Chapter 6. Optics of Solids
Part 1

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6.2 Macroscopic Fields and Maxwell's Equations

The electromagnetic state of matter at a given point is described by four quantities:

(1) The volume density of electric charge ρ

(2) The volume density of electric dipoles, *polarization* **P**

(3) The volume density of magnetic dipoles, *magnetization* **M**

(4) The electric current per unit area, *current density* **J**.

$$\begin{cases} \nabla \times \vec{E} = -\mu_o \dfrac{\partial \vec{H}}{\partial t} - \mu_o \dfrac{\partial \vec{M}}{\partial t} \\ \nabla \times \vec{H} = \varepsilon_o \dfrac{\partial \vec{E}}{\partial t} + \dfrac{\partial \vec{P}}{\partial t} + \vec{J} \\ \nabla \cdot \vec{E} = -\dfrac{1}{\varepsilon_o} \nabla \cdot \vec{P} + \dfrac{\rho}{\varepsilon_o} \\ \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \end{cases}$$

electric displacement

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

magnetic induction

$$\vec{B} = \mu_o (\vec{H} + \vec{M})$$

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J} \\ \nabla \cdot \vec{D} = \rho \\ \nabla \cdot \vec{B} = 0 \end{cases}$$

6.2 Macroscopic Fields and Maxwell's Equations

Ohm's law $\vec{J} = \sigma\vec{E}, \quad \sigma = \text{conductivity}$

$$\vec{D} = \varepsilon\vec{E}, \quad \text{constitutive relation}$$

$$\vec{B} = \mu\vec{H}$$

$$\vec{P} = (\varepsilon - \varepsilon_o)\vec{E} = \chi\varepsilon_o\vec{E}$$

$$\chi = \frac{\varepsilon}{\varepsilon_o} - 1, \quad \text{electric susceptibility}$$

$$D_i = \sum_j \varepsilon_{ij} E_j$$

For nonisotropic media, such as most crystals, the magnitude of the polarization varies with the direction of the applied field, consequently, χ must be expressed as a tensor.

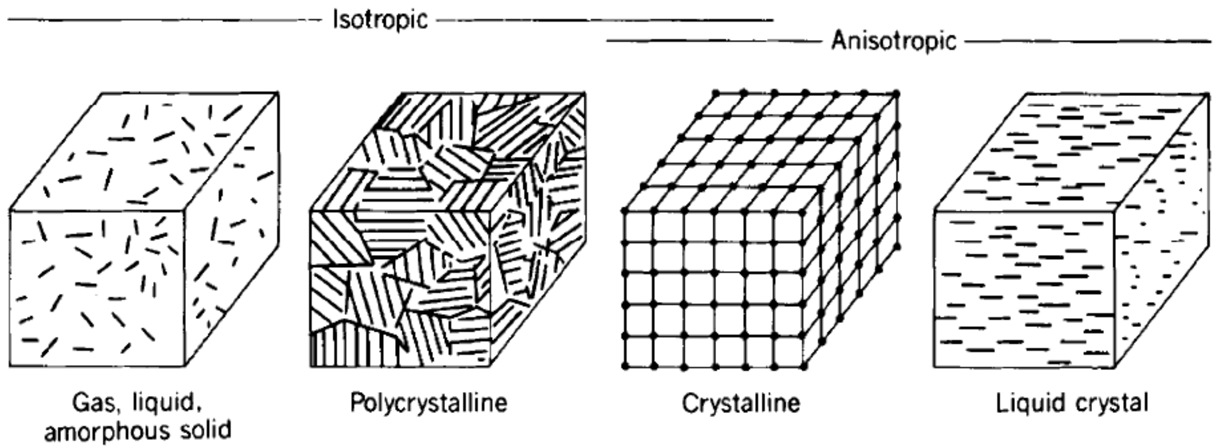


Figure 6.3-1 Positional and orientational order in different kinds of materials.

Bahaa E. A. Saleh, Malvin Carl Teich, Fundamentals of Photonics (1991)

6.3 The General Wave Equation

Consider nonmagnetic, electrically neutral media where **M** and ρ are both zero.

