Chapter 6. Optics of Solids
Part 3 – Reflection and absorption in metals

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Reflection and refraction at the boundary of an absorbing medium

Let a plane wave be incident on the boundary of a medium having a complex index of refraction.

$$\tilde{N} = n + i\kappa, \qquad \tilde{K} = \mathbf{k} + i\boldsymbol{\alpha},$$

$$e^{i(\mathbf{k_o}\cdot\mathbf{r}-\omega t)} \quad \text{(incident wave)}$$

$$e^{i(\mathbf{k'_o}\cdot\mathbf{r}-\omega t)} \quad \text{(reflected wave)}$$

$$e^{i(\tilde{K}\cdot\mathbf{r}-\omega t)} = e^{-\boldsymbol{\alpha}\cdot\mathbf{r}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \text{(refracted wave)}$$

At the boundary plane,

$$\mathbf{k_o}\cdot\mathbf{r} = \mathbf{k'_o}\cdot\mathbf{r} \quad \text{(at boundary)} \rightarrow \text{law of reflection}$$

$$\mathbf{k_o}\cdot\mathbf{r} = \tilde{K}\cdot\mathbf{r} = (\mathbf{k} + i\boldsymbol{\alpha})\cdot\mathbf{r} \quad \text{(at boundary)}$$

$$\rightarrow \begin{cases} \mathbf{k_o}\cdot\mathbf{r} = \mathbf{k}\cdot\mathbf{r} \\ 0 = \boldsymbol{\alpha}\cdot\mathbf{r} \end{cases}$$

in general, $\mathbf{k}$ and $\boldsymbol{\alpha}$ have different directions; ***inhomogeneous*** wave

Reflection and refraction at the boundary of an absorbing medium

$$\mathbf{k_o} \cdot \mathbf{r} = \mathbf{k'_o} \cdot \mathbf{r} \qquad \text{(at boundary)}$$

$$\mathbf{k_o} \cdot \mathbf{r} = (\mathbf{k} + i\boldsymbol{\alpha}) \cdot \mathbf{r} \qquad \text{(at boundary)}$$

$$\rightarrow \begin{cases} \mathbf{k_o} \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{r} \\ 0 = \boldsymbol{\alpha} \cdot \mathbf{r} \end{cases}$$

$$\mathbf{k_o} \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{r} \rightarrow k_o \sin\theta = k \sin\varphi$$

$\boldsymbol{\alpha}$, which defines the direction of planes of constant amplitudes, is always normal to the boundary.

$\mathbf{k}$, which defines the planes of constant phase, may have any direction.

