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소성재료역학
(Metal Plasticity)

Chapter 12: Yield Function

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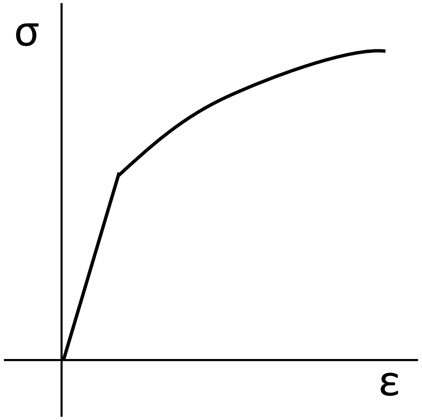
Three-dimensional plasticity

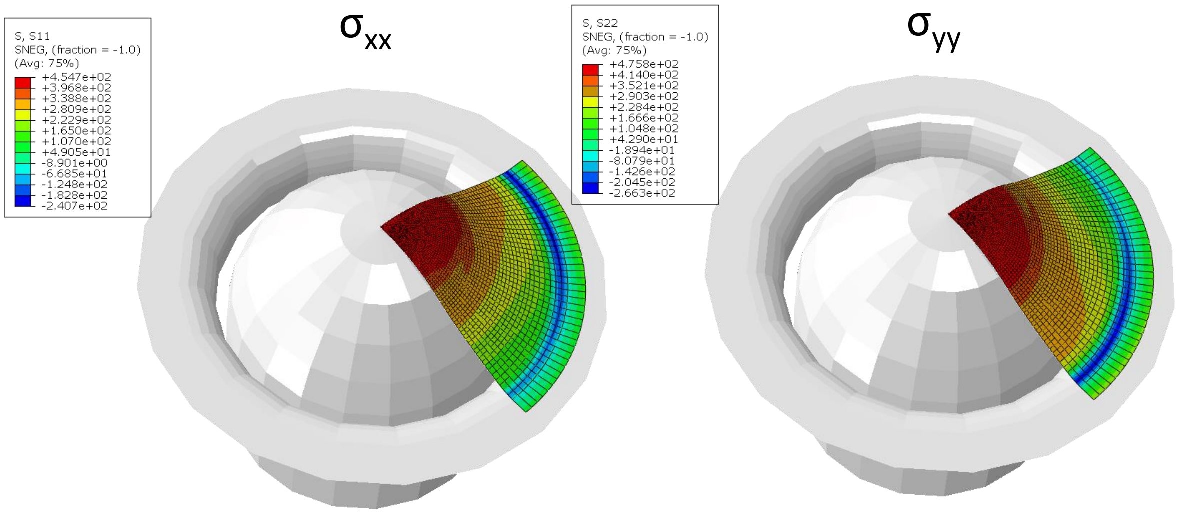
1-D constitutive law

Tension
Compression
Bending
....



Stresses and strains in 3-D

General state : multiple components exist



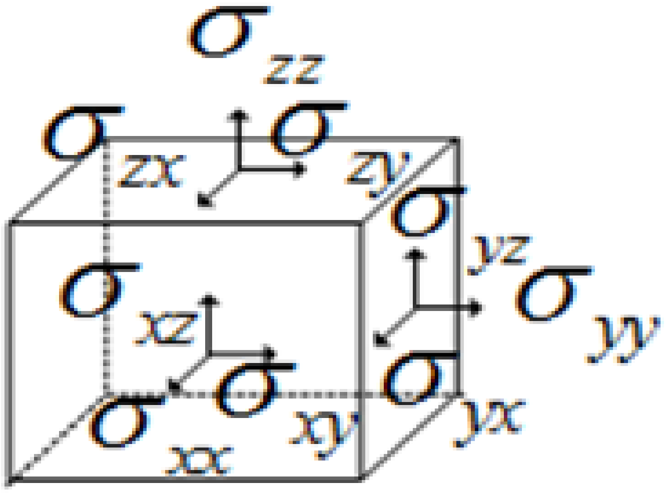
Three-dimensional plasticity

Stress and Strain tensor

Tensor : mathematical notation of physical quantities

- 0 th order Tensor : scalar value – energy, etc
- 1st order Tensor : vector - force, displacement, velocity, etc
- 2nd order Tensor : linear transformation from vector to the other vector - stress, strain, etc.

Stress : symmetric 2nd order tensor

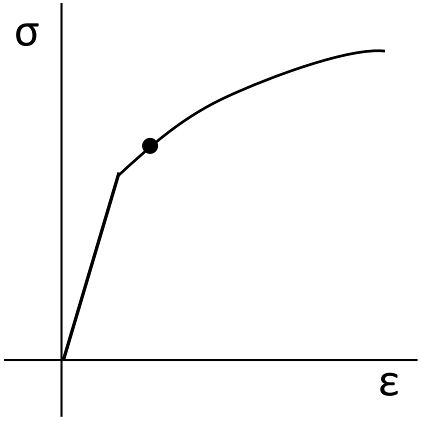


$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}$$

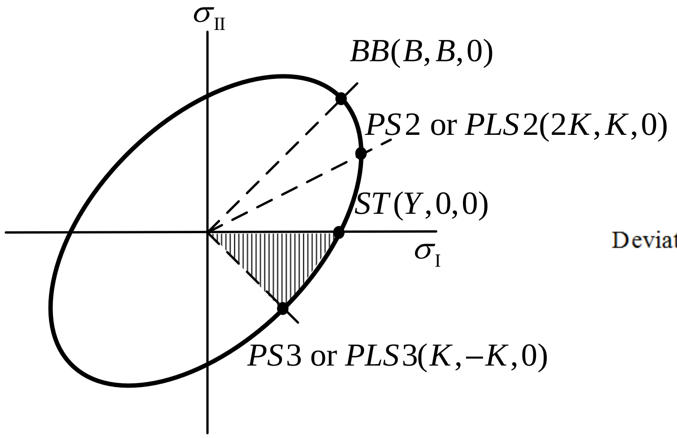
Three-dimensional plasticity

For isotropic case,

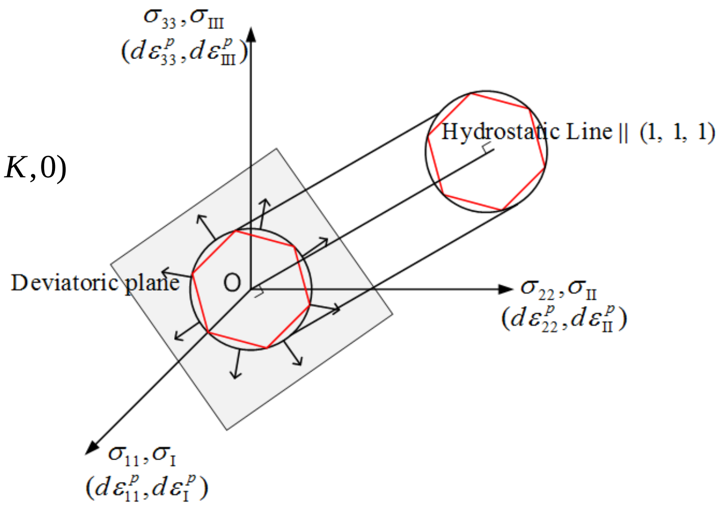
1-D : point



2-D : line



3-D : surface



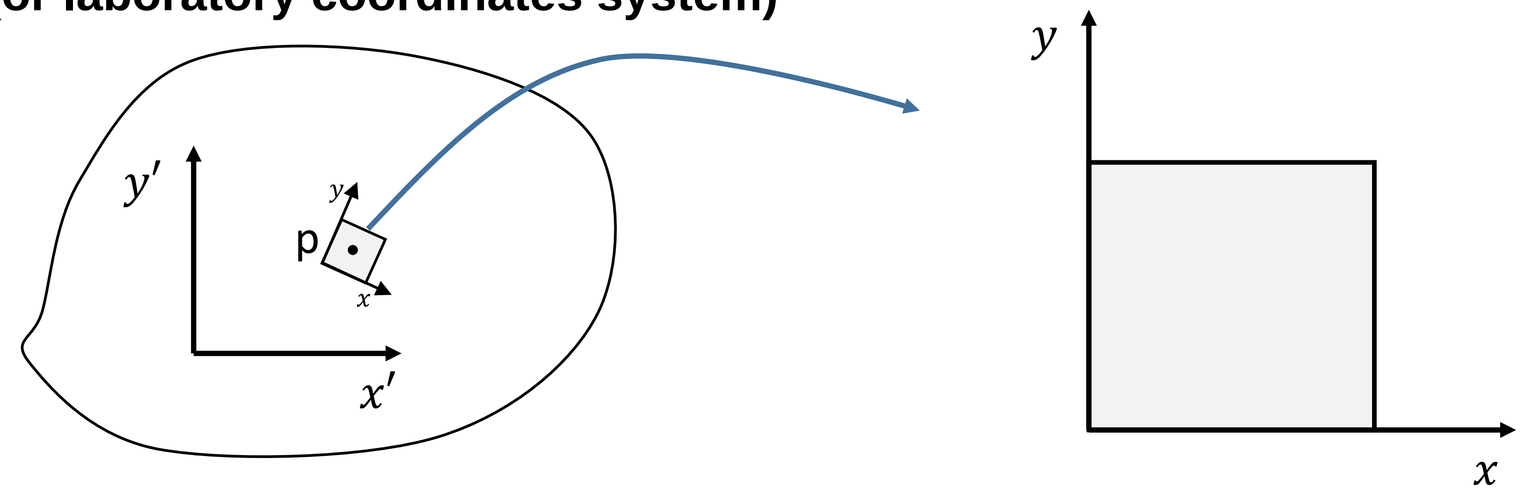
Basic features of the yield surface

<u>Concept of material coordinates</u>
<u>(or materially embedded coordinates)</u>

- Since the constitutive law describes material properties, components of vector or tensor quantities are defined with respect to material coordinates

Global coordinates system
(or laboratory coordinates system)

Material coordinates system

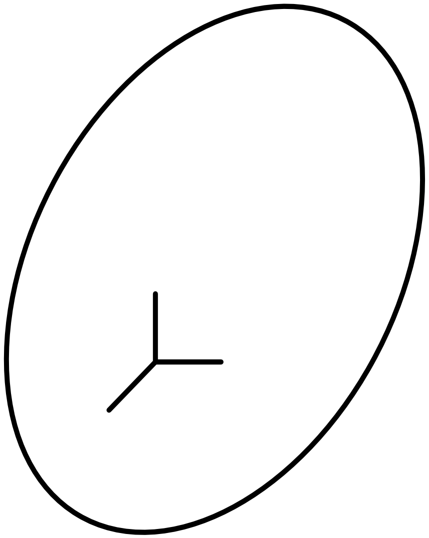


Basic features of the yield surface

Convexity of yield surface

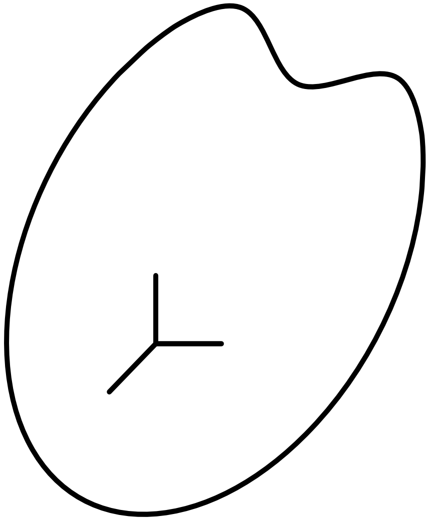
- The yield surface is considered to be convex
- In other words, yield surface should be bulged out
- Or, any straight line connecting two points located inside the surface stays inside the surface

Convex



(a)

Non-convex



(b)

Basic features of the yield surface

- For the simplicity, consider a 2-D imaginary yield surface

$$f(\boldsymbol{\sigma}) = f(\sigma_{xx}, \sigma_{yy}) = constant = C$$

- Size of yield function f(σ) is defined by constant C
- For the isotropic case, yield function can be assumed as a circle

$$\begin{cases} f_1(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}| = (\sigma_{xx}^2 + \sigma_{yy}^2)^{1/2} = \mathrm{c} \\ f_2(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}|^2 = (\sigma_{xx}^2 + \sigma_{yy}^2) = \mathrm{c}^2 \end{cases}$$

- For the anisotropic case, ellipse can be an yield function

$$\begin{cases} f_1(\boldsymbol{\sigma}) = \bar{\sigma}_1(\boldsymbol{\sigma}) = (\sigma_{xx}^2 + (\frac{a\sigma_{yy}}{b})^2)^{\frac{1}{2}} = a \\ f_2(\boldsymbol{\sigma}) = \bar{\sigma}_2(\boldsymbol{\sigma}) = ((\frac{b\sigma_{xx}}{a})^2 + \sigma_{yy}^2)^{\frac{1}{2}} = b \\ f_3(\boldsymbol{\sigma}) = (\sigma_{xx}^2 + (\frac{a\sigma_{yy}}{b})^2) = a^2 \\ f_4(\boldsymbol{\sigma}) = ((\frac{b\sigma_{xx}}{a})^2 + \sigma_{yy}^2) = b^2 \end{cases}$$

Basic features of the yield surface

n-th order homogeneous function

$$f(\alpha \boldsymbol{x}) = \alpha^n f(\boldsymbol{x})$$ Bold : vector or tensor

Two properties of n-th order homogeneous function

i) $\dfrac{\partial f}{\partial \boldsymbol{x}}$ → (n-1)-th order homogeneous function

or $\mathbf{g} = \dfrac{\partial f}{\partial \boldsymbol{x}} = \begin{pmatrix} nx^{n-1} \\ ny^{n-1} \end{pmatrix}$ and $\mathbf{g}(\alpha \boldsymbol{x}) = \begin{pmatrix} n(\alpha x)^{n-1} \\ n(\alpha y)^{n-1} \end{pmatrix} = \alpha^{n-1} \begin{pmatrix} nx^{n-1} \\ ny^{n-1} \end{pmatrix} = \alpha^{n-1}\mathbf{g}$

ii) $\dfrac{\partial f}{\partial \boldsymbol{x}} \cdot \boldsymbol{x} \left(= \dfrac{\partial f}{\partial x_i} x_i \ or \ \dfrac{\partial f}{\partial x_{ij}} x_{ij} \right) = nf(\boldsymbol{x})$

