445.204

Introduction to Mechanics of Materials

(재료역학개론)

Chapter 7: Generalized Hooke's law

(Ch. 6 in Shames)

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Hooke's law in 3D

The stress σ_x in the x-direction produces 3 strains

1) Longitudinal strain (extension) along the **x-axis**,

$$\boxed{\varepsilon_x = \frac{\sigma_x}{E}}$$

2) Transverse strains (contraction) along the **y and z -axes**, due to the Poisson's effect,

$$\boxed{\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\frac{\nu\sigma_x}{E}}$$

Hooke's law in 3D

The total strain produced along a particular direction can be determined by the principle of superposition

That is, the resultant strain along the x-axis, comes from the **strain contribution** due to the application of σ_x, σ_y and σ_z

σ_x causes: $\dfrac{\sigma_x}{E}$ in the x-direction

σ_y causes: $-\dfrac{\nu\sigma_y}{E}$ in the x-direction

σ_z causes: $-\dfrac{\nu\sigma_z}{E}$ in the x-direction

Therefore, applying the principle of superposition in the x-axis results in

$$\boxed{\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]}$$

Hooke's law in 3D

In general, by superposition of the components of strain in the x, y, and z directions, the strains can be written as

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - v\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - v\left(\sigma_x + \sigma_y\right)\right]$$

Hooke's law in 3D

Shearing stresses acting on the unit cube produce shearing Strains; i.e., no Poisson's effect

$$\tau_{xy} = G\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{xz} = G\gamma_{xz}$$

$$\boxed{\gamma_{ij} = 2\varepsilon_{ij}}$$

Why?

Hooke's law in 3D

$$\boxed{\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}}$$

In Matrix Form

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} + \underline{\begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ \alpha\Delta T \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

If non-isothermal

E vs. G vs. v

- Stress tensor (state) in x-y coordinate system

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} = \begin{bmatrix} -S & 0 \\ 0 & S \end{bmatrix}$$

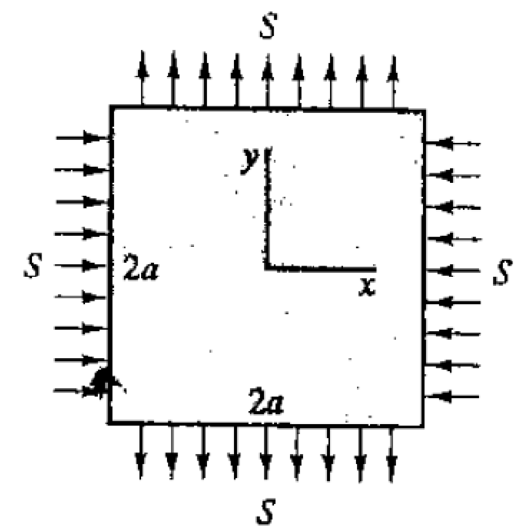


Figure 6.3. Square plate under stress.

Force equilibrium in the x' direction

$$\boxed{\tau_{y'x'}(2a)(t)\sqrt{2} - 2\left[S(2a)(t)\frac{1}{\sqrt{2}}\right] = 0 \rightarrow \tau_{y'x'} = S}$$

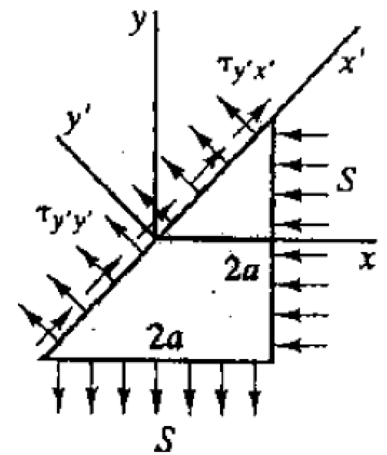


Figure 6.4. Half of square plate.