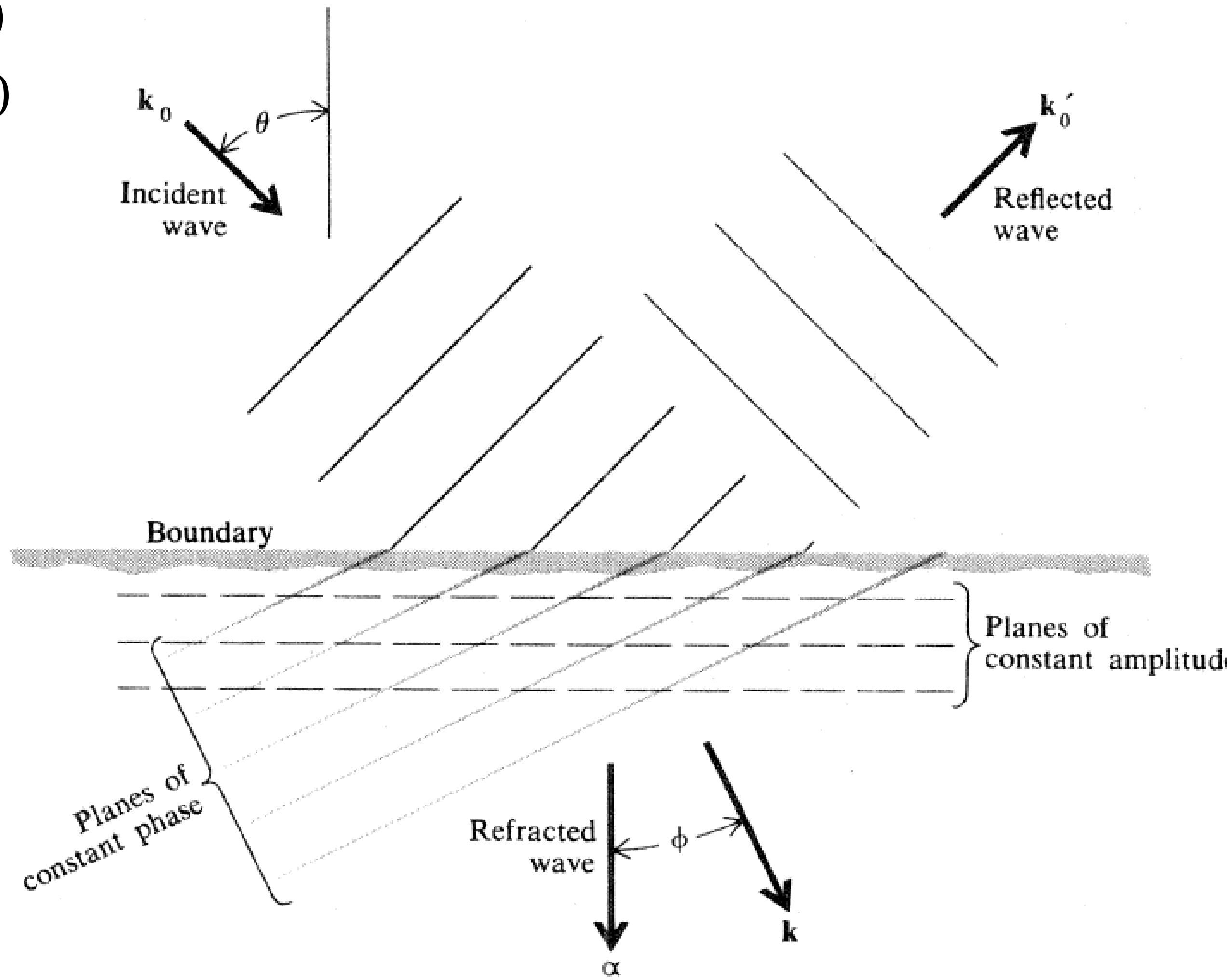


Figure 6.5 Real and imaginary parts of the wave vector in an absorbing medium for the case of oblique incidence of light at the boundary.

Reflection and refraction at the boundary of an absorbing medium

$$\nabla^2 \vec{E} = \frac{\tilde{N}^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

For plane harmonic waves, $\nabla \to i\tilde{K},\ \dfrac{\partial}{\partial t} \to -i\omega.$

$$\tilde{K}\cdot\tilde{K} = \frac{\tilde{N}^2\omega^2}{c^2} = \tilde{N}^2 k_o^2,\quad k_o = \frac{\omega}{c}$$

$$(\mathbf{k}+i\boldsymbol{\alpha})\cdot(\mathbf{k}+i\boldsymbol{\alpha}) = (n+i\kappa)^2 k_o^2,$$

$$k^2-\alpha^2 = (n^2-\kappa^2)k_o^2$$

$$\mathbf{k}\cdot\boldsymbol{\alpha} = k\alpha\cos\varphi = n\kappa k_o^2$$

$$k^2-\alpha^2 = k^2(\cos^2\varphi+\sin^2\varphi)-\alpha^2 = (k^2\cos^2\varphi-\alpha^2)+k^2\sin^2\varphi = (n^2-\kappa^2)k_o^2$$

$$k^2\cos^2\varphi-\alpha^2 = (n^2-\kappa^2)k_o^2 - k_o^2\sin^2\theta$$

$$(k\cos\varphi+i\alpha)^2-2ik\alpha\cos\varphi = (k\cos\varphi+i\alpha)^2-2in\kappa k_o^2 = (n^2-\kappa^2)k_o^2-k_o^2\sin^2\theta$$

$$(k\cos\varphi+i\alpha)^2 = (n^2-\kappa^2+2in\kappa)k_o^2-k_o^2\sin^2\theta = k_o^2(\tilde{N}^2-\sin^2\theta)$$

$$\therefore k\cos\varphi+i\alpha = k_o\sqrt{\tilde{N}^2-\sin^2\theta}$$

$$k\cos\varphi+i\alpha = k_o\sqrt{\tilde{N}^2-\sin^2\theta}$$

$$\to k\cos\varphi+i\alpha = k_o\tilde{N}\quad \text{for normal incidence}\quad (\theta=0)$$

Reflection and refraction at the boundary of an absorbing medium

Now express the law of refraction in a purely formal way.

$$\tilde{N} = \frac{\sin\theta}{\sin\varphi}, \quad \cos\varphi = \sqrt{1 - \frac{\sin^2\theta}{\tilde{N}^2}}$$

$$\tilde{N} = \frac{k\cos\varphi + i\alpha}{k_o\cos\varphi}$$

$$\mathbf{E}, \mathbf{H} = \frac{1}{\mu_0\omega}\mathbf{k}_0 \times \mathbf{E} \qquad \text{(incident)}$$

$$\mathbf{E}', \mathbf{H}' = \frac{1}{\mu_0\omega}\mathbf{k}_0' \times \mathbf{E}' \qquad \text{(reflected)}$$

$$\mathbf{E}'', \mathbf{H}'' = \frac{1}{\mu_0\omega}\mathscr{K} \times \mathbf{E}'' = \frac{1}{\mu_0\omega}(\mathbf{k} \times \mathbf{E}'' + i\boldsymbol{\alpha} \times \mathbf{E}'') \qquad \text{(refracted)}$$

The boundary conditions giving the continuity of the tangential components of the E and H fields for TE polarization are

$$E + E' = E''$$

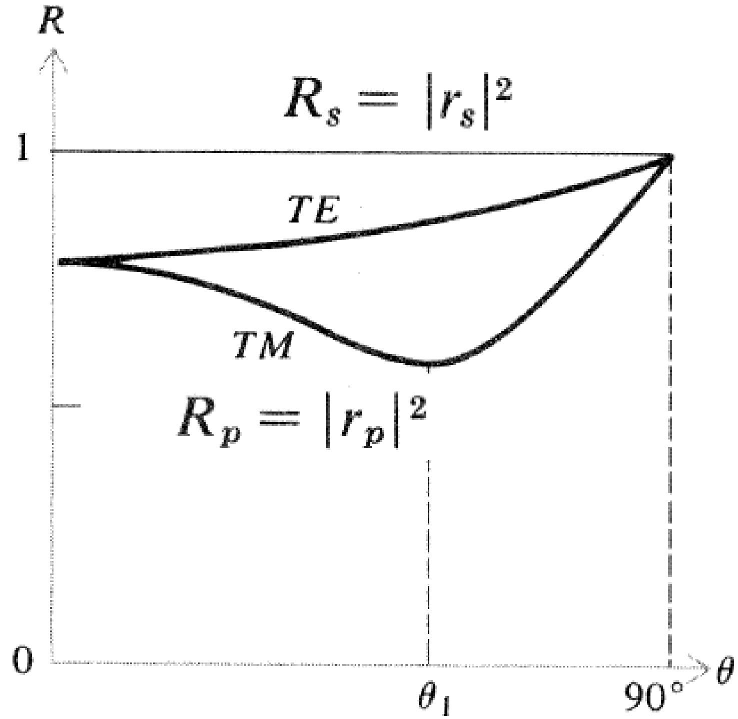
$$-H\cos\theta + H'\cos\theta = H''_{\text{tangential}}$$

$$-k_0E\cos\theta + k_0E'\cos\theta = -(kE''\cos\phi + i\alpha E'')$$
$$= -\mathscr{N} k_0 E''\cos\phi$$

Reflection and refraction at the boundary of an absorbing medium

$$r_s = \frac{\cos\theta - \mathcal{N}\cos\phi}{\cos\theta + \mathcal{N}\cos\phi} \qquad (TE\ polarization)$$

$$r_p = \frac{-\mathcal{N}\cos\theta + \cos\phi}{\mathcal{N}\cos\theta + \cos\phi} \qquad (TM\ polarization)$$



θ_1= *principal angle of incidence* for a metal, corresponding to the Brewster angle for dielectrics

Figure 6.6. Reflectance as a function of angle of incidence for a typical metal.

