directed field lines for vector fields: Flux lines or streamlines



*What if there a **vortex source** that causes a circulation of a vector field around it?*

Net circulation (or circulation) of a vector field around a closed path:

$$\rightarrow \text{Circulation of } \mathbf{A} \text{ around contour } C \equiv \oint_C \mathbf{A} \cdot d\mathbf{l}$$

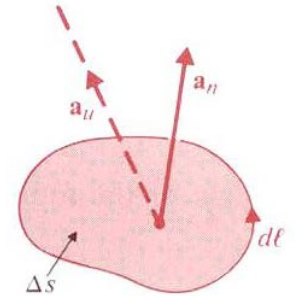
# Curl of a Vector Field (2)

Definition of the curl of a vector field  $\mathbf{A}$  at a point:

A vector whose magnitude is **the maximum net circulation** of  $\mathbf{A}$  per unit area as the area tends to zero and whose direction is **the normal direction of the area (right-hand rule)** when the area is oriented to make the net circulation maximum:

$$\text{curl } \mathbf{A} \equiv \nabla \times \mathbf{A}$$

$$\equiv \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \left[ \mathbf{a}_n \oint_C \mathbf{A} \cdot d\mathbf{l} \right]_{\text{max}}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Component of  $\nabla \times \mathbf{A}$  in any other direction  $\mathbf{a}_u$ :

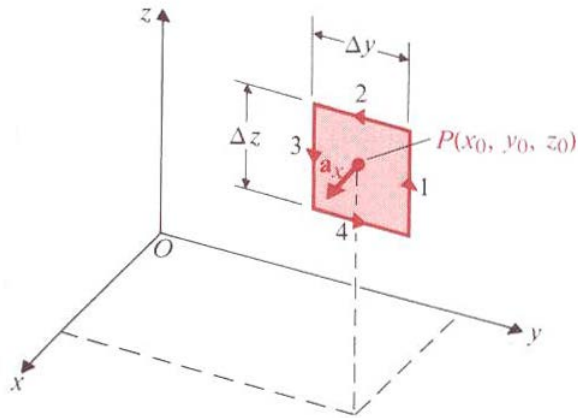
$$(\nabla \times \mathbf{A})_u = \mathbf{a}_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \rightarrow 0} \frac{1}{\Delta s_u} \left( \oint_{C_u} \mathbf{A} \cdot d\mathbf{l} \right)$$

# Curl of a Vector Field (3)

Consider a differential rectangle in Cartesian coordinates:

$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

$$(\nabla \times \mathbf{A})_x = \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \rightarrow 0} \frac{1}{\Delta y \Delta z} \left( \oint_{\text{sides } 1,2,3,4} \mathbf{A} \cdot d\mathbf{l} \right)$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

**Side 1:**  $d\mathbf{l} = \mathbf{a}_z \Delta z$  ← Taylor series expansion

$$\rightarrow \mathbf{A} \cdot d\mathbf{l} = A_z \left( x_0, y_0 + \frac{\Delta y}{2}, z_0 \right) \Delta z$$

$$= \left\{ A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y} \Big|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta z$$

**Side 3:**  $d\mathbf{l} = -\mathbf{a}_z \Delta z$  ← Taylor series expansion

$$\rightarrow \mathbf{A} \cdot d\mathbf{l} = A_z \left( x_0, y_0 - \frac{\Delta y}{2}, z_0 \right) (-\Delta z)$$

$$= \left\{ A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y} \Big|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta z)$$

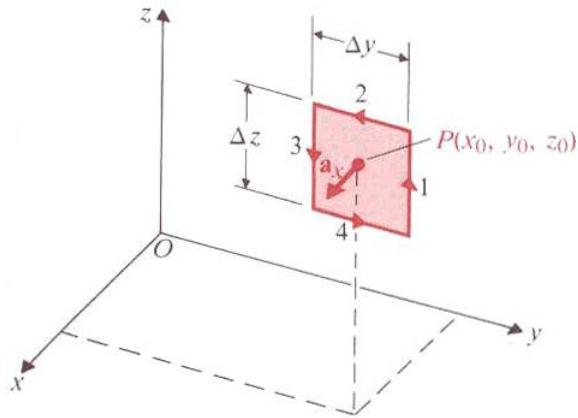
$$\rightarrow \int_{\text{sides } 1 \& 3} \mathbf{A} \cdot d\mathbf{l} = \Delta y \Delta z \left( \frac{\partial A_z}{\partial y} + \text{H.O.T.} \right) \Big|_{(x_0, y_0, z_0)}$$

