Introduction to Electromagnetism
Static Electric Fields
(3-1, 3-2, 3-3)

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Introduction to Electrostatics

Hypotheses in electrostatics:

- Electric charges are at rest.
- Electric fields do not change with time.
- There are no magnetic fields.

Charles-Augustin de Coulomb
(1736 - 1806)

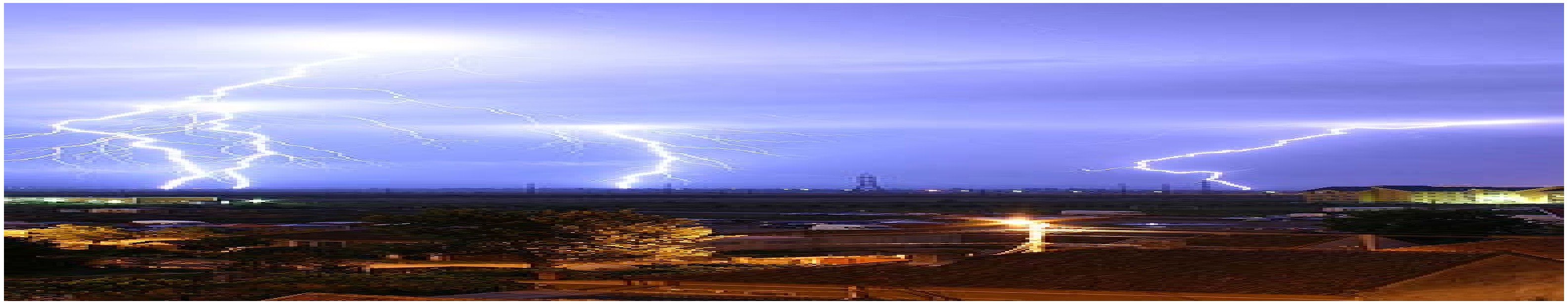
Coulomb's law:

$$\mathbf{F}_{12} = \mathbf{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2}$$

Helmholtz's theorem:

A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

Electrostatics in nature:



http://en.wikipedia.org/wiki/File:Lightning_over_Oradea_Romania_3.jpg

Fundamental Postulates of Electrostatics (1)

Electric field intensity:

The force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists:

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad \text{(V/m)}$$

$$\mathbf{F} = q\mathbf{E} \quad \text{(N)}$$

Two fundamental postulates (via Helmholtz's theorem):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

ρ : Volume charge density

ε_0 : Permittivity of free space

$$\nabla \times \mathbf{E} = 0$$

$\rightarrow$ Irrotational (conservative) but not solenoidal unless $\rho = 0$

Fundamental Postulates of Electrostatics (2)

Integral form:

$$\int_V \nabla \cdot \mathbf{E} dv = \oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} \int_V \rho dv = \frac{Q}{\varepsilon_0}$$

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}}$$ $\rightarrow$ Gauss's law

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0}$$ $\rightarrow$ Kirchhoff's voltage law

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

The work done by the electric field while a unit charge is moving from P_1 to P_2 remains constant regardless of path! $\rightarrow$ *"Conservation" of work or energy*

$\rightarrow$ Irrotational or conservative

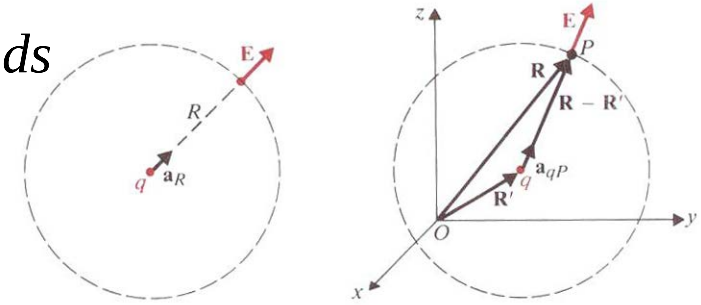
Coulomb's Law

For a single point charge q at the origin:

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{1}{\varepsilon_0}\int_V \rho dv = \frac{q}{\varepsilon_0} = \oint_S (\mathbf{a}_R E_R)\cdot \mathbf{a}_R ds$$

$$\rightarrow \quad E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\varepsilon_0}$$

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\varepsilon_0 R^2} \quad \text{(V/m)}}$$



(a) Point charge at the origin. (b) Point charge not at the origin.