Normal incidence

$$r_s = r_p = \frac{1-\mathcal{N}}{1+\mathcal{N}} = \frac{1-n-i\kappa}{1+n+i\kappa}$$

$$R = \left|\frac{1-\mathcal{N}}{1+\mathcal{N}}\right|^2 = \frac{(1-n)^2+\kappa^2}{(1+n)^2+\kappa^2}$$

For a metal, the extinction coefficient κ is large, resulting in a high value of the reflectance.

$$R \to \infty \text{ as } \kappa \to \infty.$$

$$\tilde{K}^2 = \frac{\omega^2}{c^2} + \frac{i\omega\mu_o\sigma}{1-i\omega\tau}, \quad \tilde{K}^2 \approx i\omega\mu_o\sigma \text{ (for very low frequencies)}$$

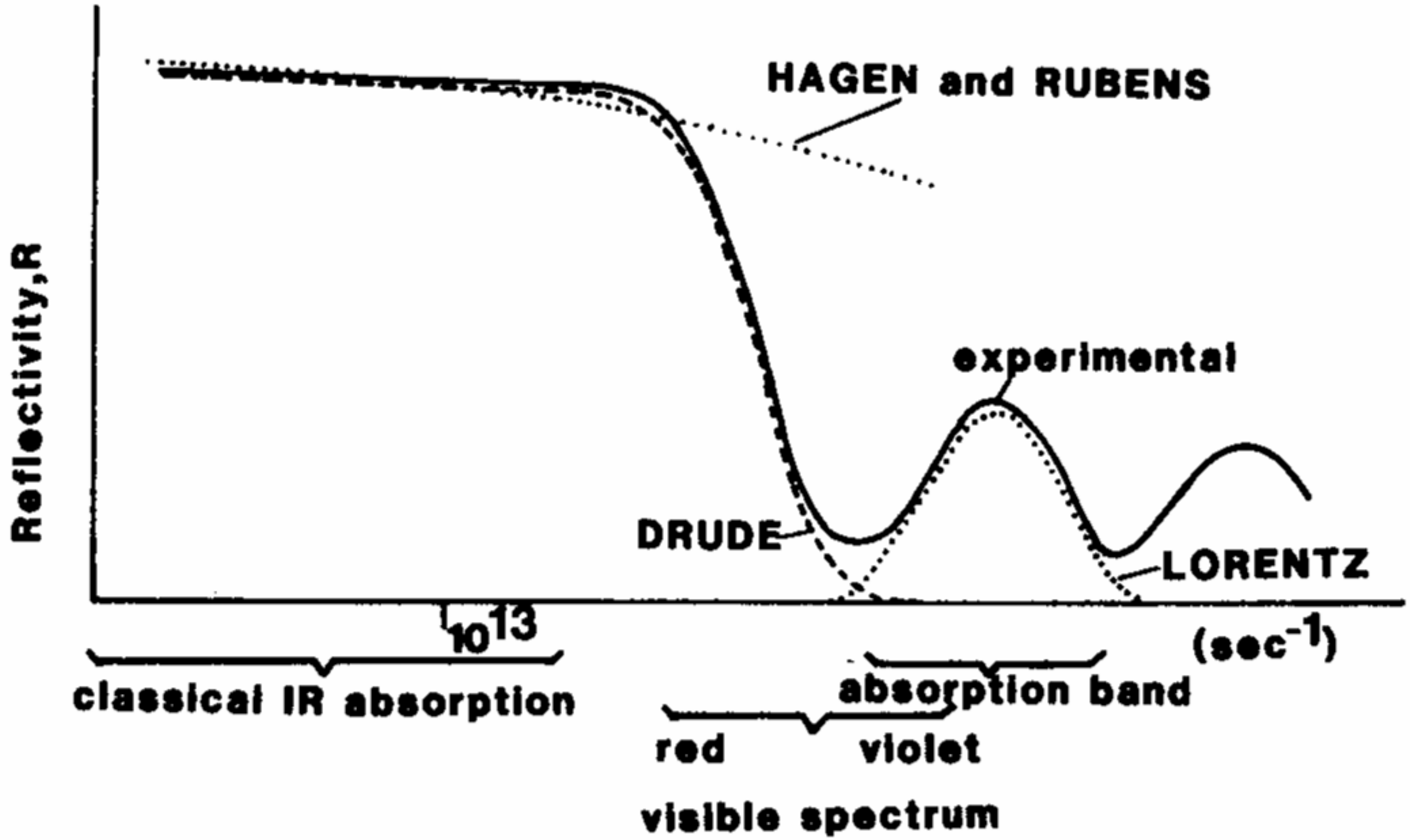
$$\tilde{K} = k + i\alpha \approx \sqrt{i\omega\mu_o\sigma} = (1+i)\sqrt{\omega\mu_o\sigma/2}, \quad k \approx \alpha \approx \sqrt{\frac{\omega\mu_o\sigma}{2}}$$