Basic features of the yield surface

Proof (1)

$$f(\alpha \boldsymbol{x}) = \alpha^n f(\boldsymbol{x})$$

Left hand side

$$\frac{\partial f(\alpha x)}{\partial \boldsymbol{x}} = \frac{\partial f(\alpha \boldsymbol{x})}{\partial(\alpha \boldsymbol{x})} \frac{\partial(\alpha \boldsymbol{x})}{\partial \boldsymbol{x}} = \alpha \boldsymbol{I} \frac{\partial f(\alpha \boldsymbol{x})}{\partial(\alpha \boldsymbol{x})} = \alpha \mathbf{g}(\alpha \boldsymbol{x})$$

$$\textit{where} \quad \mathbf{g}(\mathrm{x}) = \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}$$

Right hand side

$$\alpha^n \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \alpha^n \mathbf{g}(\boldsymbol{x})$$

$$\therefore \mathbf{g}(\alpha x) = \alpha^{n-1} \mathbf{g}(x)$$

Basic features of the yield surface

Proof (2)

$$f(\alpha \boldsymbol{x}) = \alpha^n f(\boldsymbol{x})$$

Left hand side

$$\frac{\partial f(\alpha x)}{\partial \alpha} = \frac{\partial f(\alpha \boldsymbol{x})}{\partial(\alpha \boldsymbol{x})} \frac{\partial(\alpha \boldsymbol{x})}{\partial \alpha} = \frac{\partial f(\alpha \boldsymbol{x})}{\partial(\alpha \boldsymbol{x})} \boldsymbol{x} = \mathbf{g}(\alpha \boldsymbol{x}) \cdot \boldsymbol{x}$$

Right hand side

$$\frac{\partial\left(\alpha^n f(\boldsymbol{x})\right)}{\partial \alpha} = n\alpha^{n-1} f(\boldsymbol{x})$$

$$\therefore \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{x} = nf(\boldsymbol{x})$$

Basic features of the yield surface

- Effective stress or equivalent stress –

the yield function of the first order homogeneous function

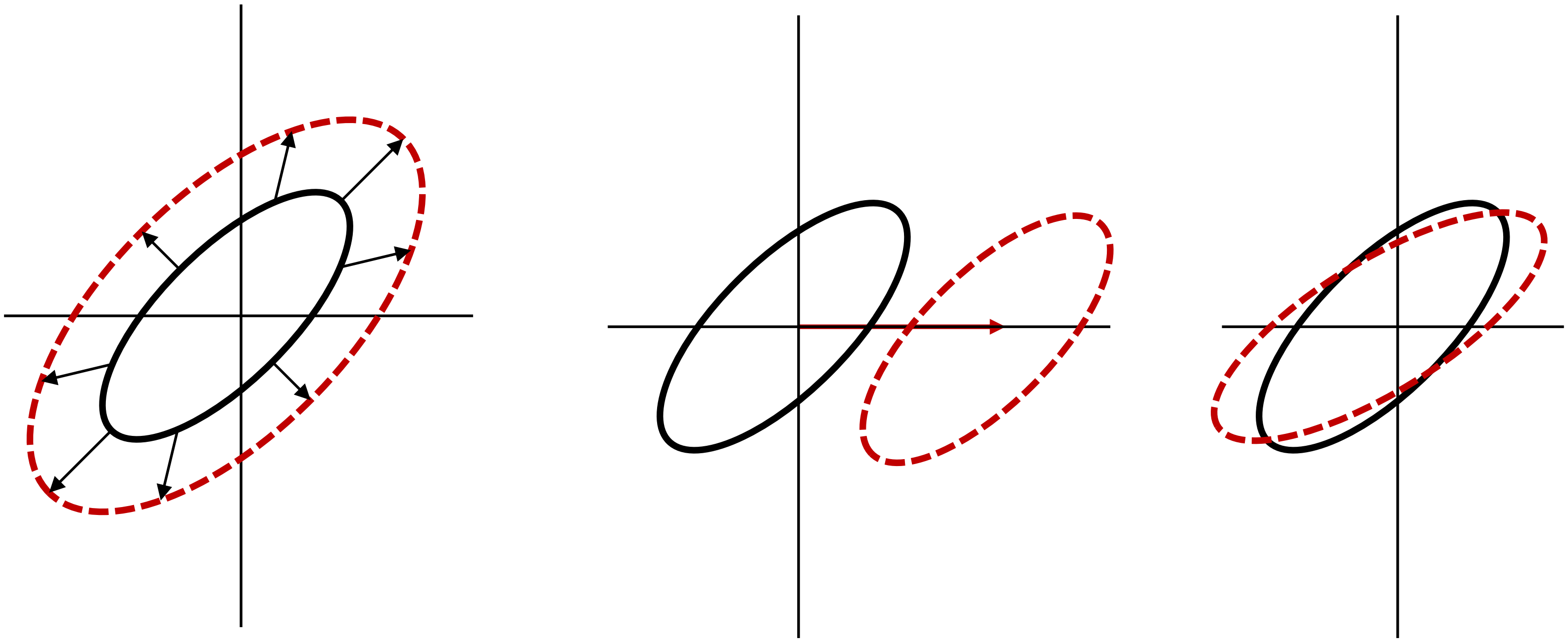
- Linear transformation of isotropic yield surface into anisotropic yield surface – preserve the convexity

$$\begin{pmatrix} \sigma'_{xx} \\ \sigma'_{yy} \end{pmatrix} = \begin{pmatrix} c/a & 0 \\ 0 & c/b \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix}$$

Basic features of the yield surface

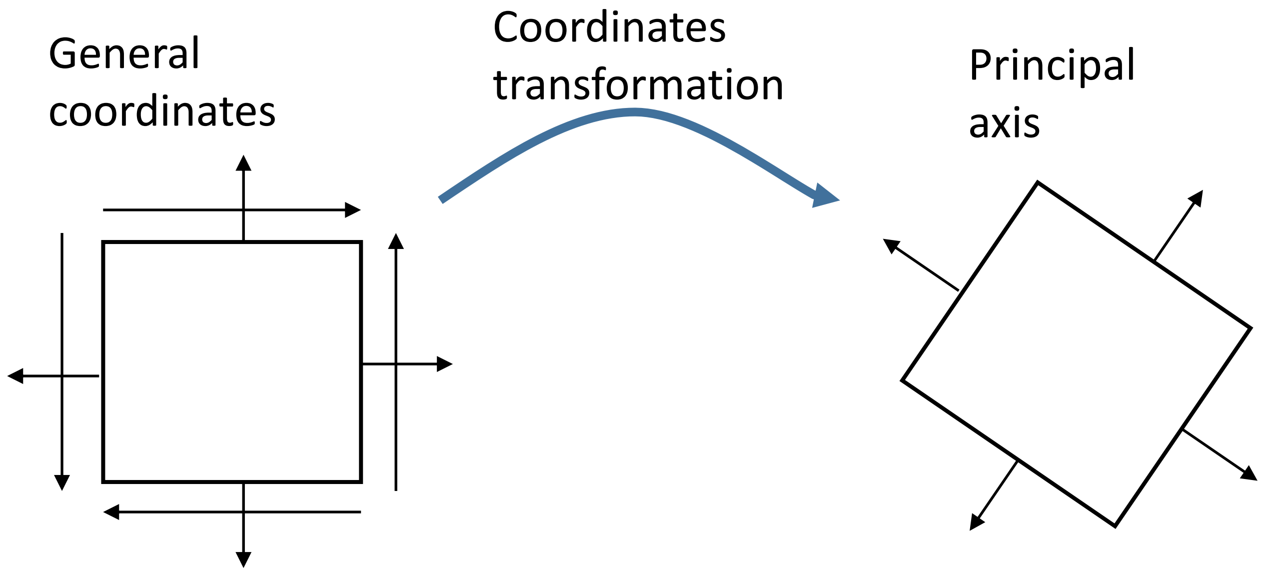
Expansion, translation, shape change of yield surface

