445.204

Introduction to Mechanics of Materials

(재료역학개론)

Chapter 6: Torsion

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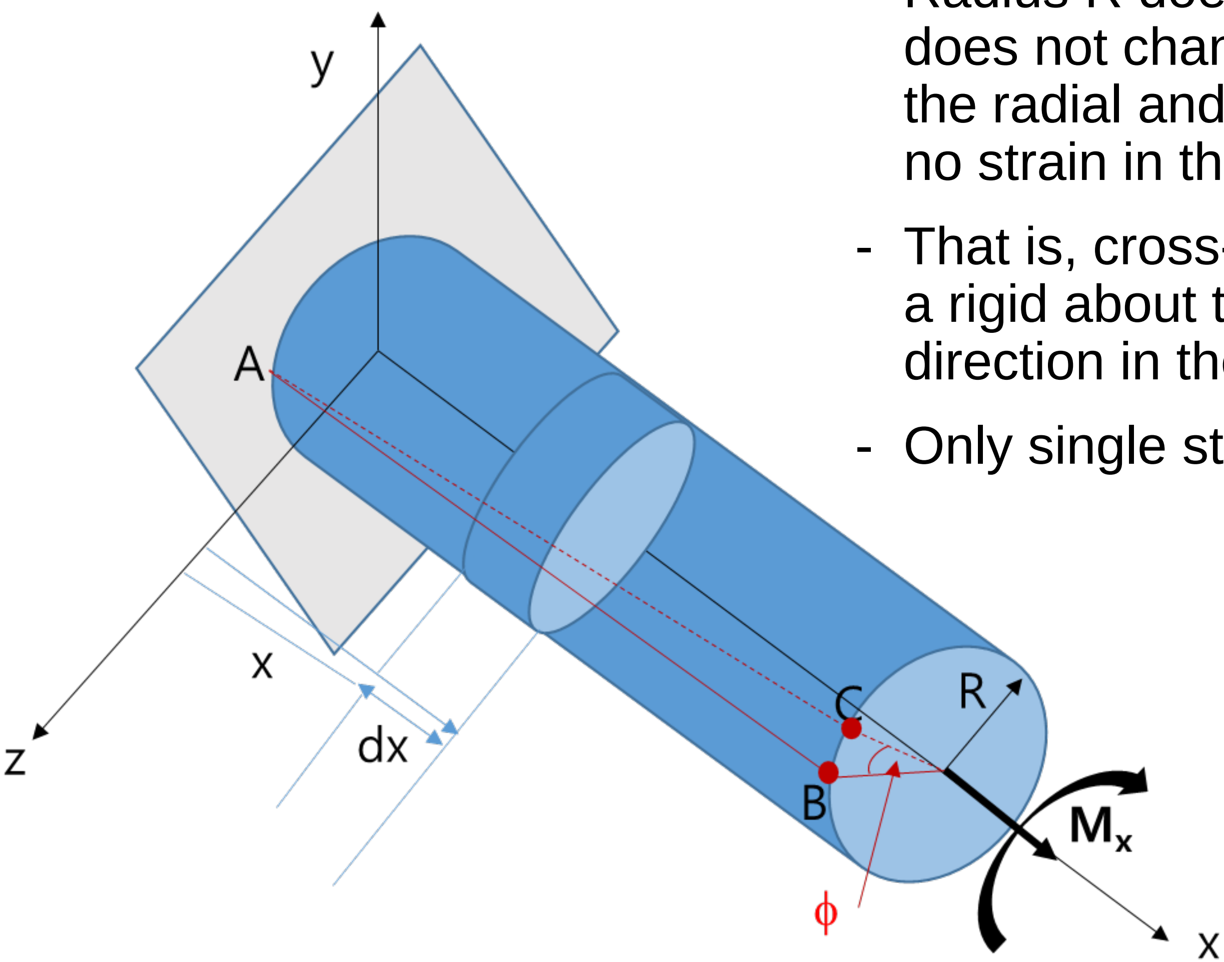
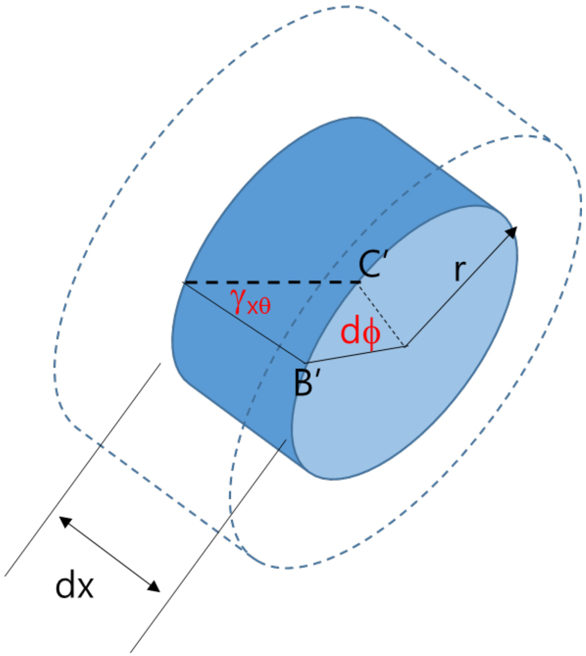
Torsion of circular shafts