$$\tilde{N} = n + i\kappa, \quad n \approx \kappa \approx \sqrt{\frac{\sigma}{2\omega\varepsilon_o}}$$

$$R \approx 1 - \frac{2}{n} \approx 1 - \sqrt{\frac{8\omega\varepsilon_o}{\sigma}} \quad \text{(Hagen - Rubens formula)}$$

This relation is named after German physicists Ernst Bessel Hagen and Heinrich Rubens in 1903.
Good metals with a high electrical conductivity reflect very well in the infrared region.

Reflectivity of a metal



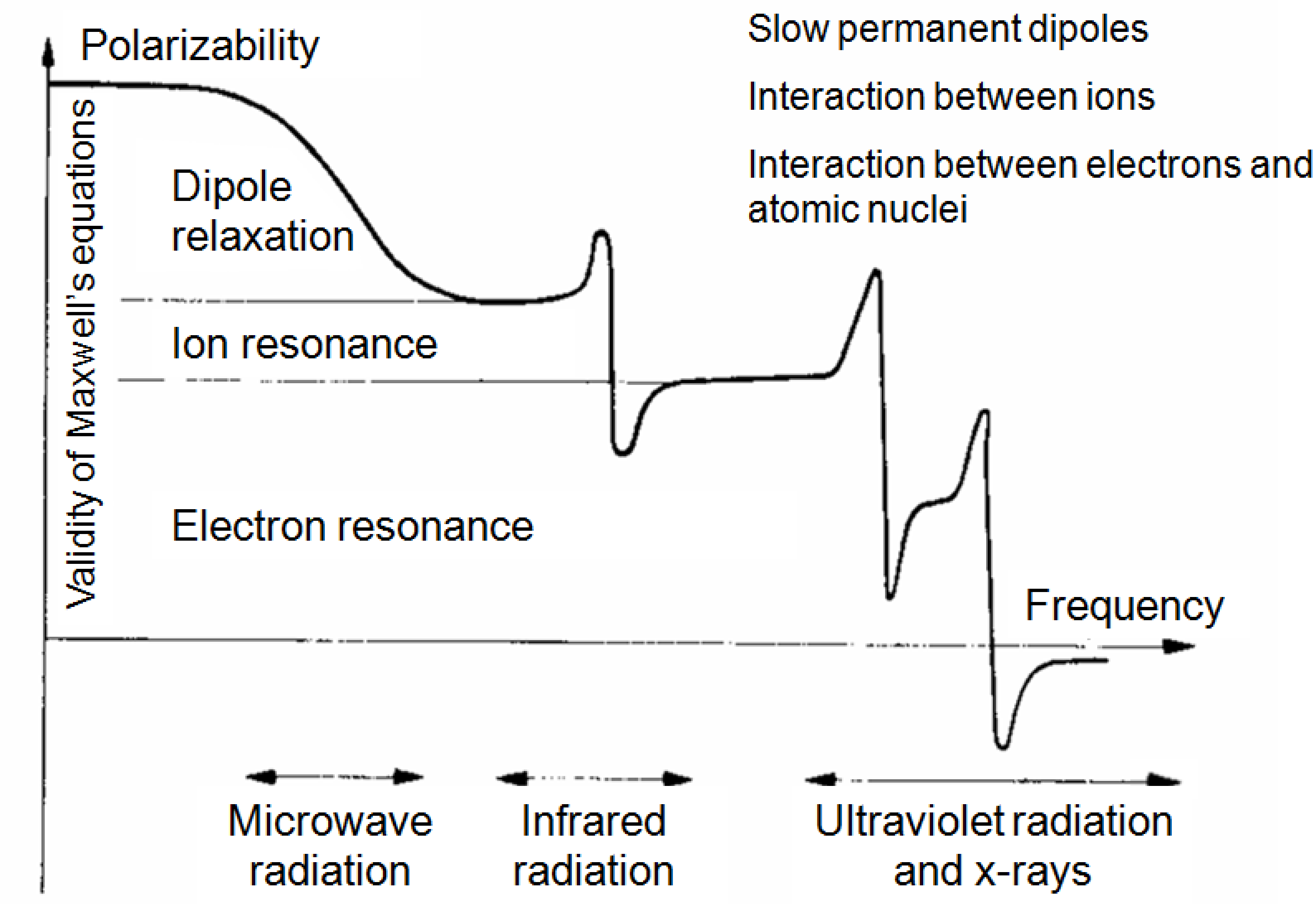