# Curl of a Vector Field (4)

Continued:

$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

$$(\nabla \times \mathbf{A})_x = \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \rightarrow 0} \frac{1}{\Delta y \Delta z} \left( \oint_{\text{sides } 1,2,3,4} \mathbf{A} \cdot d\mathbf{l} \right)$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

**Side 2:**  $d\mathbf{l} = -\mathbf{a}_y \Delta y$

$$\begin{aligned} \rightarrow \mathbf{A} \cdot d\mathbf{l} &= A_y \left( x_0, y_0, z_0 + \frac{\Delta z}{2} \right) (-\Delta y) \\ &= \left\{ A_y(x_0, y_0, z_0) + \frac{\Delta z}{2} \frac{\partial A_y}{\partial z} \Big|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta y) \end{aligned}$$

**Side 4:**  $d\mathbf{l} = \mathbf{a}_y \Delta y$

$$\begin{aligned} \rightarrow \mathbf{A} \cdot d\mathbf{l} &= A_y \left( x_0, y_0, z_0 - \frac{\Delta z}{2} \right) \Delta y \\ &= \left\{ A_y(x_0, y_0, z_0) - \frac{\Delta z}{2} \frac{\partial A_y}{\partial z} \Big|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta y \end{aligned}$$

$$\rightarrow \int_{\text{sides } 2 \& 4} \mathbf{A} \cdot d\mathbf{l} = -\Delta y \Delta z \left( \frac{\partial A_y}{\partial z} + \text{H.O.T.} \right) \Big|_{(x_0, y_0, z_0)}$$

# Curl of a Vector Field (5)

Recall:

$$\begin{aligned}
 (\nabla \times \mathbf{A})_x &= \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \rightarrow 0} \frac{1}{\Delta y \Delta z} \left( \oint_{\text{sides } 1,2,3,4} \mathbf{A} \cdot d\mathbf{l} \right) \\
 &= \lim_{\Delta y \Delta z \rightarrow 0} \frac{1}{\Delta y \Delta z} \left\{ \Delta y \Delta z \left( \frac{\partial A_z}{\partial y} + \text{H.O.T.} \right) \Big|_{(x_0, y_0, z_0)} - \Delta y \Delta z \left( \frac{\partial A_y}{\partial z} + \text{H.O.T.} \right) \Big|_{(x_0, y_0, z_0)} \right\} \\
 &\rightarrow (\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}
 \end{aligned}$$

Similarly:

$$\left. \begin{aligned}
 (\nabla \times \mathbf{A})_y &= \mathbf{a}_y \cdot (\nabla \times \mathbf{A}) = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\
 (\nabla \times \mathbf{A})_z &= \mathbf{a}_z \cdot (\nabla \times \mathbf{A}) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}
 \end{aligned} \right\}$$

In result:

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

# Curl of a Vector Field (6)

Recall:

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Determinantal form:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial & \partial & \partial \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In general orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \partial & \partial & \partial \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

→ HW

# Stokes's Theorem

For a very small differential area  $\Delta s_j$  bounded by a contour  $C_j$ :

$$(\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \oint_{C_j} \mathbf{A} \cdot d\mathbf{l}$$

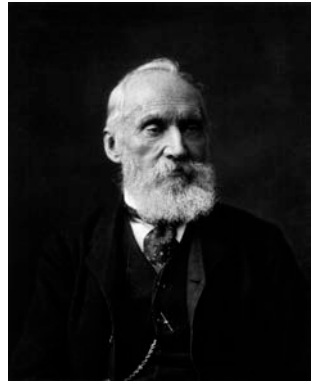
←  $\mathbf{A}$ : Smooth or piecewise smooth!