- Stress tensor (state) in x'-y' coordinate system

$$\begin{bmatrix} \tau_{x'x'} & \tau_{x'y'} \\ \tau_{x'y'} & \tau_{y'y'} \end{bmatrix} = \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix}$$

E vs. G vs. v

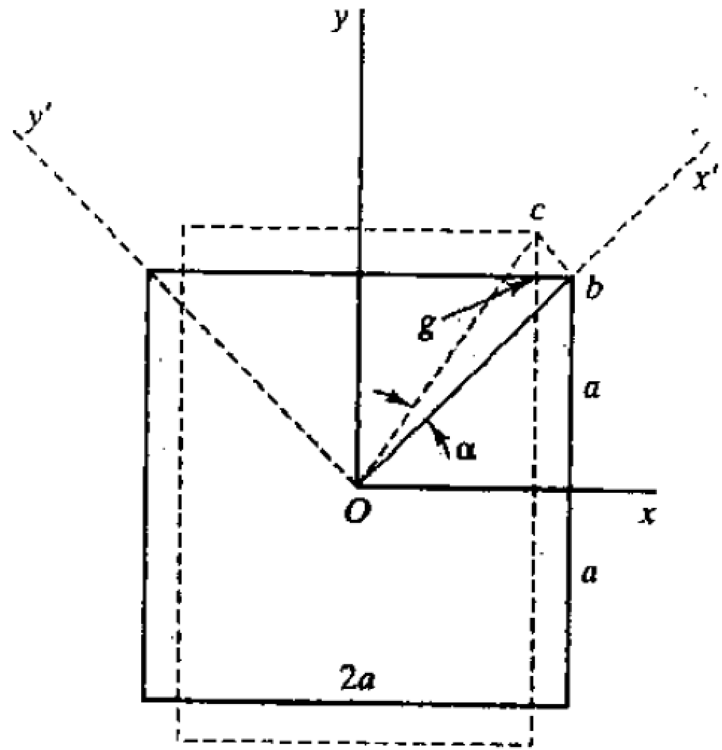


Figure 6.5. Deformation of square plate.

$$\varepsilon_{x'y'} = \frac{cb}{Ob} \qquad cb = \frac{gb}{\cos 45^o} = \sqrt{2}gb$$

$$Ob = \frac{a}{\cos 45^o} = \sqrt{2}a$$

$$gb = \frac{1}{2}\varepsilon_{xx}(2a) + \frac{1}{2}v\varepsilon_{yy}(2a) = \frac{S}{E}a + v\frac{S}{E}a = \frac{Sa}{E}(1+v)$$

$$\boxed{\varepsilon_{x'y'} = \frac{S}{E}(1+v)}$$

$$\tau_{x'y'} = G(2\varepsilon_{x'y'}) \leftrightarrow S = G(2)(\frac{S}{E}(1+v))$$

$$G = \frac{E}{2(1+v)}$$

Strain energy

- Strain energy increment in the uniaxial tension for infinitesimal volume dv

$$\tau_{xx} d\varepsilon_{xx} dv$$

- More generally for normal stresses

$$\left(\tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz}\right) dv$$

$$\tau_{xy} d\gamma_{xy} = \tau_{xy}\left(2d\varepsilon_{xy}\right) = \tau_{xy} d\varepsilon_{xy} + \tau_{yx} d\varepsilon_{yx}$$

- More generally for normal and shear stresses

$$\left(\tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz} + \boxed{\tau_{xy} d\gamma_{xy}} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx}\right) dv$$

- Energy per unit volume

$$du = \tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz} + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} = \boxed{\tau_{ij} d\varepsilon_{ij}}$$

- Then,

$$u = \int_0^{\varepsilon_{ij}} \tau_{ij} d\varepsilon_{ij} \qquad U = \int_V \int_0^{\varepsilon_{ij}} \tau_{ij} d\varepsilon_{ij} dv$$