Metal = Free electrons with damping and bound electrons with damping and natural frequency

Hagen-Rubens: from the solution of Maxwell's equations ($\sigma = n\kappa\nu$) for small frequencies

Drude: free damped electrons (classical electron theory), determines the color of materials

Lorentz: strongly bound electrons (classical electron theory for dielectric materials)

Dispersion curve



Optical Absorption

Conducting electrons

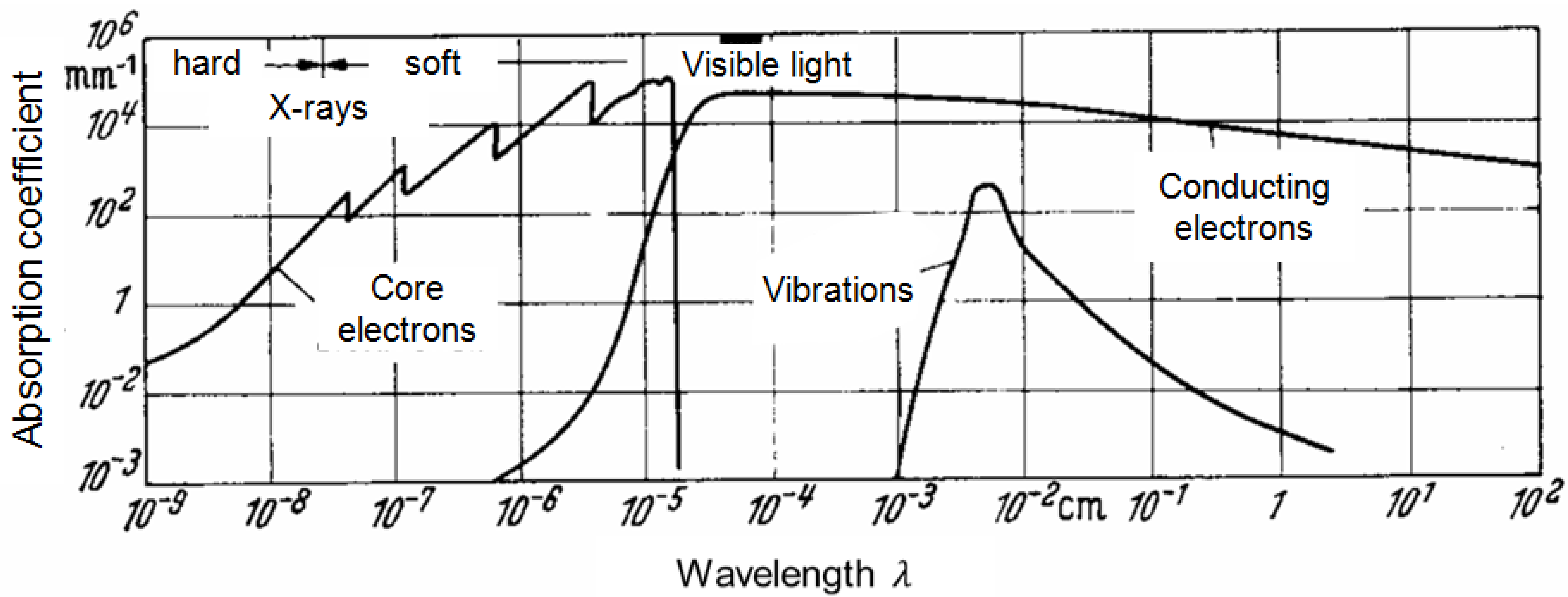
- Especially in metals
- Ionic crystals and insulators are normally transparent