Observation and assumptions

- Plane section remains plane during deformation; no warpage
- Radius R does not change, length does not change; i.e., normal strains in the radial and length are zeros; also no strain in the plane of cross section
- That is, cross-section simply rotates as a rigid about the axis of the shaft (or x-direction in the left figure)
- Only single strain component

Torsion of circular shafts

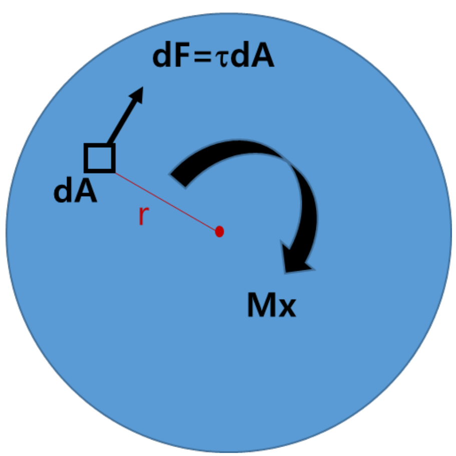


Angle of twist per unit length

$$\gamma_{x\theta} = \frac{B'C'}{dx} = \frac{r\,d\phi}{dx} = \frac{d\phi}{dx} r$$

Constitutive equation

$$\tau_{x\theta} = G\gamma_{x\theta} = G\frac{d\phi}{dx} r$$



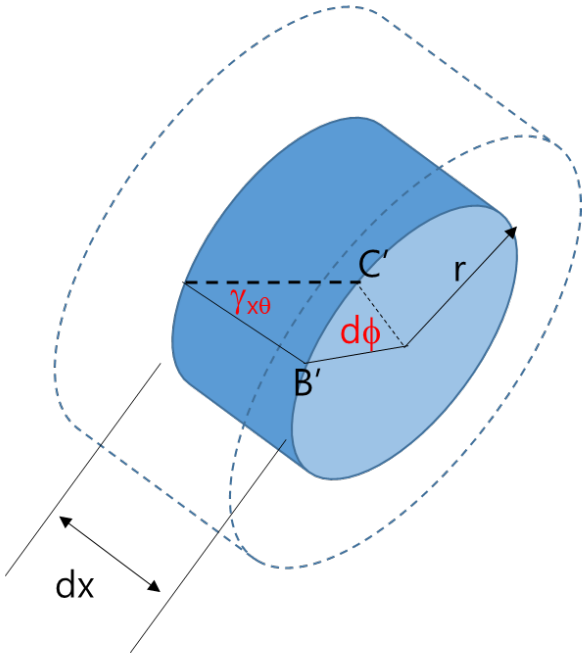
Equilibrium

$$M_x = \iint_A r\tau_{x\theta} dA = \iint_A r^2 G \frac{d\phi}{dx} dA = G\frac{d\phi}{dx}\iint_A r^2 dA = G\frac{d\phi}{dx} J$$