Strain energy for linear isotropic elastic solids

$$du = \frac{\tau_{xx}}{E}\left[d\tau_{xx} - v(\tau_{yy} + \tau_{zz})\right] + \frac{\tau_{yy}}{E}\left[d\tau_{yy} - v(\tau_{zz} + \tau_{xx})\right] + \frac{\tau_{zz}}{E}\left[d\tau_{zz} - v(\tau_{xx} + \tau_{yy})\right]$$

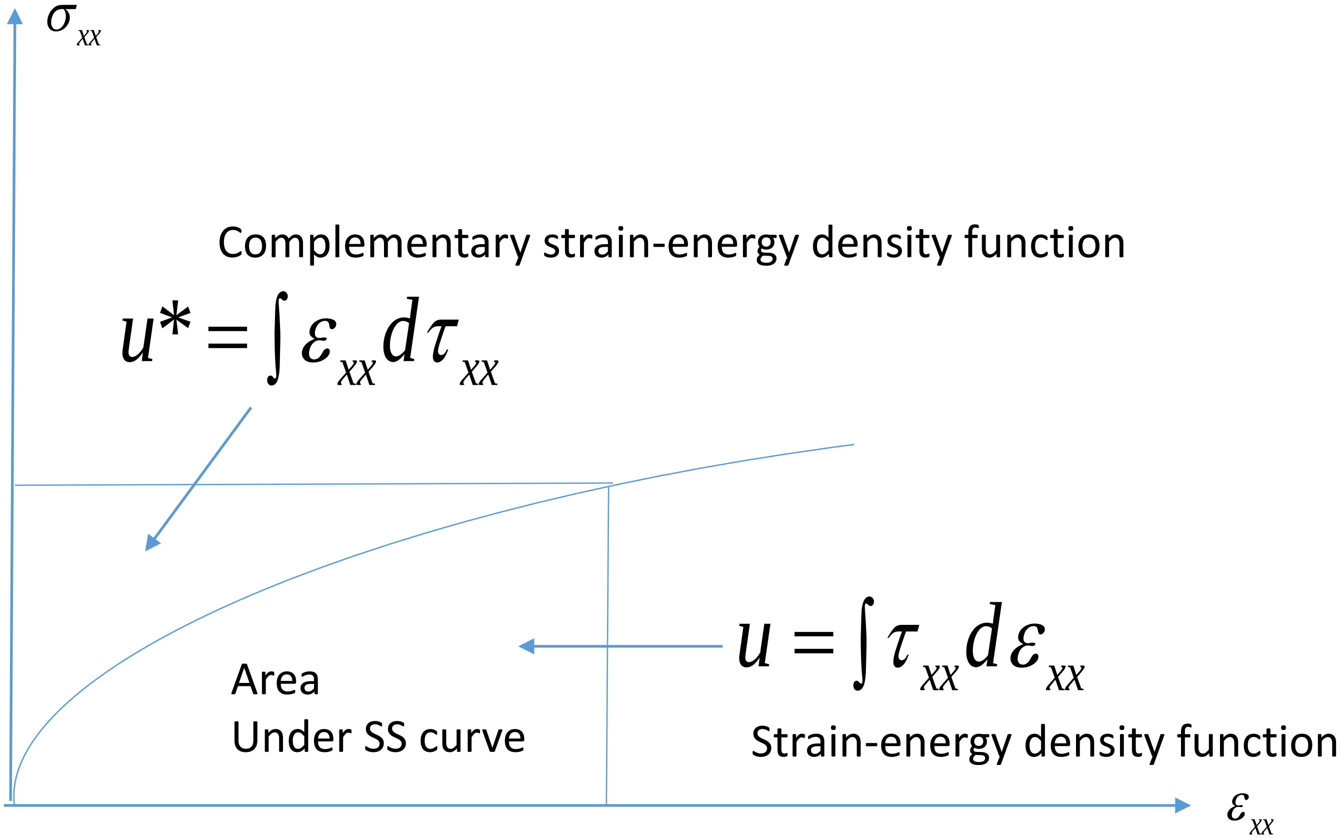
$$+ \frac{\tau_{xy}}{G}d\tau_{xy} + \frac{\tau_{yz}}{G}d\tau_{yz} + \frac{\tau_{zx}}{G}d\tau_{xz}$$

$$u = \frac{1}{2E}(\tau_{xx}^2 + \tau_{yy}^2 + \tau_{zz}^2) - \frac{v}{E}\left(\tau_{xx}\tau_{yy} + \tau_{yy}\tau_{zz} + \tau_{zz}\tau_{xx}\right) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$$