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For a single point charge q not at the origin:

$$\mathbf{E}_p = \mathbf{a}_{qP}\frac{q}{4\pi\varepsilon_0 |\mathbf{R}-\mathbf{R}'|^2} \quad \leftarrow \mathbf{a}_{qP} = \frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|}$$

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R}-\mathbf{R}')}{4\pi\varepsilon_0 |\mathbf{R}-\mathbf{R}'|^3}}$$

Recall:

$$\mathbf{F}_{12} = \mathbf{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2}$$

Force $\mathbf{F}_{12}$ experienced by q_2 due to electric field by q_1:

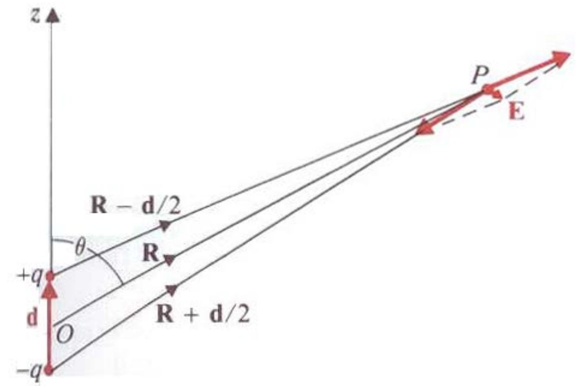
$$\boxed{\mathbf{F}_{12} = q_2 \mathbf{E}_{12} = \mathbf{a}_{R_{12}} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2} \quad \text{(N)}}$$

→ Coulomb's law

Electric Field Due to a System of Discrete Charges

Principle of superposition:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{R}-\mathbf{R}'_k)}{|\mathbf{R}-\mathbf{R}'_k|^3} \quad \text{(V/m)}$$



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Electric dipole:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

$$\leftarrow \left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} = \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2}$$

$$\cong R^{-3}\left[1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \quad \leftarrow \; R >> d$$

$$\cong R^{-3}\left[1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

Similarly:

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} = R^{-3}\left[1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\rightarrow \mathbf{E} \cong \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

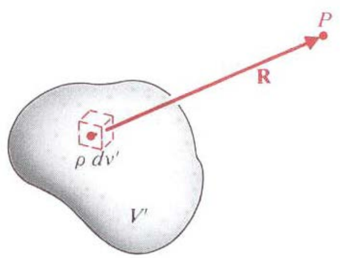
Electric dipole moment: $\mathbf{p} = q\mathbf{d}$

$$\rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R}-\mathbf{p}\right] \rightarrow \boxed{\mathbf{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left[\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right] \quad \text{(V/m)}}$$

Continuous Distribution of Charge

For a volume charge density:

Point charge: $\mathbf{E} = \mathbf{a}_R \dfrac{q}{4\pi\varepsilon_0 R^2}$ *Intuitively:* $\rightarrow\ d\mathbf{E} = \mathbf{a}_R \dfrac{\rho dv'}{4\pi\varepsilon_0 R^2}$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv'$$

$$\rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho \frac{\mathbf{R}}{R^3} dv'$$

For a surface charge density:

$$\rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2} ds'$$

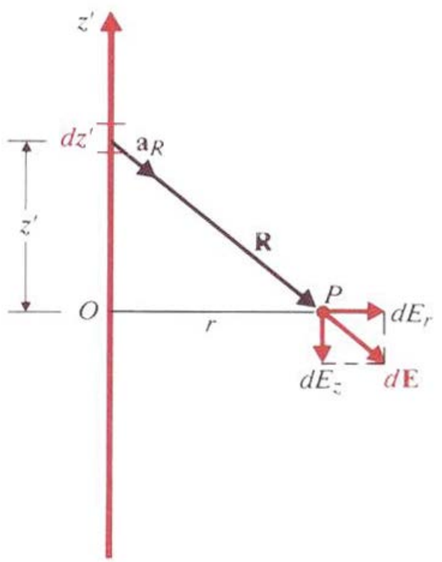
For a line charge density:

$$\rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_l}{R^2} dl'$$

Example 3-4

For an infinitely long, straight line charge of a uniform density ρ_l:

Solution:



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$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_l}{R^2} dl' = \frac{1}{4\pi\varepsilon_0}\int_{L'} \rho_l \frac{\mathbf{R}}{R^3} dl'$$

$$\leftarrow \mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

Net contribution?

$$d\mathbf{E} = \frac{\rho_l dz'}{4\pi\varepsilon_0} \frac{\mathbf{a}_r r - \mathbf{a}_z z'}{(r^2 + z'^2)^{3/2}} \rightarrow d\mathbf{E} = \mathbf{a}_r dE_r + \mathbf{a}_z dE_z$$

$$\rightarrow \mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{\rho_l r}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{dz'}{(r^2 + z'^2)^{3/2}}$$

$$= \mathbf{a}_r \frac{\rho_l r}{4\pi\varepsilon_0}\left[\frac{z}{r^2 (r^2 + z'^2)^{1/2}}\right]_{-\infty}^{\infty}$$

$$\rightarrow \boxed{\mathbf{E} = \mathbf{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r} \quad (\text{V/m})}$$

Any simpler approach?