Expansion | **Translation** | **Shape change(distortion)**



Appendix

Principal stress and invariants



Appendix

Principal values and directions of symmetric tensor
= Eigenvalues and eigenvectors

$$\boldsymbol{\sigma}\mathbf{n} = \lambda\mathbf{n} \rightarrow \left(\boldsymbol{\sigma} - \lambda\mathbf{I}\right)\mathbf{n} = \mathbf{0}$$

$$or\ (\sigma_{ij} - \lambda\delta_{ij})n_i = 0$$

$$for\ the\ non-trivial\ solution,$$

$$\boxed{\det\left(\boldsymbol{\sigma} - \lambda\mathbf{I}\right) = 0}$$

Where λ : principal values (or eigenvalues) of tensor
n : principal directions(or eigenvectors) of tensor

Appendix

Principal values & vectors and Invariants

= Do not change with any coordinates system

$$\textit{from}\ \det\left(\boldsymbol{\sigma}-\lambda\mathbf{I}\right)=0,\ \text{charactoristic equation is}$$

$$\lambda^3 - I_1\lambda^2 + I_2\lambda^2 - I_3 = 0$$

where I_1, I_2, I_3 are invariants of tensor

$$I_1 = trace(\boldsymbol{\sigma})$$

$$I_2 = \frac{1}{2}\left(trace(\boldsymbol{\sigma})^2 - trace(\boldsymbol{\sigma}^2) \right)$$

$$I_3 = \det(\boldsymbol{\sigma})$$

Yield function - Isotropy

Isotropic generalization

$$f(\sigma_{ij})$$

$$= f(\sigma_I, \sigma_{II}, \sigma_{III}, \tilde{\mathbf{n}}_I, \tilde{\mathbf{n}}_{II}, \tilde{\mathbf{n}}_{III})$$ ← General case

$$= \boxed{f(\sigma_I, \sigma_{II}, \sigma_{III})} \quad \textit{as a symmetric function}$$ ← Isotropic case

$$= \boxed{f(I_1, I_2, I_3)}$$ ← Denote with invariants

where $\sigma_I, \sigma_{II}, \sigma_{III}$: *pricipal stresses*

$\tilde{\mathbf{n}}_I, \tilde{\mathbf{n}}_{II}, \tilde{\mathbf{n}}_{III}$: *principal directions*

I_1, I_2, I_3 : the three invariants of $\boldsymbol{\sigma}$

Yield function - Incompressibility

Hydrostatic & deviatoric stresses

- Decomposition of the stress into two components: the hydrostatic and deviatoric components

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad \text{where } \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\boldsymbol{\sigma} = \mathbf{S} + \frac{1}{3}trace(\boldsymbol{\sigma})\mathbf{I}$$

$\mathbf{S}$: deviatric stress

$$\left(\sigma_{ij}\right) = \begin{pmatrix} \frac{2\sigma_{11} - \sigma_{22} - \sigma_{33}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \frac{2\sigma_{22} - \sigma_{33} - \sigma_{11}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \frac{2\sigma_{33} - \sigma_{11} - \sigma_{22}}{3} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \sigma_{11} + \sigma_{22} + \sigma_{33} & 0 & 0 \\ 0 & \sigma_{11} + \sigma_{22} + \sigma_{33} & 0 \\ 0 & 0 & \sigma_{11} + \sigma_{22} + \sigma_{33} \end{pmatrix}$$

Yield function - Incompressibility

Deviatoric strains

Similarly, $$d\boldsymbol{\varepsilon}^p = d\mathbf{e}^p + \frac{1}{3} trace(d\boldsymbol{\varepsilon}^p)\mathbf{I}$$

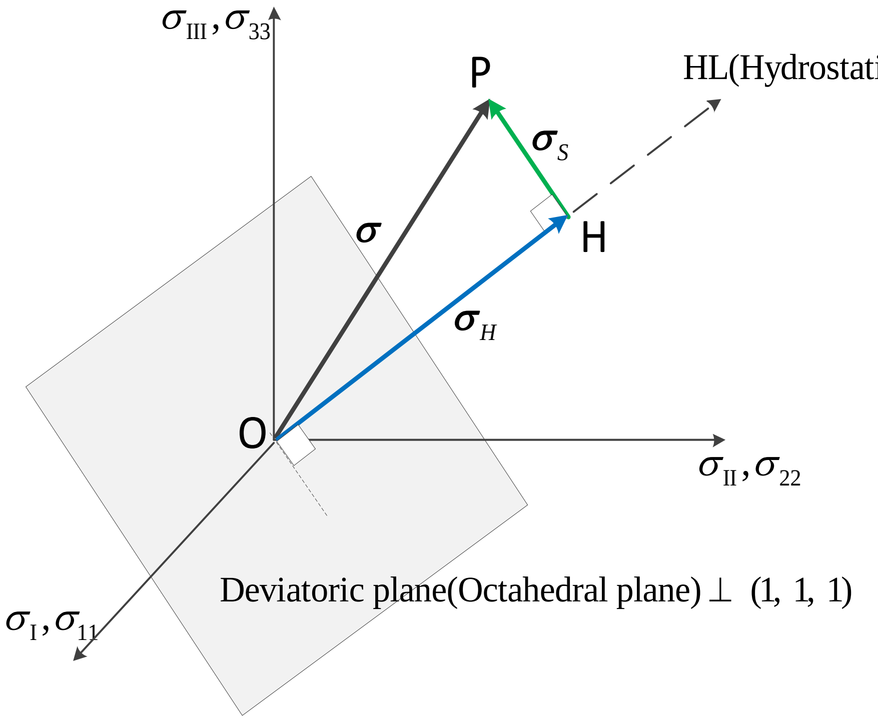
$$d\boldsymbol{\varepsilon}^p : (plastic)\, strain\, increment$$

$$d\mathbf{e}^p : deviatoric\, (plastic)\, strain\, increment$$

$$\boxed{trace(d\mathbf{e}^p) = de_{kk}^p = 0} \quad (for\ crystaline\ materials)$$

Yield function - Incompressibility

Hydrostatic & deviatoric stresses - example



$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} + A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Leftrightarrow$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} S_{11} \\ S_{22} \\ S_{33} \end{pmatrix} + A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ with } \begin{cases} \sigma_{23} = S_{23} = S_{32} \\ \sigma_{31} = S_{31} = S_{13} \\ \sigma_{12} = S_{12} = S_{21} \end{cases}$$