- As functions strain

$$u = \frac{Ev}{2(1+v)(1-2v)}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + G\left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2\right) + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)$$

Strain energy density: physical interpretation



Energy method – Castigliano theorem II
* Derivation in the textbook

$$\frac{\partial U}{\partial P_k} = \Delta_k$$

The rate of change of the strain energy for a body with respect to any statically independent force P_k gives the deflection component of the point of application of this force in the direction of force

$$\frac{\partial U}{\partial M} = \theta$$

The rate of change of the strain energy for a body with respect to any statically independent couple M gives the amount of rotation at the point of application of the point couple about an axis collinear with the couple moment

Volume strain

For a rectangular parallelepiped with edges *dx*, *dy* and *dz*

The volume in the strained condition is:

$(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)\, dx\, dy\, dz$

The dilation (or volume strain) Δ is given as

$$\boxed{\begin{aligned} \Delta &= \frac{(1+\varepsilon_x)(1+\varepsilon_x)(1+\varepsilon_x)\,\mathrm{dx\,dy\,dz} - \mathrm{dx\,dy\,dz}}{\mathrm{dx\,dy\,dz}} \\ &= (1+\varepsilon_x)(1+\varepsilon_x)(1+\varepsilon_x) - 1 \\ & \\ &\text{for small strain} \\ \Delta &= \varepsilon_x + \varepsilon_y + \varepsilon_z \end{aligned}}$$

Plane stress

Plane Stress ($\sigma_3 = 0$)

This stress state is typically observed in

- a thin sheet loaded in the plane of the sheet, or

- a thin wall tube loaded by internal pressure where there is no stress normal to a free surface.

Set $\sigma_z = \sigma_3 = 0$.

Therefore,

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\sigma_2\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\sigma_3\right]$$

$$\varepsilon_3 = -\frac{1}{E}\nu\left[\sigma_1 + \sigma_2\right]$$

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\varepsilon_2 E + \nu\sigma_1\right)\right]$$

$$= \frac{1}{E}\left[\sigma_1\left(1 - \nu^2\right)\right] - \frac{1}{E}\left(\nu\varepsilon_2 E\right)$$

$$\varepsilon_1 + \nu\varepsilon_2 = \frac{1 - \nu^2}{E}\sigma_1$$

$$\boxed{\begin{gathered}\sigma_1 = \frac{E}{1 - \nu^2}\left[\varepsilon_1 + \nu\varepsilon_2\right] \\ \textit{Similarly,} \\ \sigma_2 = \frac{E}{1 - \nu^2}\left[\varepsilon_2 + \nu\varepsilon_1\right]\end{gathered}}$$

Plane strain

Plane Strain ($\varepsilon_3 = 0$): This occurs typically when
One dimension is much greater than the other two

Examples are a long rod or a cylinder with restrained ends.

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] = 0$$

but

$$\sigma_3 = \nu[\sigma_1 + \sigma_2]$$

This shows that a stress exists along direction-3 (z-axis) even though the strain is zero.

$$\boxed{\begin{aligned} \varepsilon_1 &= \frac{1}{E}\left[\left(1 - \nu^2\right)\sigma_1 - \nu(1 + \nu)\sigma_2\right] \\ \varepsilon_2 &= \frac{1}{E}\left[\left(1 - \nu^2\right)\sigma_2 - \nu(1 + \nu)\sigma_1\right] \\ \varepsilon_3 &= 0 \end{aligned}}$$

Questions ?