Polar moment

$$\frac{d\phi}{dx} = \frac{M_x}{GJ}$$

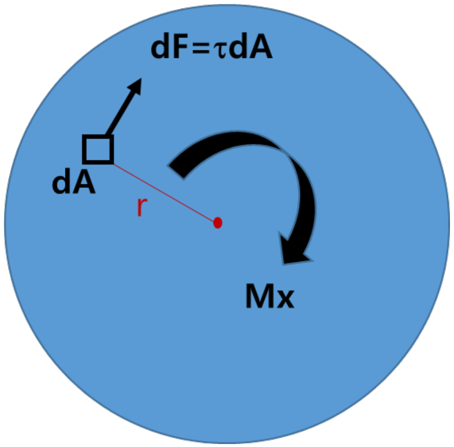
Torsion of circular shafts



$$\frac{d\phi}{dx} = \frac{M_x}{GJ}$$

Angle of twist for constant torque

$$\phi = \int_0^L d\phi = \int_0^L \frac{M_x}{GJ} dx = \frac{M_x L}{GJ}$$



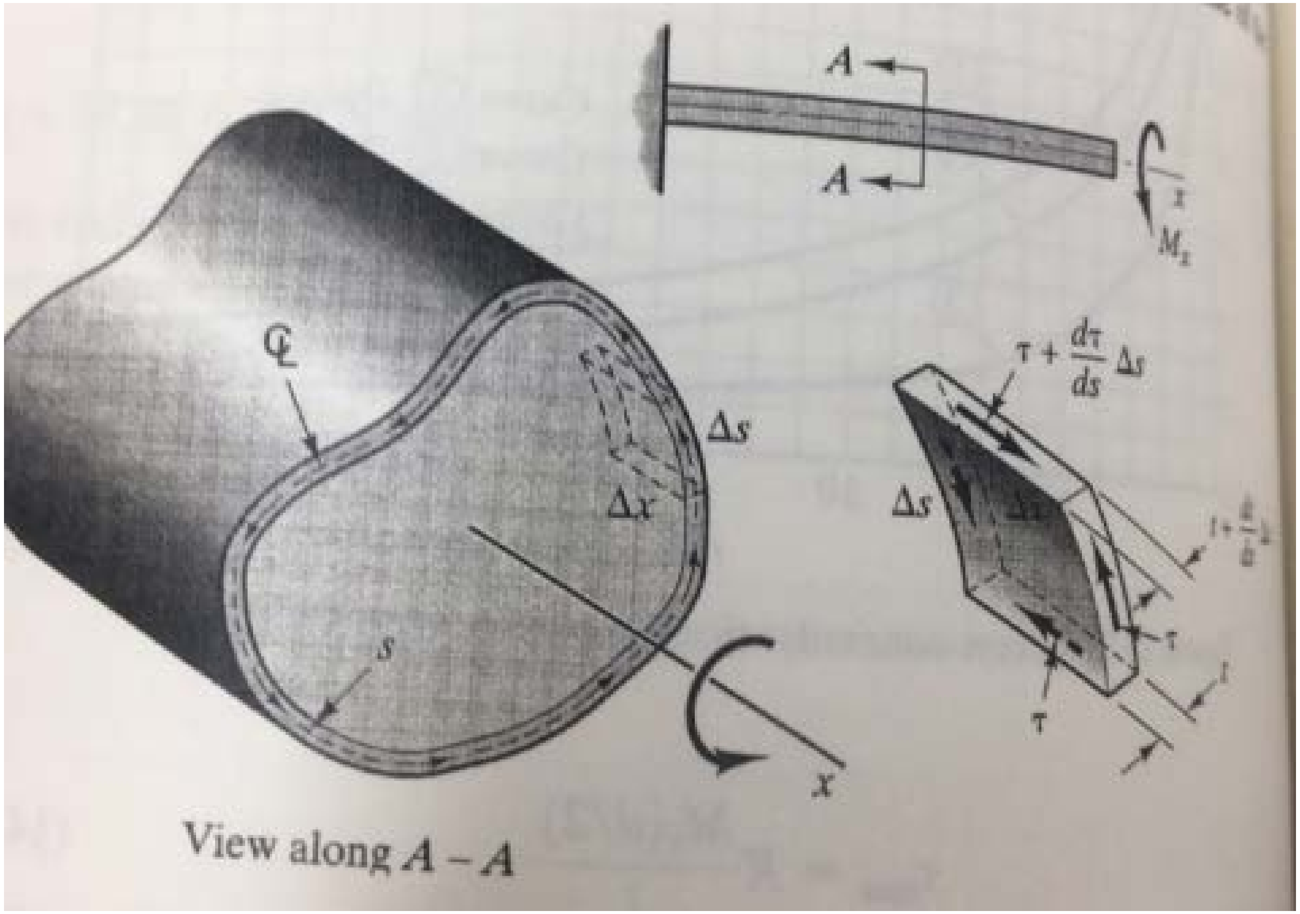
c.f. $\delta = \frac{PL}{EA}$

$$\tau_{x\theta} = \frac{M_x r}{J}$$

Torsion of thin walled, non-circular closed shafts

Assumptions

- Tube is prismatic (or constant cross section)
- Wall thickness can be variable
- Define middle surface or midline (along this coordinate s is defined)