$\Leftrightarrow$

$$Now,\ from\ the\ figure,\ \boldsymbol{\sigma} = \boldsymbol{\sigma}_S + \boldsymbol{\sigma}_H$$

$$\boldsymbol{\sigma}_H = \left[\boldsymbol{\sigma}^T \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\sigma_{ii}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{\sigma}_S = \boldsymbol{\sigma} - \boldsymbol{\sigma}_H = \frac{1}{3} \begin{pmatrix} 2\sigma_{11} - \sigma_{22} - \sigma_{33} \\ -\sigma_{11} + 2\sigma_{22} - \sigma_{33} \\ -\sigma_{11} - \sigma_{22} + 2\sigma_{33} \end{pmatrix} = \begin{pmatrix} S_{11} \\ S_{22} \\ S_{33} \end{pmatrix}$$

$$\therefore A = \frac{1}{3} trace(\boldsymbol{\sigma}) \qquad \boldsymbol{\sigma}_H \parallel [111]$$

$$\therefore S_{11} + S_{22} + S_{33} = 0 \Leftrightarrow \boldsymbol{\sigma}_S^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \boldsymbol{\sigma}_S \perp [111]$$

Yield function - Incompressibility

Yield surface for incompressible case

- For crystal materials, plastic deformation is incompressible and the hydrostatic stress does not affect the plastic deformation since the plastic deformation is incurred by shear stress.

$$d\varepsilon^{p}{}_{kk} = 0 \quad for\ arbitrary\ \sigma_{kk}$$

$$\therefore dW^{p} = S_{ij} de^{p}{}_{ij}$$

$$f = f(\sigma_{ij}) \text{ is independent of } \sigma_{kk}$$

$$f = f(\sigma_{ij}) = f(S_{ij}, \sigma_{kk}) \Rightarrow f = f(S_{ij})$$

Isotropic & Incompressible yield surface

$f(\sigma_{ij})$ ← General case

$= f(S_{ij})$ ← incompressible case

$= f(S_I, S_{II}, S_{III}, n_I^S, n_{II}^S, n_{III}^S)$ ← Isotropic case

$= f(S_I, S_{II}, S_{III})$ as a symmetric function

$= f(J_1, J_2, J_3)$ ← Denote with invariants

$= f(J_2, J_3)$ ← Since $J_1 = 0$

where J_1, J_2, J_3 are invariants of deviatoric tensor

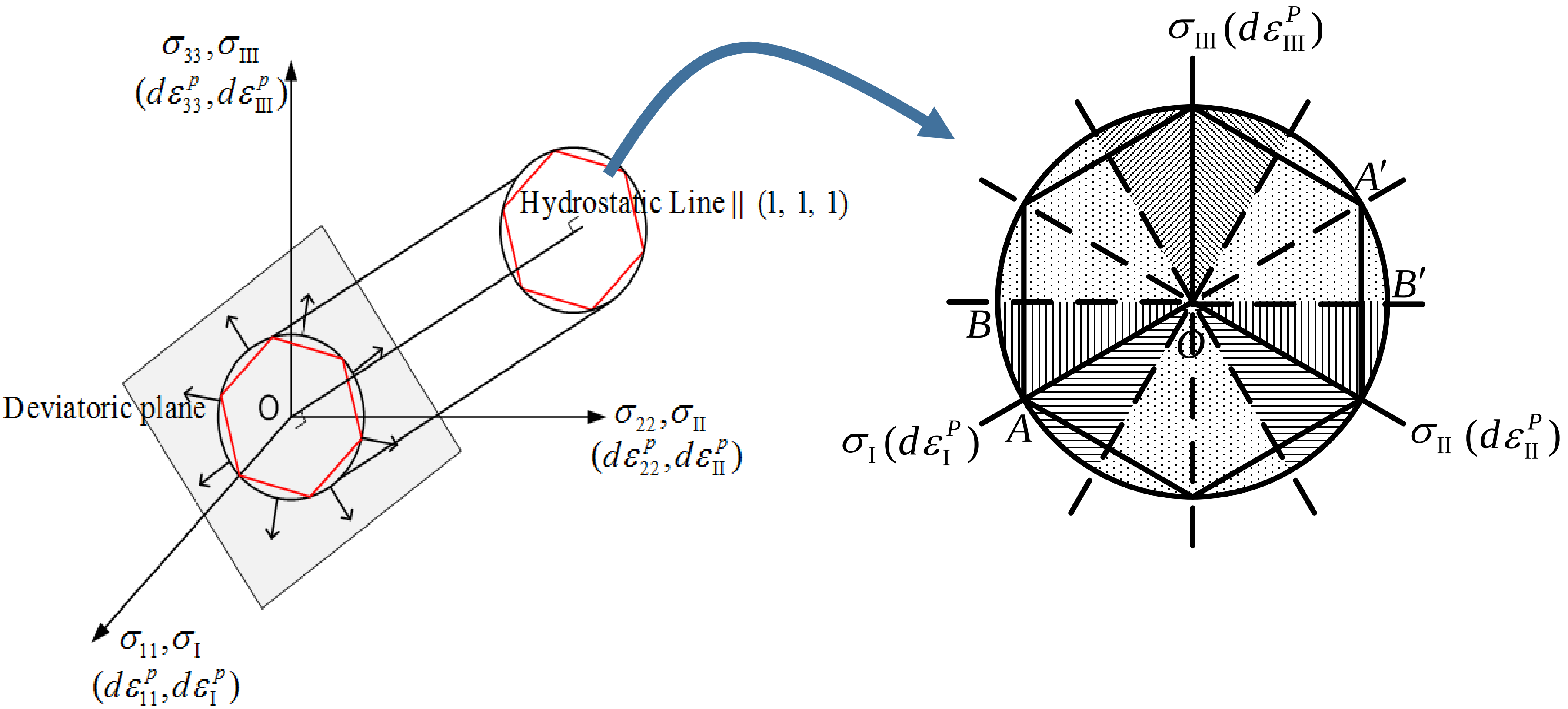
$$J_1 = trace(\mathbf{S}) = 0$$

$$J_2 = \frac{1}{2}\left(trace(\mathbf{S})^2 - trace(\mathbf{S}^2) \right)$$

$$J_3 = \det(\mathbf{S})$$

Incompressibility

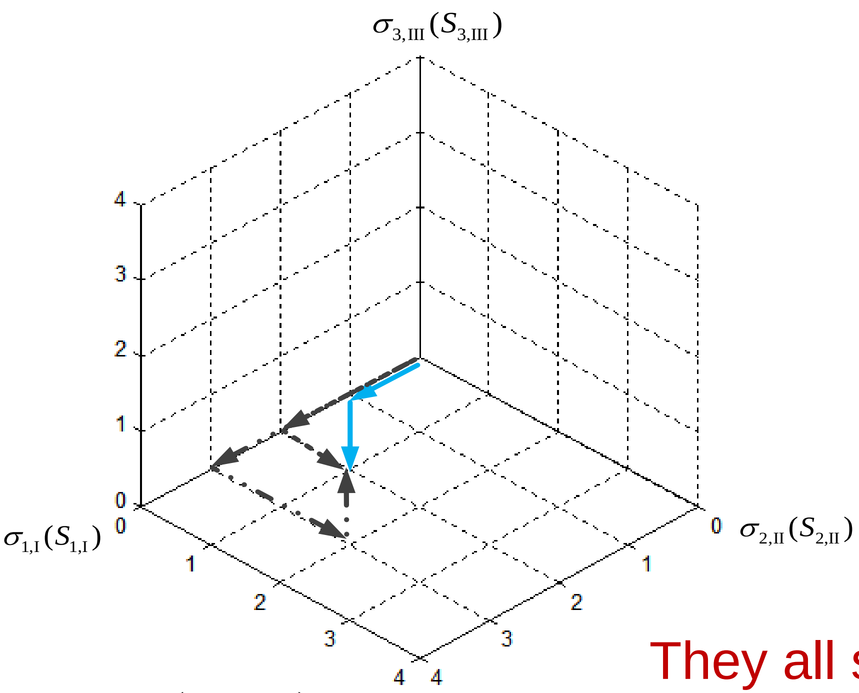
π-diagram



Incompressibility

π-diagram

A way to find the deviatoric stress component: S_1, S_2 and S_3 (with the condition $S_{ii}=0$):