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{E} = -\frac{1}{\varepsilon_o} \nabla \cdot \vec{P}$$

$$\nabla \cdot \vec{H} = 0$$

Source terms

$$\nabla \times (\nabla \times \vec{E}) + \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_o \frac{\partial^2 \vec{P}}{\partial t^2} - \mu_o \frac{\partial \vec{J}}{\partial t}$$

6.4 Propagation of light in isotropic dielectrics. Dispersion

In a nonconducting, isotropic medium, the electrons are permanently bound to the atoms comprising the medium and there is no preferential direction.
Suppose that the number of electrons per unit volume is *N* and each electron of charge –e is displaced a distance **r** from its equilibrium position. The macroscopic polarization **P** of the medium is $\vec{P} = -Ne\vec{r}$

If the electron is elastically bound to its equilibrium position with a force constant *K*, $-e\vec{E} = K\vec{r}$

$$\vec{P} = \frac{Ne^2}{K}\vec{E}$$

Consider the bound electrons as classical damped harmonic oscillators under the applied electric field with the harmonic time dependence $e^{-i\omega t}$.

$$m\frac{d^2\vec{r}}{dt^2} + m\gamma\frac{d\vec{r}}{dt} + K\vec{r} = -e\vec{E}$$

$$(-m\omega^2 - i\omega m\gamma + K)\vec{r} = -e\vec{E}$$

$$\vec{P} = \frac{Ne^2}{-m\omega^2 - i\omega m\gamma + K}\vec{E}$$

6.4 Propagation of light in isotropic dielectrics. Dispersion

$$\vec{P} = \frac{Ne^2/m}{\omega_o^2 - \omega^2 - i\omega\gamma}\vec{E}, \quad \omega_o = \sqrt{\frac{K}{m}} = \text{effective resonance frequency}$$

$$\nabla\times(\nabla\times\vec{E}) + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\mu_o\frac{\partial^2\vec{P}}{\partial t^2} - \mu_o\frac{\partial\vec{J}}{\partial t}$$

Dielectric material

$$\nabla\times(\nabla\times\vec{E}) + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{-\mu_o Ne^2}{m}\left(\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}\right)\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E} = \frac{1}{c^2}\left(1 + \frac{Ne^2}{m\varepsilon_o}\cdot\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}\right)\frac{\partial^2\vec{E}}{\partial t^2}$$

Seek a solution of the form (*homogeneous* plane harmonic waves)

$$\vec{E} = \vec{E}_o e^{i(\tilde{K}z - \omega t)}$$

$$\tilde{K}^2 = \frac{\omega^2}{c^2}\left(1 + \frac{Ne^2}{m\varepsilon_o}\cdot\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}\right)$$

6.4 Propagation of light in isotropic dielectrics. Dispersion

$$\tilde{K} = k + i\alpha, \quad \tilde{N} = n + i\kappa, \quad \tilde{K} = \frac{\omega}{c}\tilde{N}$$

$$\tilde{N}^2 = (n + i\kappa)^2 = 1 + \frac{Ne^2}{m\varepsilon_o} \cdot \frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}$$

$$n^2 - \kappa^2 = 1 + \frac{Ne^2}{m\varepsilon_o}\left(\frac{\omega_o^2 - \omega^2}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2}\right)$$

$$2n\kappa = \frac{Ne^2}{m\varepsilon_o}\left(\frac{\gamma\omega}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2}\right)$$

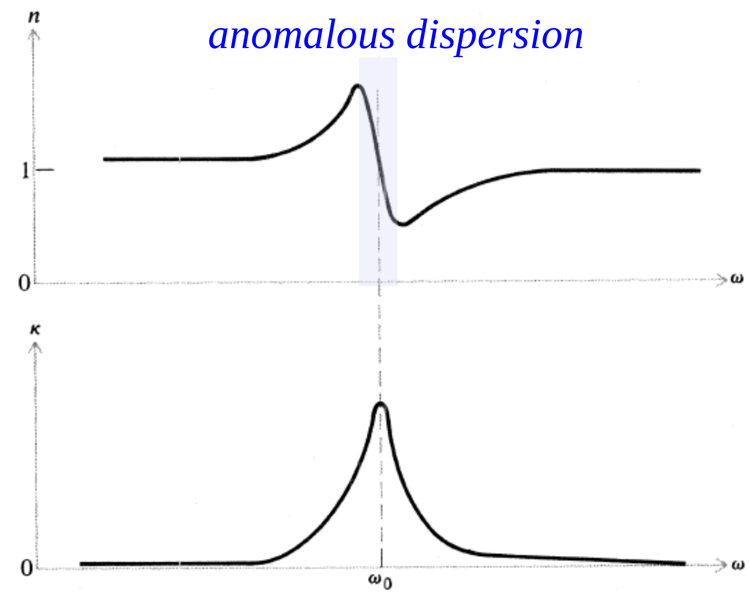


Figure 6.1. Graphs of the index of refraction and extinction coefficient versus frequency near a single resonance line.

$$\vec{E} = \vec{E}_o e^{-\alpha z} e^{i(kz - \omega t)}, \quad \text{energy} \sim \left|\vec{E}\right|^2 \sim \left|\vec{E}_o\right|^2 e^{-2\alpha z}$$

2α = *coefficient of absorption* of the medium
κ = *extinction index* of the medium

$$\alpha = \frac{\omega}{c}\kappa, \quad \text{phase velocity } u = \frac{\omega}{k} = \frac{c}{n}$$

The absorption is strongest at the resonance frequency ω_o and the index of refraction is greater than unity for small frequencies and increases with frequency as the resonance frequency is approached (***normal dispersion***). At or near the resonance frequency, the dispersion becomes **anomalous**.

6.4 Propagation of light in isotropic dielectrics. Dispersion

If a certain fraction f_1 of electrons has a resonance frequency ω_1 and a fraction f_2 has the resonance frequency ω_2, and so on,

$$\tilde{N}^2 = 1 + \frac{Ne^2}{m\varepsilon_o}\sum_j\left(\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma\omega}\right)$$

$$f_j = \text{oscillator strengths}$$

In the limit of zero frequency, the static dielectric constant becomes

$$n^2 \approx 1 + \frac{Ne^2}{m\varepsilon_o}\sum_j \frac{f_j}{\omega_j^2} = \varepsilon_r$$

In the high frequency region, the index should dip below unity and approach unity from below as ω becomes infinite.

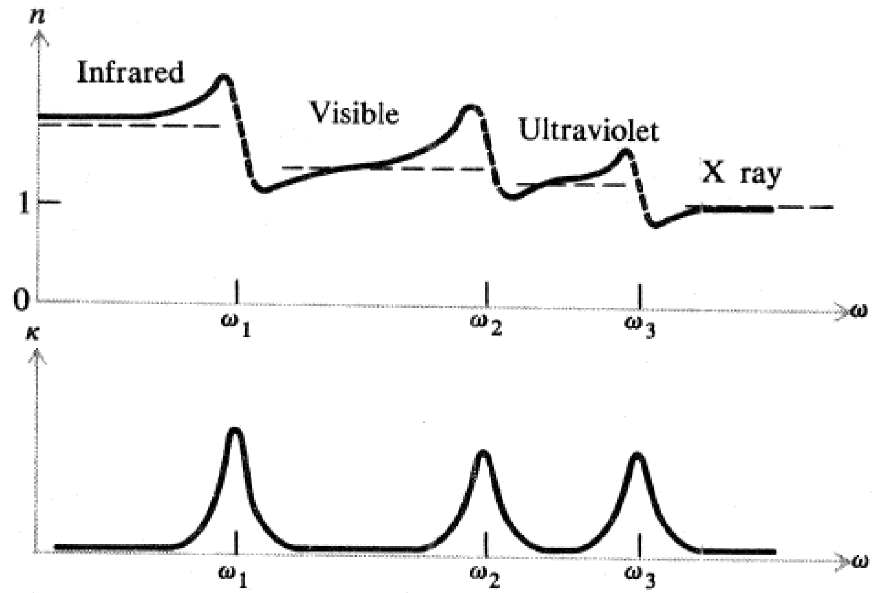


Figure 6.2. Index of refraction and extinction index for a hypothetical substance with absorption bands in the infrared, visible, and ultraviolet regions of the spectrum.

