Introduction to Electromagnetism

Static Electric Fields

(3-4, 3-5, 3-6)

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Gauss's Law and Applications

Gauss's law:

The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0.

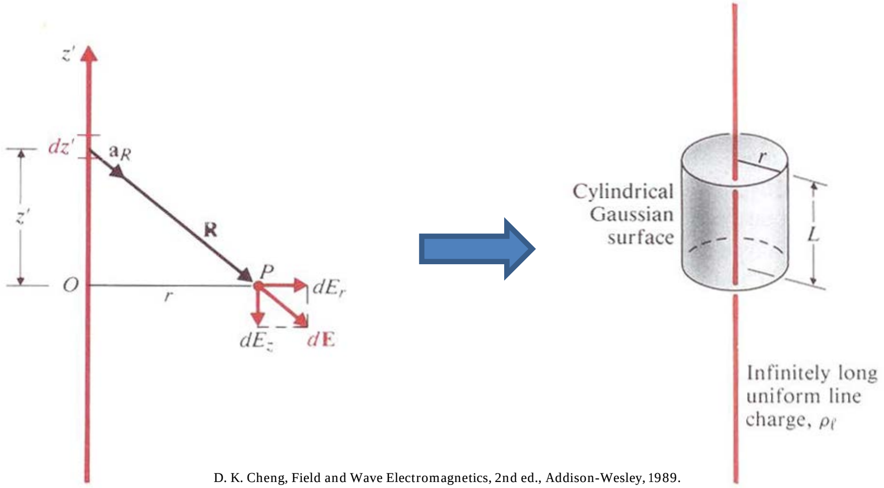
$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}$$

Carl Friedrich Gauss
(1777 - 1855)

Gaussian surface:

With some symmetry conditions, the normal component of the electric field intensity is constant over an enclosed surface.

e.g.



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

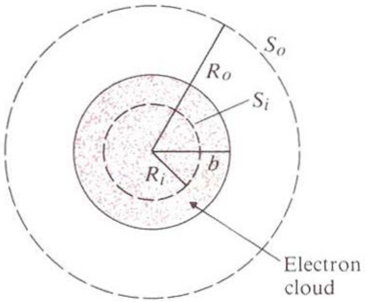
$$\rightarrow 2\pi r L E_r = \frac{\rho_l L}{\varepsilon_0}$$

$$\rightarrow \mathbf{E} = \mathbf{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{(V/m)}$$

Example 3-7

A spherical cloud of electrons with a volume charge density:

$$\rho = \begin{cases} -\rho_0 & \text{for } 0 \le R \le b \\ 0 & \text{for } R > b \end{cases}$$



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$$\mathbf{E} = \mathbf{a}_R E_R, \quad d\mathbf{s} = \mathbf{a}_R ds$$

a) $0 \le R \le b$

$$\rightarrow \oint_{S_i} \mathbf{E} \cdot d\mathbf{s} = E_R \int_{S_i} ds = E_R 4\pi R^2$$

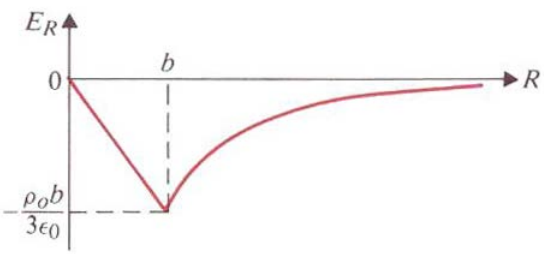
$$\rightarrow Q = \int_V \rho dv = -\rho_0 \int_V dv = -\rho_0 \frac{4\pi}{3} R^3$$

$$\rightarrow \mathbf{E} = -\mathbf{a}_R \frac{\rho_0}{3\varepsilon_0} R$$

a) $R > b$

$$\rightarrow Q = -\rho_0 \frac{4\pi}{3} b^3$$

$$\rightarrow \mathbf{E} = -\mathbf{a}_R \frac{\rho_0 b^3}{3\varepsilon_0 R^2}$$



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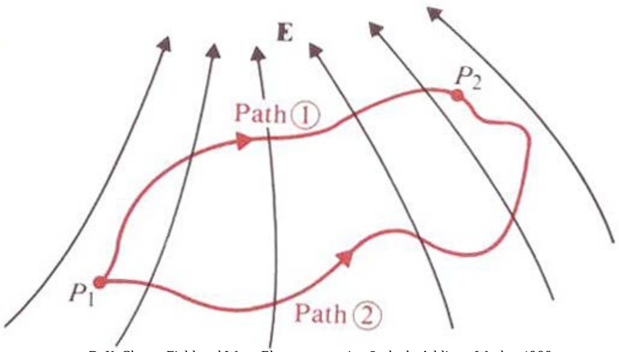
Electric Potential

Electric potential:

A curl-free vector field can always be expressed as the gradient of a scalar field.

$$\mathbf{E} = -\nabla V$$



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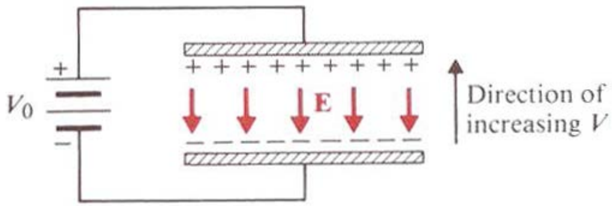
Physical significance: Work done per unit charge "against" the field

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \quad \text{(J/C or V)}$$

Potential difference:

$$\rightarrow -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} = \int_{P_1}^{P_2} (\nabla V) \cdot (\mathbf{a}_l dl) = \int_{P_1}^{P_2} dV = V_2 - V_1$$



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→ In most cases the zero-potential point is taken at infinity unless otherwise stated.

Electric Potential Due to a Charge Distribution

For a point charge:

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\varepsilon_0 R^2} \right) \cdot \left(\mathbf{a}_R dR \right)$$

$$\rightarrow \boxed{V = \frac{q}{4\pi\varepsilon_0 R} \quad \text{(V)}}$$

Potential difference:

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1} \right)$$

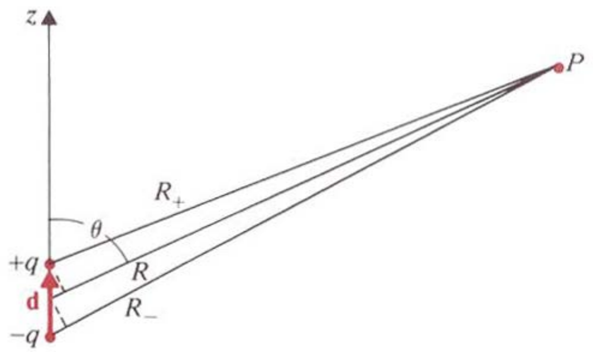
Electric potential due to a system of *n* discrete charges:

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{\left| \mathbf{R} - \mathbf{R}_k \right|} \quad \textbf{(V)}}$$

Electric Potential: An Electric Dipole

Electric potential at a distance R from an electric dipole $\mathbf{p}$:

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$



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If $d << R$:

$$\frac{1}{R_+} = \left(R^2 + (\frac{d}{2})^2 - Rd\cos\theta\right)^{-1/2} \cong \frac{1}{R}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} = \left(R^2 + (\frac{d}{2})^2 + Rd\cos\theta\right)^{-1/2} \cong \frac{1}{R}\left(1 - \frac{d}{2R}\cos\theta\right)$$

$$\rightarrow V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} \rightarrow \boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\varepsilon_0 R^2} \quad \text{(V)}}$$

Electric field:

$$\mathbf{E} = -\nabla V = -\mathbf{a}_R\frac{\partial V}{\partial R} - \mathbf{a}_\theta\frac{1}{R}\frac{\partial V}{\partial \theta}$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right)$$

Q:

Equipotential lines
$\perp$ Electric field lines

Electric Potential: Continuous Charge Distribution

For a volume charge density:

Point charge: $V = \dfrac{q}{4\pi\varepsilon_0 R}$ *Intuitively:* $\rightarrow V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{V'} \frac{\rho}{R} dv'$