Torsion of thin walled, non-circular closed shafts

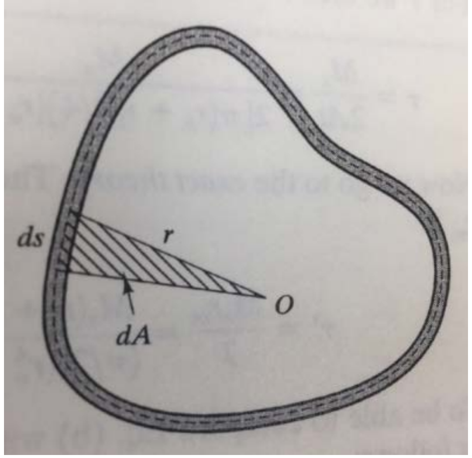
Applying equilibrium into the infinitesimal element

$$-\tau t \Delta x + \left(\tau + \frac{d\tau}{ds} \Delta s \right) \left(t + \frac{dt}{ds} \Delta s \right) \Delta x = 0$$

$$\left(\tau \frac{dt}{ds} + t \frac{d\tau}{ds} \right) \Delta s = 0 \qquad \frac{d}{ds}(\tau t) = 0 \qquad \tau t = constant$$

"Shear flow" is constant

Moment (torque) equilibrium



$$M_x \mathbf{i} = \oint \mathbf{r} \times (\tau t) \, d\mathbf{s} = (\tau t) \oint \mathbf{r} \times d\mathbf{s} = (\tau t)(2A\mathbf{i})$$

$$M_x = 2\tau t A \qquad \tau = \frac{M_x}{2tA}$$

A: total plane area enclosed by the midline s

Torsion of thin walled, non-circular closed shafts

Applying conservation of energy law,

For the net rotation

$$\frac{1}{2}M_x\phi = \frac{1}{2}\int_0^L \tau\frac{\tau}{G}tdsdx$$

$$\phi = \frac{d\phi}{dx}L$$

$$M_x\left(\frac{d\phi}{dx}\right) = \frac{1}{G}\oint\tau^2 tds = \frac{1}{G}\oint\frac{M_x^{\;2}}{4t^2A^2}tds$$

$$\left(\frac{d\phi}{dx}\right) = \frac{M_x}{4GA^2}\oint\frac{ds}{t} \Rightarrow \left(\frac{d\phi}{dx}\right) = \frac{M_x\,S}{4GA^2t}$$

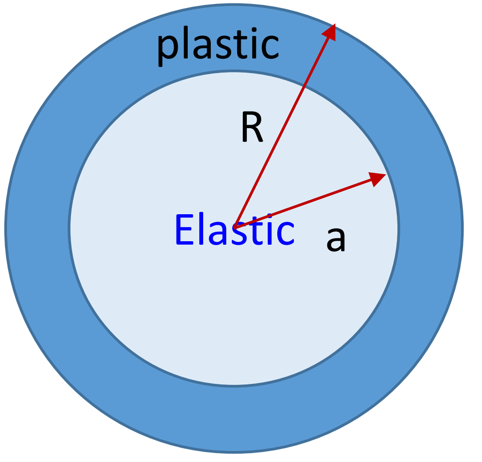
Constant thickness

Elastic-perfectly plastic torsion

From elastic circular shaft

$$\boxed{\tau_{x\theta} = \frac{M_x r}{J}} \quad \text{or} \quad \boxed{M_x = \frac{\tau_{x\theta} J}{r}}$$