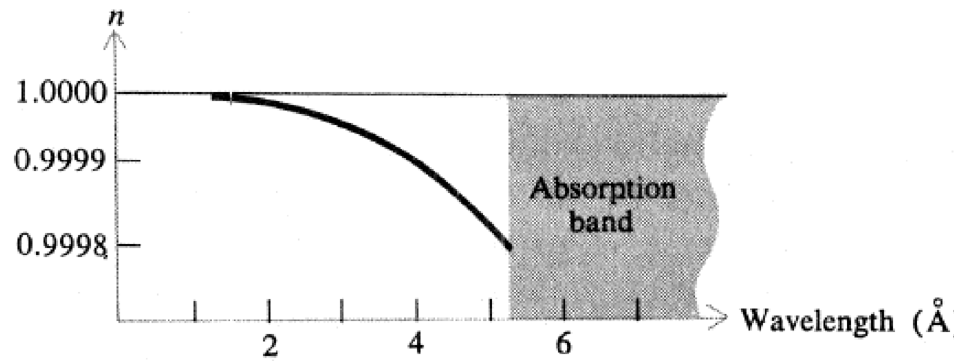


Figure 6.3. Measured index of refraction of quartz in the x-ray region.

6.4 Propagation of light in isotropic dielectrics. Dispersion

If the damping constants γ_j are sufficiently small, then the index of refraction is essentially real.

$$n^2 \approx 1 + \frac{Ne^2}{m\varepsilon_o} \sum_j \left(\frac{f_j}{\omega_j^2 - \omega^2} \right)$$

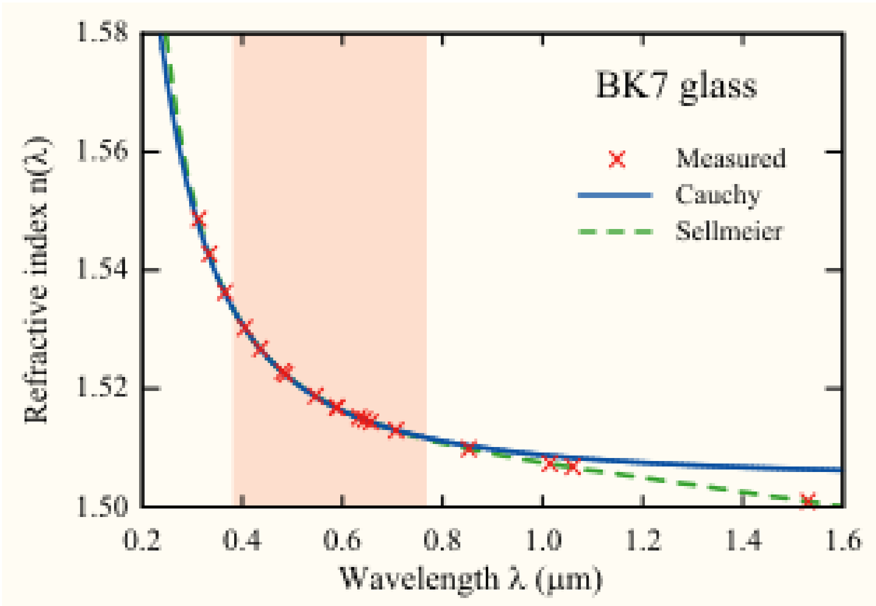
The ***Sellmeier formula***, first proposed in 1871 by Wilhelm Sellmeier, is an empirical relationship between refractive index and wavelength for a particular transparent medium. The equation is used to determine the dispersion of light in the medium.

$$n^2(\lambda) = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3}$$

Cauchy's equation was introduced by Augustin-Louis Cauchy in 1836.

$$n(\lambda) = B + \frac{C}{\lambda^2} + \frac{D}{\lambda^4} + \cdots$$

The *Sellmeier equation* handles anomalously dispersive regions, and more accurately models a material's refractive index across the ultraviolet, visible, and infrared spectrum.



https://en.wikipedia.org/wiki/Sellmeier_equation

6.5 Propagation of light in conducting media

Since the conduction electrons are not bound, there is no elastic restoring force as there was in the case of polarization.

$$m\frac{d\vec{v}}{dt}+\frac{m}{\tau}\vec{v}=-e\vec{E}$$

Current density $\vec{J}=-Ne\vec{v}$

$$\frac{d\vec{J}}{dt}+\frac{1}{\tau}\vec{J}=\frac{Ne^2}{m}\vec{E}$$

The decay of a transient current is governed by

$$\frac{d\vec{J}}{dt}+\frac{1}{\tau}\vec{J}=0, \quad \vec{J}=\vec{J}_o e^{-t/\tau}, \quad \tau=\text{relation time}$$

For a static electric field,

$$\vec{J}=\frac{Ne^2\tau}{m}\vec{E}=\sigma\vec{E}, \quad \text{static conductivity } \sigma=\frac{Ne^2\tau}{m}$$