$\boldsymbol{S}(1,0,-1)$

$\boldsymbol{\sigma}(2,1,0)$

$\boldsymbol{\sigma}'(3,2,1)$

They all share the same position on the deviatoric plane (therefore, in the π diagram), but with different hydrostatic component

Von Mises isotropic yield function

(or J_2 plasticity)

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = f(J_2,\ J_3) = J_2 = \frac{1}{2}\left|\boldsymbol{S}\right|^2 = \frac{1}{2}S_{ij}S_{ij} = \frac{1}{2}(S_I^2 + S_{II}^2 + S_{III}^2) = Const.$$

From above, yield function can be defined like in below

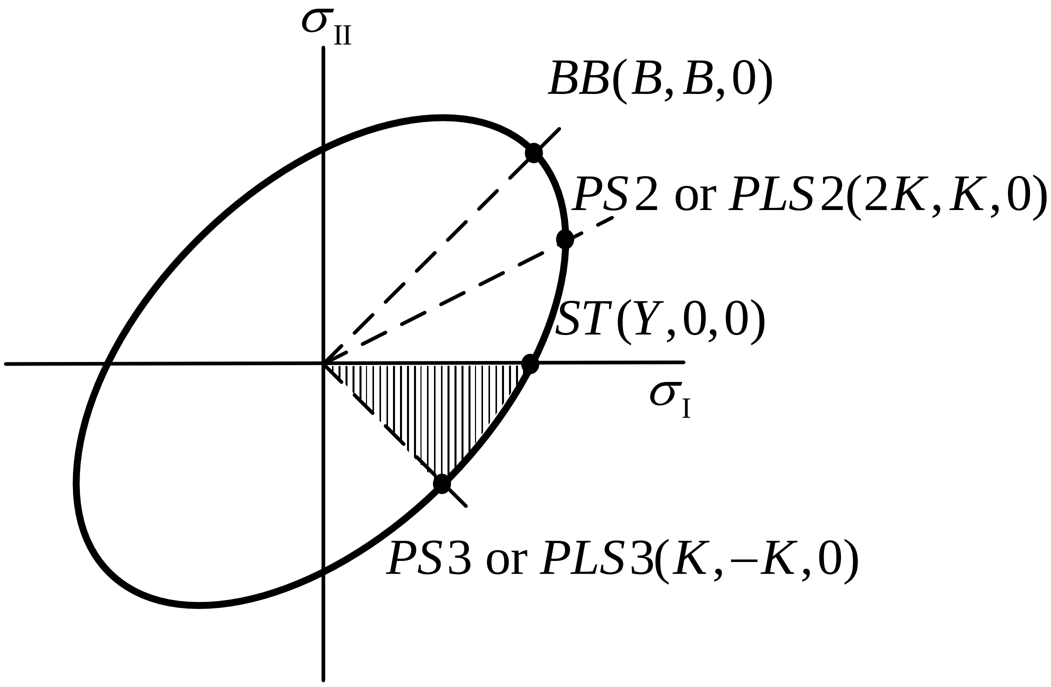
$$f(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{S}) = \sqrt{\alpha S_{ij}S_{ij}} = c$$

Where $\bar{\sigma}$: equivalent(or effective) stress

Reference stress state

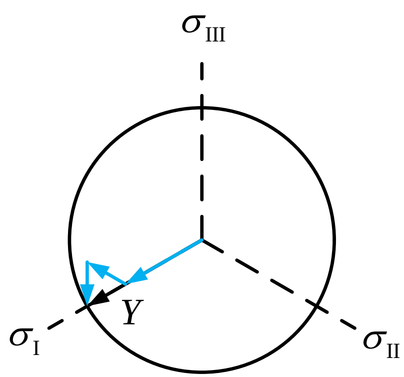
How can we define the size of yield function c?
→ From reference state

$$f(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{S}) = \sqrt{\alpha S_{ij} S_{ij}} = c$$



Reference stress state

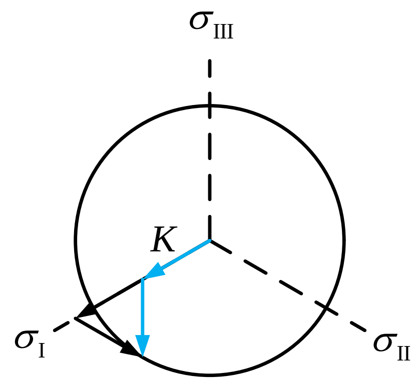
Simple tension



$S:(\frac{2Y}{3},-\frac{Y}{3},-\frac{Y}{3})$

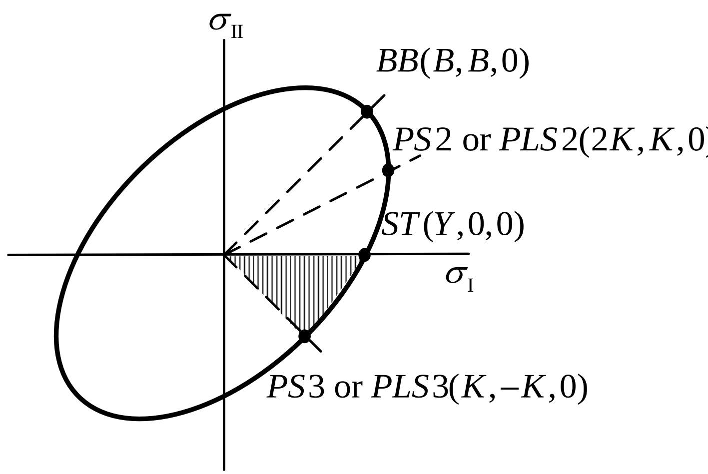
$\sigma:(Y,0,0)$

Pure shear(PS2)

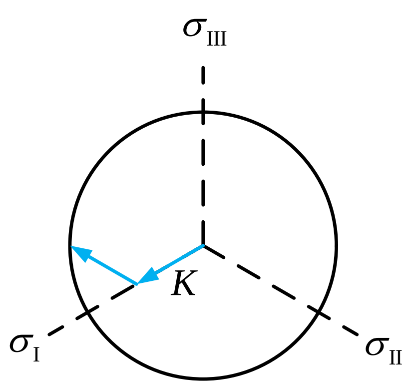


$S:(K,0,-K)$

$\sigma:(2K,K,0)$



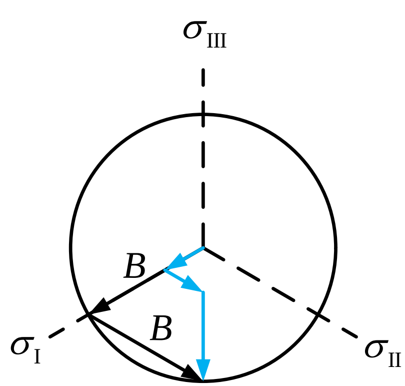
Pure shear(PS3)