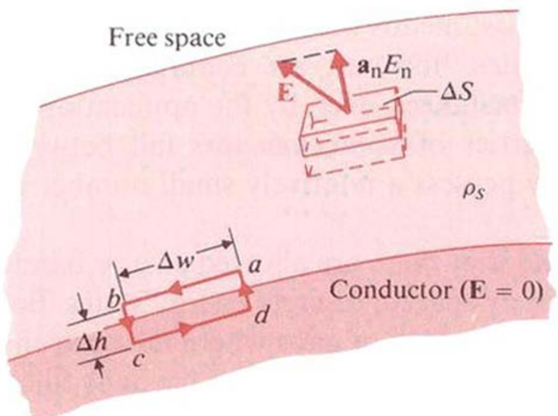
For a surface charge density:

$$\rightarrow V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\rho_s}{R} ds'$$

For a line charge density:

$$\rightarrow V = \frac{1}{4\pi\varepsilon_0}\int_{L'} \frac{\rho_l}{R} dl'$$

Conductors in Static Electric Field



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Classification of Materials:

- Conductors
- Insulators (or Dielectrics)
- Semiconductors

Inside a conductor (under "static" conditions):

$$\rightarrow \rho = 0$$

$$\rightarrow \mathbf{E} = 0$$

Charge distribution on the boundary surface:

$$\rightarrow \oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = E_t \Delta w = 0 \quad \rightarrow E_t = 0$$

Normal component of the E-field at the boundary surface:

$$\rightarrow \oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{Q_{tot}}{\varepsilon_0} = \frac{\rho_s \Delta S}{\varepsilon_0}$$

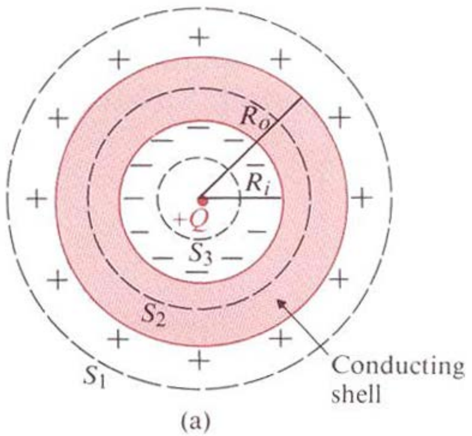
Boundary conditions at a conductor/free space interface:

$$\rightarrow E_t = 0$$

$$\rightarrow E_n = \frac{\rho_s}{\varepsilon_0}$$

Example 3-11

For a charge Q at the center of a spherical conducting shell:



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a) $R > R_o$

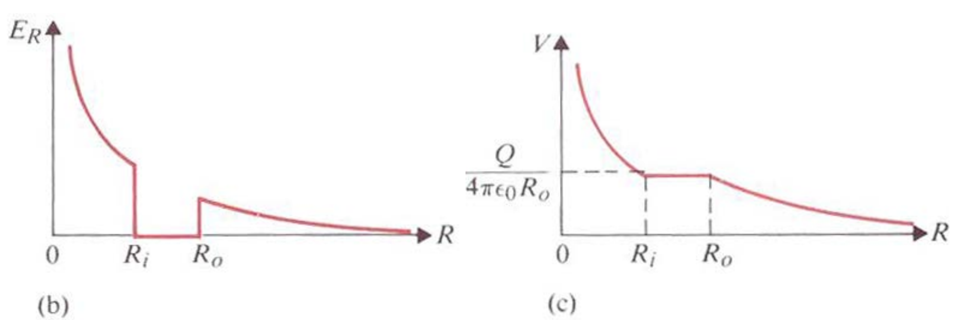
$$\rightarrow \oint_S \mathbf{E} \cdot d\mathbf{s} = E_{R_1} 4\pi R^2 = \frac{Q}{\varepsilon_0}$$

$$\rightarrow E_{R_1} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad \rightarrow V_1 = -\int_\infty^R E_{R_1} dR = \frac{Q}{4\pi\varepsilon_0 R}$$

b) $R_i < R < R_o$

$$\rightarrow E_{R_2} = 0$$

$$\rightarrow V_2 = V_1\big|_{R=R_o} = \frac{Q}{4\pi\varepsilon_0 R_o}$$



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c) $R < R_i$

$$\rightarrow E_{R_3} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad \rightarrow V_3 = -\int E_{R_3} dR + C = \frac{Q}{4\pi\varepsilon_0 R} + C$$

$$\rightarrow V_3\big|_{R=R_i} = V_2 \quad \rightarrow V_3 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R_i} + \frac{1}{R_o}\right)$$