Lattice vibrations

- Absorption in IR range – small natural frequencies of lattice vibrations
- IR and Raman spectroscopy – investigation of lattice dynamics

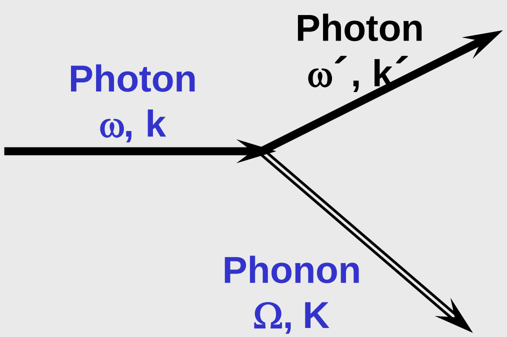
Core electrons

- Interaction between electron and atomic nucleus
- High natural frequency
- Absorption and emission of radiation in the x-ray range (selective filters, fluorescence spectroscopy)



Interaction between elementary particles in solids

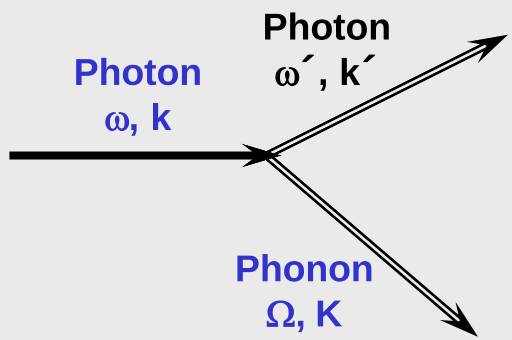
Raman process



$$\hbar\omega = \hbar\omega' \pm \hbar\Omega$$

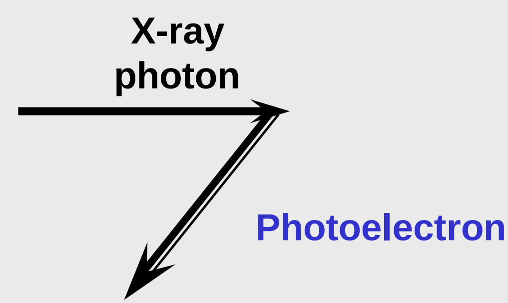
$$\hbar\vec{k} = \hbar\vec{k}' \pm \hbar\vec{K}$$

IR absorption with two phonons



Electron spectroscopy with x-rays

XPS



Photon – light quantum
Phonon – quasiparticle to describe lattice vibrations