6.5 Propagation of light in conducting media

Let us assume a harmonic time dependence $e^{-i\omega t}$ for both applied electric field **E** and the resulting current **J**,

$$(-i\omega+\tau^{-1})\vec{J}=\frac{Ne^2}{m}\vec{E}=\tau^{-1}\sigma\vec{E}, \quad \therefore \vec{J}=\frac{\sigma}{1-i\omega\tau}\vec{E}$$

conducting media

$$\nabla\times(\nabla\times\vec{E})+\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=-\mu_o\frac{\partial^2\vec{P}}{\partial t^2}-\mu_o\frac{\partial\vec{J}}{\partial t}$$

$$\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}+\frac{\mu_o\sigma}{1-i\omega\tau}\frac{\partial\vec{E}}{\partial t}$$

For a trial solution, we take a simple *homogeneous* plane-wave type

$$\vec{E}=\vec{E}_o e^{i(\tilde{K}z-\omega t)}$$

$$\tilde{K}^2=\frac{\omega^2}{c^2}+\frac{i\omega\mu_o\sigma}{1-i\omega\tau}$$

$$\tilde{K}^2\approx i\omega\mu_o\sigma \quad \text{(for very low frequencies)}$$

6.5 Propagation of light in conducting media

$$\tilde{K} = k + i\alpha \approx \sqrt{i\omega\mu_o\sigma} = (1+i)\sqrt{\omega\mu_o\sigma/2}$$

$$k \approx \alpha \approx \sqrt{\frac{\omega\mu_o\sigma}{2}}$$

$$\tilde{N} = n + i\kappa$$

$$n \approx \kappa \approx \sqrt{\frac{\sigma}{2\omega\varepsilon_o}}$$

$$\delta = \frac{1}{\alpha} \approx \sqrt{\frac{2}{\omega\mu_o\sigma}} = \sqrt{\frac{\lambda_o}{c\pi\sigma\mu_o}} \quad \text{(skin depth)}$$

So good conductors are also highly opaque. A high value of the conductivity gives a large coefficient of absorption a and a correspondingly small skin depth.

6.5 Propagation of light in conducting media

$$\tilde{N}^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}}$$

$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_o}} = \sqrt{\frac{\mu_o \sigma c^2}{\tau}}$$ (plasma frequency for the metal)

$$n^2 - \kappa^2 = 1 - \frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

$$2n\kappa = \frac{\omega_p^2}{\omega^2 + \tau^{-2}}\left(\frac{1}{\omega\tau}\right)$$

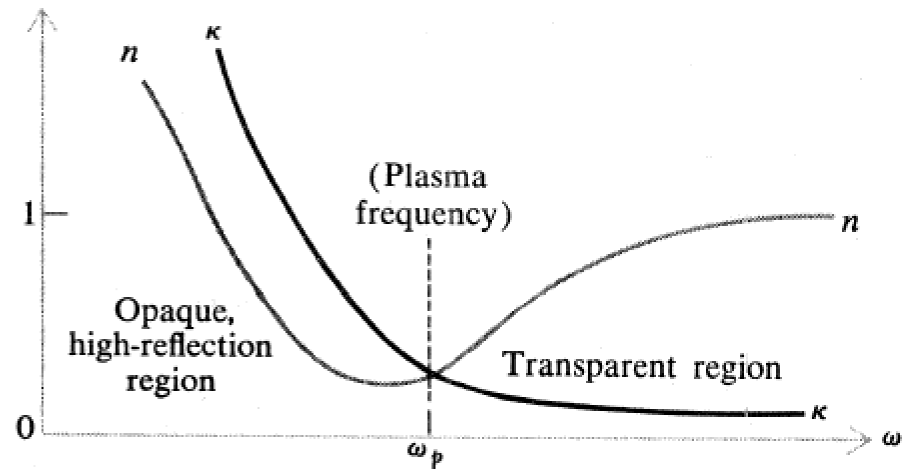


Figure 6.4. Index of refraction and extinction index versus frequency for a metal.

- Typical relaxation times for metals are of the order of 10^{-13} s (infrared region)
- Typical plasma frequencies of metals are around 10^{15} s^{-1} (visible and near UV region).

$$\tilde{N}^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}} + \frac{Ne^2}{m\varepsilon_o}\sum_j \left(\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$ (for poor conductors and semiconductors)