$S:(K,-K,0)$

$\sigma:(K,-K,0)$

Balanced biaxial



$S:(\frac{B}{3},\frac{B}{3},-\frac{2B}{3})$

$\sigma:(B,B,0)$

Reference stress state

Reference state : simple tension case

$$f(\boldsymbol{\sigma}) = \sqrt{\alpha S_{ij} S_{ij}} = \bar{\sigma}$$

For the reference state to calibrate the yield criterion is the simple tension,

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\delta_{kk}\sigma_{ij}$$

$$\Leftrightarrow \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2Y/3 & 0 & 0 \\ 0 & -Y/3 & 0 \\ 0 & 0 & -Y/3 \end{pmatrix} + \begin{pmatrix} Y/3 & 0 & 0 \\ 0 & Y/3 & 0 \\ 0 & 0 & Y/3 \end{pmatrix}$$

$f = \bar{\sigma} = Y$ (yield stress of the simple tension)

$$f = \sqrt{\alpha\left(\frac{4}{9}Y^2 + \frac{1}{9}Y^2 + \frac{1}{9}Y^2\right)} = Y$$

$$\therefore \alpha = \frac{3}{2}$$

$$\boxed{\sqrt{\frac{3}{2} S_{ij} S_{ij}}\ (= Y) = \bar{\sigma}}$$

Reference stress state

Reference state : pure shear case

$$f(\boldsymbol{\sigma}) = \sqrt{\alpha S_{ij} S_{ij}} = \bar{\sigma}$$

For the reference state to calibrate the yield criterion is the pure shear,

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\delta_{kk}\sigma_{ij}$$

$$\Leftrightarrow \begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$f = \bar{\sigma} = K$ (yield stress of the pure shear)

$$f = \sqrt{\alpha\left(K^2 + K^2\right)} = K$$

$$\therefore \alpha = \frac{1}{2}$$

$$\boxed{\sqrt{\frac{1}{2} S_{ij} S_{ij}} \left(= K = \frac{1}{\sqrt{3}} Y \right) = \bar{\sigma}}$$

Reference stress state

Reference state : balanced biaxial case

$$f(\boldsymbol{\sigma}) = \sqrt{\alpha S_{ij} S_{ij}} = \bar{\sigma}$$

For the reference state to calibrate the yield criterion is the balanced biaxial,

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\delta_{kk}\sigma_{ij}$$

$$\Leftrightarrow \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B/3 & 0 & 0 \\ 0 & B/3 & 0 \\ 0 & 0 & -2B/3 \end{pmatrix} + \begin{pmatrix} 2B/3 & 0 & 0 \\ 0 & 2B/3 & 0 \\ 0 & 0 & 2B/3 \end{pmatrix}$$

$f = \bar{\sigma} = B$ (yield stress of balanced biaxial tension)

$$f = \sqrt{\alpha\left(\frac{1}{9}B^2 + \frac{1}{9}B^2 + \frac{4}{9}B^2\right)} = B$$

$$\therefore \alpha = \frac{3}{2}$$

$$\sqrt{\frac{3}{2} S_{ij} S_{ij}} \left(= B = Y = \sqrt{3}K\right) = \bar{\sigma}$$

Von Mises yield criterion

In summary,

$$\sqrt{\frac{1}{2}\left[\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{22}-\sigma_{33}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2+6\sigma_{12}{}^2+6\sigma_{23}{}^2+6\sigma_{31}{}^2\right]}\left(=Y\right)=\bar{\sigma}$$

$$f = f\ (\sigma_{ij}) = f\left(S_{ij}\right)$$

$$f = \sqrt{\frac{3}{2}\ S_{ij}S_{ij}} = \bar{\sigma}$$

$$\Leftrightarrow\ f = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \sqrt{\frac{3}{2}(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij})(\sigma_{ij}-\frac{1}{3}\sigma_{pp}\delta_{ij})} = \bar{\sigma}$$

$$\Leftrightarrow \sqrt{\frac{3}{2}\left(\sigma_{ij}\sigma_{ij}-\frac{1}{3}\sigma_{kk}\sigma_{pp}\right)} = \bar{\sigma}$$

$$\Leftrightarrow \sqrt{\frac{3}{2}\left(\sigma_{11}{}^2+\sigma_{22}{}^2+\sigma_{33}{}^2+2\sigma_{12}{}^2+2\sigma_{23}{}^2+2\sigma_{31}{}^2-1/3\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)^2\right)} = \bar{\sigma}$$

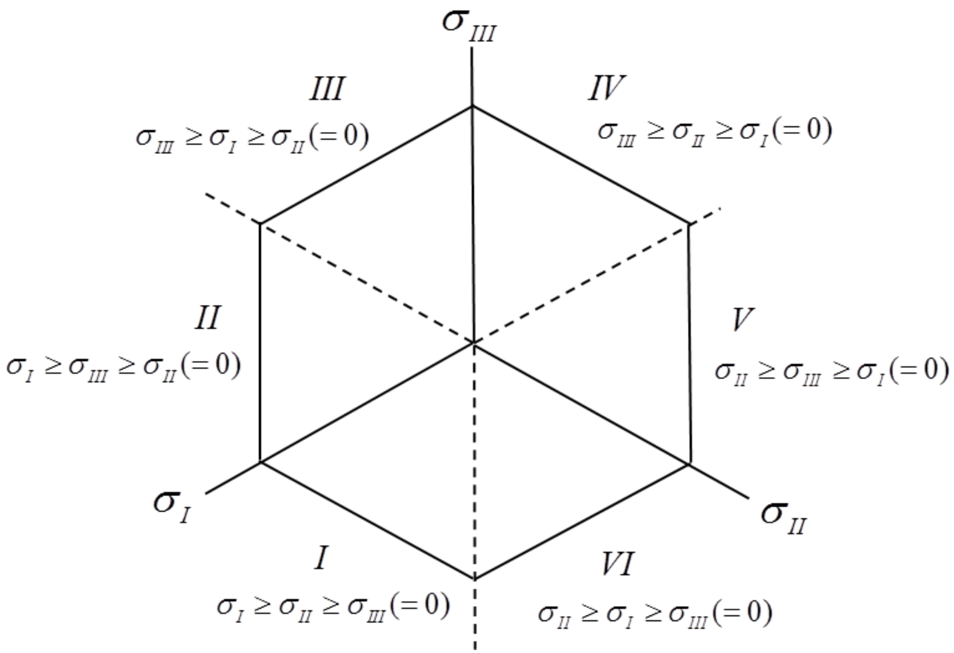
$$\therefore \sqrt{\frac{1}{2}\ \left\{\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{22}-\sigma_{33}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2+6\sigma_{12}{}^2+6\sigma_{23}{}^2+6\sigma_{31}{}^2\right\}} = \bar{\sigma}$$

Tresca isotropic yield function

(or maximum shear criterion)

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = \bar{\sigma}(S_I, S_{II}, S_{III}) = \frac{S_{\max} - S_{\min}}{2} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = K$$

$$or \quad \frac{1}{16}\mathrm{J}_2^3 - \frac{27}{64}\mathrm{J}_3^2 - \frac{9}{16} K^2 J_2^2 + \frac{3}{2} K^4 J_2 - K^6 = 0$$