Add the contributions of all the differential area to the flux:

$$\lim_{\Delta s_j \rightarrow 0} \sum_{j=1}^N (\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

Sum up the line integrals around the contours of all the differential elements:

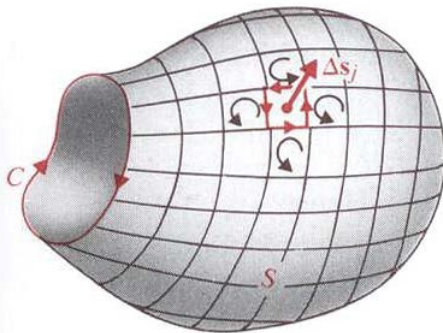
$$\rightarrow \lim_{\Delta s_j \rightarrow 0} \sum_{j=1}^N \left( \oint_{C_j} \mathbf{A} \cdot d\mathbf{l} \right) = \oint_C \mathbf{A} \cdot d\mathbf{l}$$



William Thomson Kelvin  
(1824 - 1907)



George Gabriel Stokes  
(1819 - 1903)



Stokes's theorem:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$



# Two Null Identities

Identity I:  $\nabla \times (\nabla V) \equiv 0$

Proof:

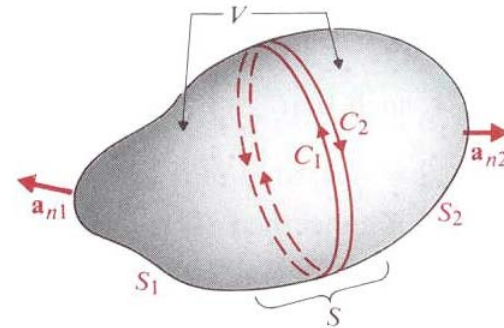
$$\begin{aligned} \int_S [\nabla \times (\nabla V)] \cdot d\mathbf{s} &= \oint_C (\nabla V) \cdot d\mathbf{l} \quad \text{Recall: } dV = (\nabla V) \cdot d\mathbf{l} \rightarrow \text{Eq. (2-88)} \\ &= \oint_C dV = 0 \quad \rightarrow \text{For any } S \end{aligned}$$

Curl-free vector:  $\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V \rightarrow \text{Irrotational (conservative)}$

Identity II:  $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$

Proof:

$$\begin{aligned} \int_V \nabla \cdot (\nabla \times \mathbf{A}) dv &= \oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_{S_1} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n_1} ds + \int_{S_2} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n_2} ds \\ &= \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{l} = 0 \quad \rightarrow \text{For any } V \end{aligned}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Divergenceless vector:  $\nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{B} = \nabla \times \mathbf{A} \rightarrow \text{Solenoidal}$

# Helmholtz's Theorem (1)

1. Solenoidal and irrotational if:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} = 0$$

2. Solenoidal but not irrotational if:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0$$

3. Irrotational but not solenoidal if:

$$\nabla \times \mathbf{F} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F} \neq 0$$

4. Neither solenoidal nor irrotational if:

$$\nabla \cdot \mathbf{F} \neq 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0$$

# Helmholtz's Theorem (2)

## Helmholtz's theorem:

A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.



Hermann von Helmholtz  
(1824 - 1894)

**Divergence of a vector:** A measure of the strength of the flow source

**Curl of a vector:** A measure of the strength of the vortex source

Decomposition of a vector into an irrotational part and a solenoidal part:

$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s \quad \text{with} \quad \begin{cases} \nabla \times \mathbf{F}_i = 0 \\ \nabla \cdot \mathbf{F}_i = g \end{cases} \quad \& \quad \begin{cases} \nabla \cdot \mathbf{F}_s = 0 \\ \nabla \times \mathbf{F}_s = \mathbf{G} \end{cases}$$

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_i = g$$

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_s = \mathbf{G}$$

→ When  $g$  and  $\mathbf{G}$  are specified, the vector function  $\mathbf{F}$  is determined!

$$\rightarrow \mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$