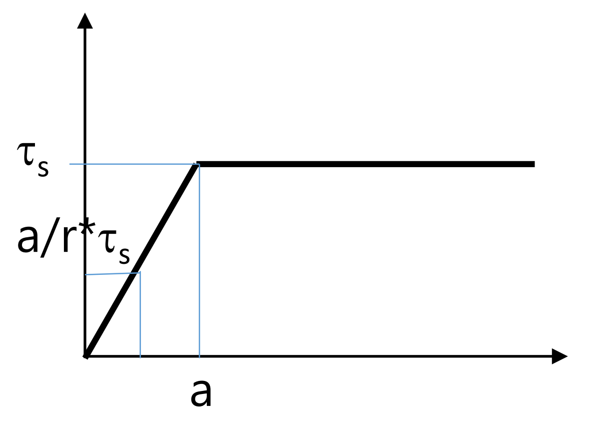
$$T_{\max,el} = \frac{\tau_s J}{R} = \frac{\tau_s}{R}\left(\frac{\pi}{2}R^4\right) = \frac{\pi}{2}R^3\tau_s$$



$$T_{hinge} = \int_0^{2\pi}\int_0^R r\tau_s\left(rdrd\theta\right) = 2\pi\tau_s\frac{R^3}{3}$$

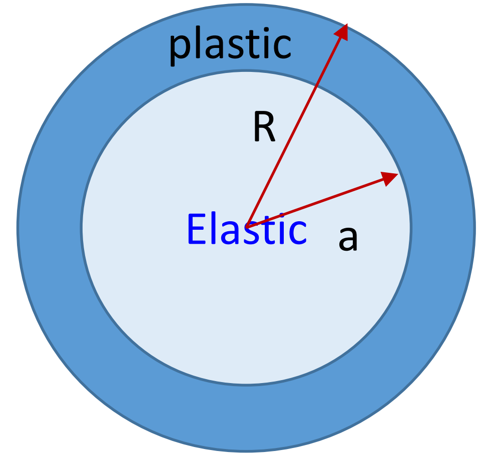
$$\frac{T_{hinge}}{T_{\max,el}} = \frac{4}{3}$$

$$\boxed{\begin{aligned} T &= \int_0^{2\pi}\int_0^a r\left(\frac{r}{a}\tau_s\right)\left(rdrd\theta\right) + \\ &\int_0^{2\pi}\int_a^R r\left(\tau_s\right)\left(rdrd\theta\right) = \frac{\pi\tau_s}{3}\left(2R^3 - \frac{1}{2}a^3\right) \end{aligned}}$$

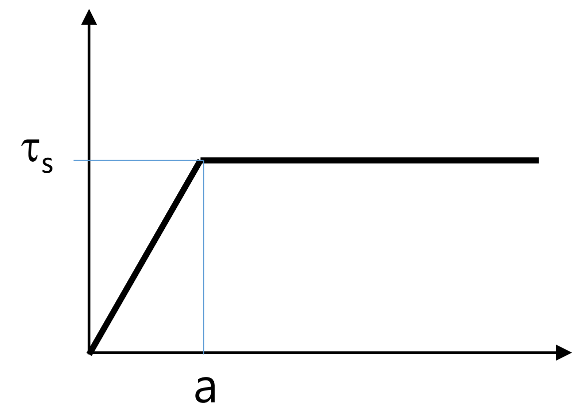


Elastic-perfectly plastic torsion

$$T = \int_0^{2\pi}\int_0^{a} r\left(\frac{r}{a}\tau_s\right)(rdrd\theta) + \int_0^{2\pi}\int_a^{R} r(\tau_s)(rdrd\theta) = \frac{\pi\tau_s}{3}\left(2R^3 - \frac{1}{2}a^3\right)$$



$$a = \left(4R^3 - \frac{6T}{\pi\tau_s}\right)^{1/3}$$ Radius of elastic core



$$\gamma_{x\theta} = \frac{d\phi}{dx} r \Rightarrow \frac{d\phi}{dx} = \frac{\gamma_{x\theta}}{r}$$

At r=a, $\gamma_{x\theta} = \frac{\tau_s}{G}$ Thus, $\boxed{\frac{d\phi}{dx} = \frac{\tau_s}{Ga}}$

Elastic-perfectly plastic torsion

Residual rate of twist

From, $$\frac{d\phi}{dx} = \frac{M_x}{GJ}$$

$$\boxed{\left(\frac{d\phi}{dx}\right)_{res} = \left(\frac{d\phi}{dx}\right)_{plastic} - \frac{T}{GJ}}$$

Questions ?