재료의 기계적 거동
(Mechanical Behavior of Materials)

Lecture 6 – Viscoelasticity

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Viscoelasticity

• **Elastic materials** deform with stress and *quickly return* to their original state *if the stress is removed* due to the bond stretching along crystallographic planes in an ordered solid.

• **Viscous materials**, like honey, resist shear flow and *strain with time* when a stress is applied due to the diffusion of atoms or molecules inside an amorphous material.

• **Viscoelasticity** is the property of materials that exhibit *both viscosity and elasticity* during deformation and *time-dependent strain*. 
If the stress is held constant, the strain increases with time (creep).

If the strain is held constant, the stress decreases with time (stress relaxation).

If a cyclic loading is applied, hysteresis occurs, leading to a dissipation of mechanical energy $\int \sigma d\varepsilon$. 
Constitutive models for linear viscoelasticity

\[ \sigma = \sigma(t) \quad \varepsilon = \varepsilon(t) \]

Since its viscous component, the stress-strain relation of viscoelastic materials is time-dependent!
Constitutive models for linear viscoelasticity

Viscoelasticity can be divided to elastic components and viscous components. We can model viscoelastic materials as **linear combinations** of *springs* and *dashpots*.

The *springs* represent the *elastic* components.

\[ \sigma = E \varepsilon \]

where \( E \) is the elastic modulus of the material.

The *dashpots* represent the *viscous* components.

\[ \sigma = \eta \frac{d\varepsilon}{dt} = \eta \dot{\varepsilon} \]

where \( \eta \) is the viscosity of the material and \( d\varepsilon/dt \) is the strain rate.
Maxwell Model

A purely viscous damper and purely elastic spring connected in series.

\[ \varepsilon = \varepsilon_s + \varepsilon_d \]
\[ \sigma = \sigma_s = \sigma_d \]
\[ \varepsilon_s = \frac{\sigma}{E} \]
\[ \dot{\varepsilon}_d = \frac{\sigma}{\eta} \]

\[ \dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \]
Maxwell Model

\[
\dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\ddot{\sigma}}{E} + \frac{\sigma}{\eta}
\]

Stress Relaxation
\((\varepsilon = const. (\dot{\varepsilon} = 0))\)

\[
G(t) = \frac{\sigma(t)}{\varepsilon} = E \cdot e^{-\frac{E}{\eta} t}
\]

Relaxation Modulus \(G(t)\)

Creep
\((\sigma = const. (\ddot{\sigma} = 0))\)

\[
J(t) = \frac{\varepsilon(t)}{\sigma} = \frac{1}{E} + \frac{1}{\eta} \cdot t
\]

Creep Compliance Function \(J(t)\)

In creep, actual strain rate decreases with time!
Voigt-Kelvin (V-K) Model

A purely viscous damper and purely elastic spring connected in parallel.

\[ \varepsilon = \varepsilon_s = \varepsilon_d \]
\[ \sigma = \sigma_s + \sigma_d \]
\[ \sigma_s = E \cdot \varepsilon \]
\[ \sigma_d = \eta \cdot \dot{\varepsilon} \]

\[ \sigma = \sigma_s + \sigma_d = E\varepsilon + \eta \dot{\varepsilon} \]
V-K Model

\[ \sigma = \sigma_s + \sigma_d = E\varepsilon + \eta\dot{\varepsilon} \]

Stress Relaxation
\( (\varepsilon = \text{const.} \ (\dot{\varepsilon} = 0)) \)

\[ G(t) = \sigma(t) = E \]

Relaxation Modulus \( G(t) \)

Creep
\( (\sigma = \text{const.} \ (\dot{\sigma} = 0)) \)

\[ J(t) = \varepsilon(t) = \frac{1}{E} \left(1 - e^{-\frac{E}{\eta} t}\right) \]

Creep Compliance Function \( J(t) \)

Actual stress is not **constant** in viscoelastic materials.
Standard Linear Solid (Zener) Model

A Maxwell model and a purely elastic spring connected in parallel (three-parameter standard model)

\[ \varepsilon = \varepsilon_1 = \varepsilon_2 + \varepsilon_3 \]

\[ \sigma = \sigma_1 + \sigma_2 \]

\[ \sigma_2 = \sigma_3 \]

\[ \eta(E_1 + E_2)\dot{\varepsilon} + E_1E_2\varepsilon = \eta\dot{\sigma} + E_2\sigma \]
SLS Model

\[ \eta(E_1 + E_2)\dot{\varepsilon} + E_1E_2\varepsilon = \eta\dot{\sigma} + E_2\sigma \]

Creep Compliance Function \( J(t) \)

\[ J_{SLS}(t) = \frac{1}{E_1} \left( 1 - \frac{E_2}{E_1 + E_2} e^{-\frac{E_1E_2}{\eta(E_1+E_2)t}} \right) \]

Relaxation Modulus \( G(t) \)

\[ G_{SLS}(t) = E_1 + E_2 e^{-\frac{E_2}{\eta}t} \]

It matches well to real linear viscoelastic behaviors!
# Comparison of Several Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Creep compliance function $J(t)$</th>
<th>Relaxation modulus $G(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maxwell</td>
<td>$\frac{1}{E} \left(1 + \frac{E}{\eta} t \right)$</td>
<td>$E , e^{-\frac{E}{\eta} t}$</td>
</tr>
<tr>
<td>Voigt-Kelvin</td>
<td>$\frac{1}{E} \left(1 - e^{-\frac{E}{\eta} t} \right)$</td>
<td>$E + \eta \delta(t)$</td>
</tr>
<tr>
<td>Standard Linear Solid (Zener)</td>
<td>$\frac{1}{E_1} \left(1 - \frac{E_2}{E_1 + E_2} e^{-\frac{E_1 E_2}{\eta (E_1 + E_2)} t} \right)$</td>
<td>$E_1 + E_2 , e^{-\frac{E_2}{\eta} t}$</td>
</tr>
</tbody>
</table>
Comparison of Several Models

Figure 2.11:3 Creep functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid. A negative phase is superposed at the time of unloading.

Figure 2.11:4 Relaxation functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid.