

Introduction to Electromagnetism

Static Electric Fields

(3-7, 3-8, 3-9)

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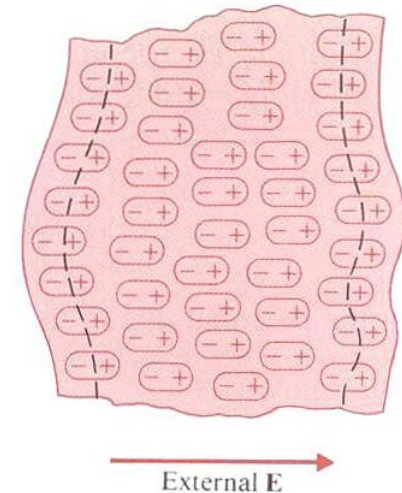
Dielectrics in Static Electric Field

Dielectrics:

No free charges but having **bound charges**

Induced electric dipoles:

Macroscopically neutral but having small displacements of positive and negative charges in opposite directions (**polarized**)

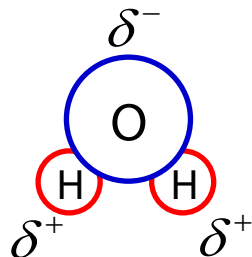


D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Permanent dipole moments:

Polar molecules, e.g. H_2O (*cf.* nonpolar molecules)

Electrets (waxes, polymers, resins), e.g. HF microphones



http://en.wikipedia.org/wiki/Electret_microphone

Equivalent Charge Distribution of Polarized Dielectrics (1)

Polarization vector (macroscopic): Volume density of electric dipole moment

$$\mathbf{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \quad (\text{C/m}^2) \quad n: \text{the number of molecules per unit volume}$$

Differential dipole moment for an elemental volume dv' :

$$\rightarrow d\mathbf{p} = \mathbf{P}dv' \quad \rightarrow dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} dv' \quad (\text{See Eq. 3-53b})$$

Electrostatic potential by \mathbf{P} :

$$\rightarrow V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} dv'$$

Note: $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 \rightarrow \nabla' \left(\frac{1}{R} \right) = -\frac{\mathbf{a}'_R}{R^2} = \frac{\mathbf{a}_R}{R^2}$

$$\rightarrow V = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{P} \cdot \nabla' \left(\frac{1}{R} \right) dv' \rightarrow V = \frac{1}{4\pi\epsilon_0} \left[\int_{V'} \nabla' \cdot \left(\frac{\mathbf{P}}{R} \right) dv' - \int_{V'} \frac{\nabla' \cdot \mathbf{P}}{R} dv' \right]$$

$$\therefore V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R} ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{-\nabla' \cdot \mathbf{P}}{R} dv'$$

Equivalent Charge Distribution of Polarized Dielectrics (2)

Electrostatic potential by \mathbf{P} :

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R} ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{-\nabla' \cdot \mathbf{P}}{R} dv'$$
$$\rightarrow V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{ps}}{R} ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_p}{R} dv'$$

Equivalent polarization surface charge density: $\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$

Equivalent polarization volume charge density: $\rho_p = -\nabla \cdot \mathbf{P}$

Separation of bound charges to the surface:

$$\Delta Q = nq(\mathbf{d} \cdot \mathbf{a}_n)(\Delta s) \rightarrow \Delta Q = \mathbf{P} \cdot \mathbf{a}_n (\Delta s)$$

$$\rightarrow \frac{\Delta Q}{\Delta s} = \mathbf{P} \cdot \mathbf{a}_n = \rho_{ps} \rightarrow \text{Surface charge density}$$

Remaining charges within the volume:

$$\rightarrow Q = -\oint_S \mathbf{P} \cdot \mathbf{a}_n ds$$
$$= \int_V (-\nabla \cdot \mathbf{P}) dv = \int_V \rho_p dv$$

$$\text{Total charge} = \oint_S \rho_{ps} ds + \int_V \rho_p dv$$
$$= 0$$

Electric Flux Density and Dielectric Constant (1)

Modified divergence postulate:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} (\rho + \rho_p) \quad \begin{array}{l} \text{"equivalent volume charge density"} \\ \text{"bounded charges"} \end{array}$$

$\rho_p = -\nabla \cdot \mathbf{P}$

$$\rightarrow \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho$$
$$\rightarrow \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{C/m}^2) \quad \leftarrow \text{Electric flux density or electric displacement}$$
$$\rightarrow \nabla \cdot \mathbf{D} = \rho \quad (\text{C/m}^3) \quad \text{"free charge density"}$$

Gauss's law:

$$\rightarrow \int_V \nabla \cdot \mathbf{D} dv = \int_V \rho dv$$
$$\rightarrow \int_S \mathbf{D} \cdot d\mathbf{s} = Q$$

→ The total outward flux of the electric displacement over any closed surface is equal to the total free charge enclosed in the surface.

Electric Flux Density and Dielectric Constant (2)

Electric susceptibility / Relative permittivity / Permittivity:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \leftarrow \quad \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\rightarrow \mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

$$= \varepsilon_0 \varepsilon_r \mathbf{E} = \boldsymbol{\varepsilon} \mathbf{E}$$

Permittivity in general:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \rightarrow \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

← Along the principal axes

Isotropic: $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$

Uniaxial: $\varepsilon_{11} = \varepsilon_{22} \neq \varepsilon_{33}$

Biaxial: $\varepsilon_{11} \neq \varepsilon_{22} \neq \varepsilon_{33}$

Example 3-12

For a charge Q at the center of a spherical dielectric shell: $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \epsilon_0 (\epsilon_r - 1) \mathbf{E}$

a) $R > R_o$

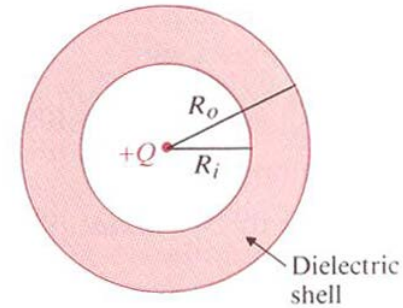
$$\begin{aligned} \rightarrow E_{R1} &= \frac{Q}{4\pi\epsilon_0 R^2} & \rightarrow V_1 &= \frac{Q}{4\pi\epsilon_0 R} \\ \rightarrow D_{R1} &= \epsilon_0 E_{R1} = \frac{Q}{4\pi R^2} & \rightarrow P_{R1} &= 0 \end{aligned}$$

b) $R_i < R < R_o$

$$\begin{aligned} \rightarrow D_{R2} &= \frac{Q}{4\pi R^2} & \rightarrow E_{R2} &= \frac{Q}{4\pi\epsilon R^2} \\ \rightarrow P_{R2} &= \left(1 - \frac{1}{\epsilon_r}\right) \frac{Q}{4\pi R^2} & \rightarrow V_2 &= -\int_{\infty}^R E dR = -\int_{\infty}^{R_o} E_{R1} dR - \int_{R_o}^R E_{R2} dR \\ & & &= V_1|_{R=R_o} - \frac{Q}{4\pi\epsilon} \int_{R_o}^R \frac{1}{R^2} dR = \frac{Q}{4\pi\epsilon_0} \left[\left(1 - \frac{1}{\epsilon_r}\right) \frac{1}{R_o} + \frac{1}{\epsilon_r R} \right] \end{aligned}$$

c) $R < R_i$

$$\begin{aligned} \rightarrow E_{R3} &= \frac{Q}{4\pi\epsilon_0 R^2} & \rightarrow D_{R3} &= \frac{Q}{4\pi R^2} \\ \rightarrow P_{R3} &= 0 & \rightarrow V_3 &= V_2|_{R=R_i} - \int_{R_i}^R E_{R3} dR = \frac{Q}{4\pi\epsilon_0} \left[\left(1 - \frac{1}{\epsilon_r}\right) \left(\frac{1}{R_o} - \frac{1}{R_i} \right) + \frac{1}{R} \right] \end{aligned}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Example 3-12

Continued:

Equivalent polarization surface charge density:

On the inner shell surface:

$$\begin{aligned}\rho_{ps}\Big|_{R=R_i} &= \mathbf{P} \cdot (-\mathbf{a}_R)\Big|_{R=R_i} = -P_{R2}\Big|_{R=R_i} \\ &= -\left(1 - \frac{1}{\epsilon_r}\right) \frac{Q}{4\pi R_i^2}\end{aligned}$$

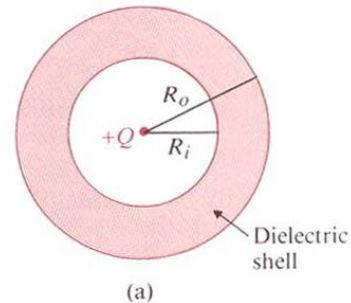
$$P_{R2} = \left(1 - \frac{1}{\epsilon_r}\right) \frac{Q}{4\pi R^2}$$

On the outer shell surface:

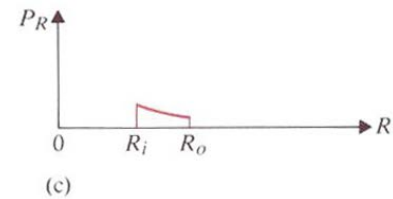
$$\begin{aligned}\rho_{ps}\Big|_{R=R_o} &= \mathbf{P} \cdot \mathbf{a}_R\Big|_{R=R_o} = P_{R2}\Big|_{R=R_o} \\ &= \left(1 - \frac{1}{\epsilon_r}\right) \frac{Q}{4\pi R_o^2}\end{aligned}$$

Inside the shell:

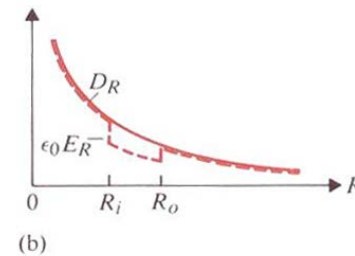
$$\begin{aligned}\rho_p &= -\nabla \cdot \mathbf{P} \\ &= -\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 P_{R2}) = 0\end{aligned}$$



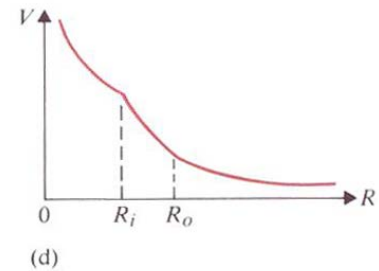
(a)



(c)



(b)



(d)

Dielectric Strength

Dielectric breakdown:

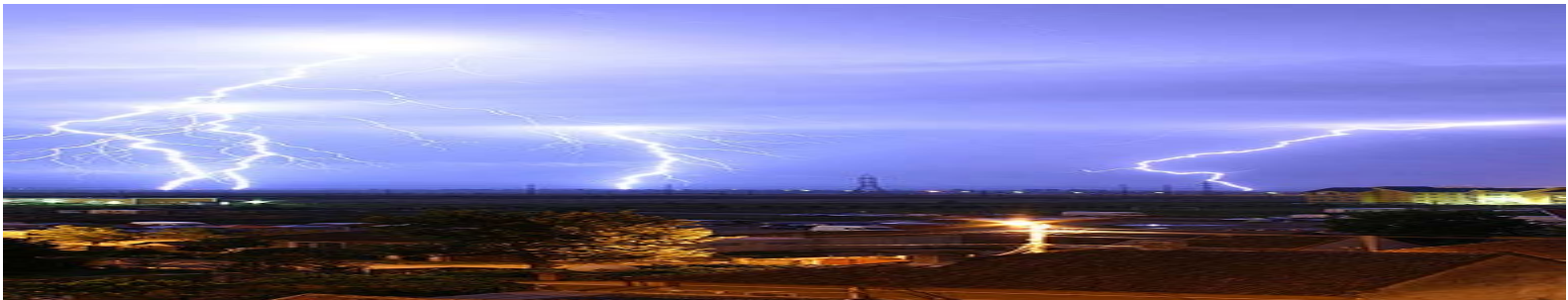
Avalanche effect of ionization due to very strong electric field

Dielectric strength:

Maximum electric field intensity that a dielectric material can withstand without breakdown

e.g. air ~ 3 kV/mm

Lightening strike: Dielectric breakdown



http://en.wikipedia.org/wiki/File:Lightning_over_Oradea_Romania_3.jpg

Principle of a lightning rod:

Electric field intensity much higher at sharp points → See: Example 3-13

Boundary Conditions for Electrostatic Fields

Tangential component of an E field:

$$\lim_{\Delta h \rightarrow 0} \oint_{abcd} \mathbf{E} \cdot d\mathbf{l} = 0 \rightarrow E_{1t} \Delta w - E_{2t} \Delta w = 0$$

$$E_{1t} = E_{2t}$$

Normal component of D field:

$$\begin{aligned} \lim_{\Delta h \rightarrow 0} \oint_S \mathbf{D} \cdot d\mathbf{s} &= (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1}) \Delta S \\ &= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \Delta S \\ &= \rho_s \Delta S \end{aligned}$$

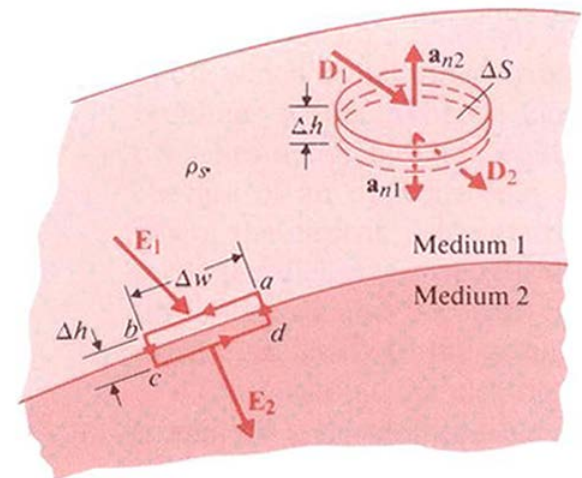
$$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

or

$$D_{1n} - D_{2n} = \rho_s$$

For dielectrics:

$$\text{No free charges} \rightarrow \rho_s = 0$$



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