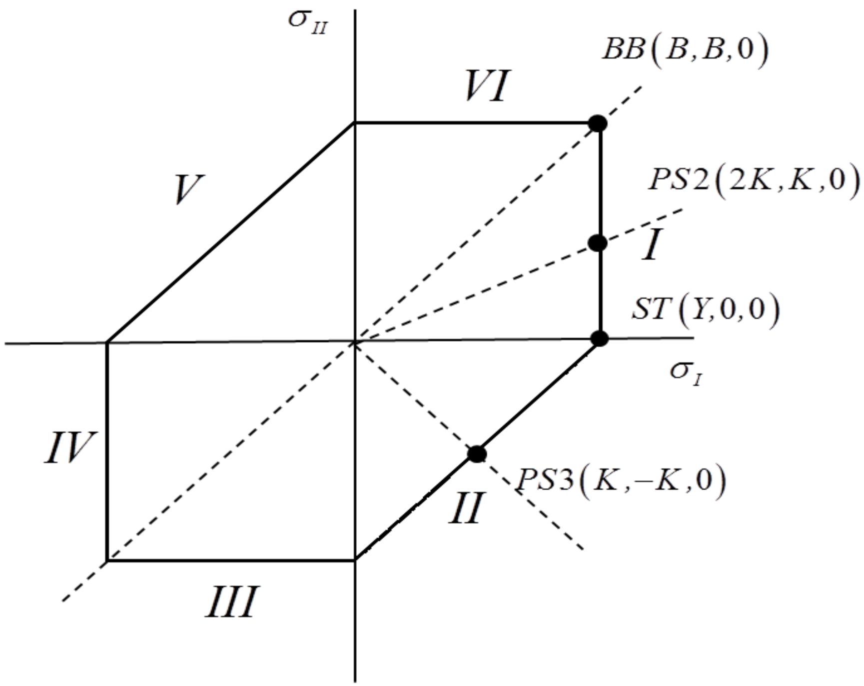


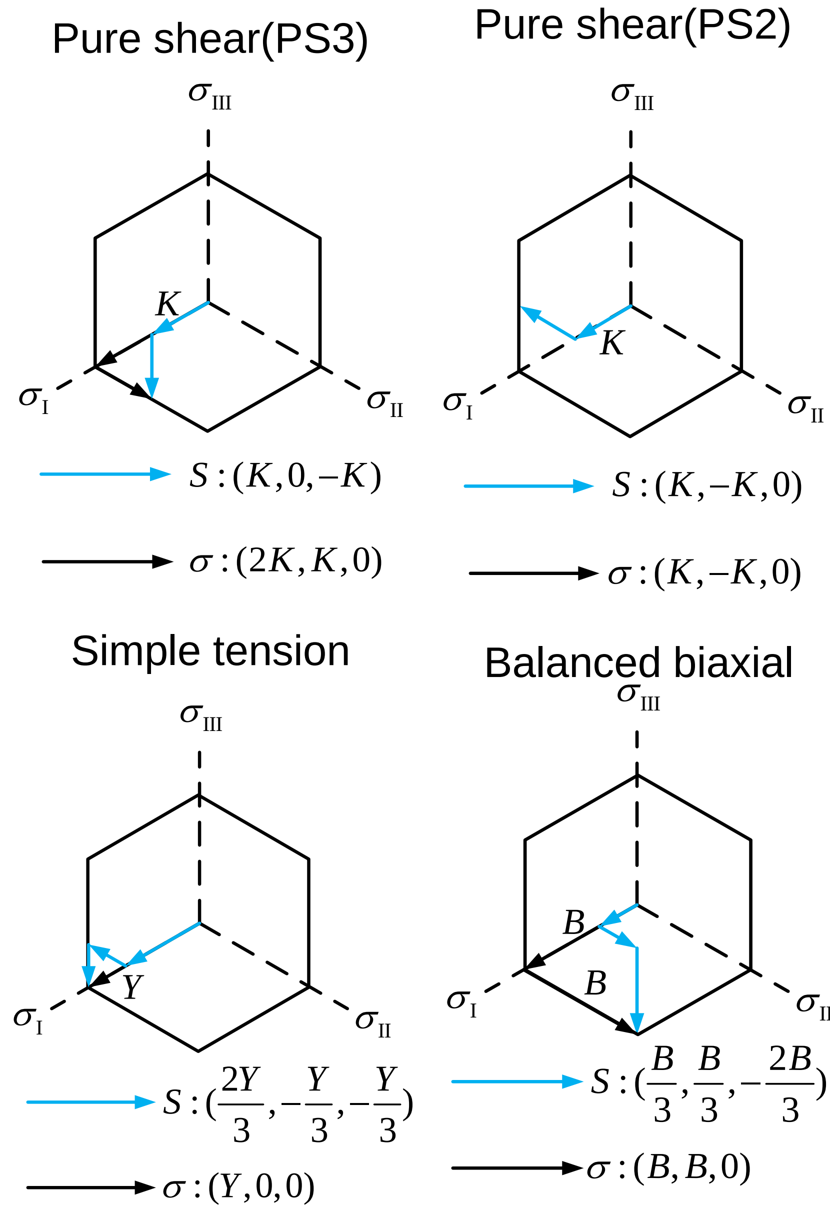
$$\begin{cases} S_I - S_{II} = \sigma_I - \sigma_{II} = \pm 2K \\ S_{II} - S_{III} = \sigma_{II} - \sigma_{III} = \pm 2K \\ S_I - S_{III} = \sigma_I - \sigma_{III} = \pm 2K \end{cases}$$

Tresca yield function

Different reference state cases

$$f = A\frac{S_{\max} - S_{\min}}{2} = A\frac{\sigma_{\max} - \sigma_{\min}}{2} = \bar{\sigma}$$





Tresca yield function

Reference state : pure shear case

$$f = A\frac{S_{\max} - S_{\min}}{2} = A\frac{\sigma_{\max} - \sigma_{\min}}{2} = \bar{\sigma}$$

For the reference state to calibrate the yield criterion is the pure shear,

$$f = \left(A \times \frac{\sigma_{\max} - \sigma_{\min}}{2} \right) = \bar{\sigma}.$$

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} K & 0 & 0 \\ 0 & -K & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{\sigma} = K \text{ (yield stress of the pure shear)}$$

$$f = \left(A \times \frac{K - (-K)}{2} \right) = K$$

$$\therefore A = 1$$

$$\frac{\sigma_{\max} - \sigma_{\min}}{2} \left(= K = \frac{Y}{2} \right) = \bar{\sigma}$$

Tresca yield function

Reference state : simple tension case

$$f = A\frac{S_{\max} - S_{\min}}{2} = A\frac{\sigma_{\max} - \sigma_{\min}}{2} = \bar{\sigma}$$

For the reference state to calibrate the yield criterion is the simple tension,

$$\boldsymbol{\sigma} = \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f = \left(A \times \frac{Y - 0}{2} \right) = Y$$

$$\therefore A = 2$$

$$\boxed{\left(\sigma_{\max} - \sigma_{\min}\right)(= Y) = \bar{\sigma}}$$

Tresca yield function

Reference state : balanced biaxial

$$f = A\frac{S_{\max} - S_{\min}}{2} = A\frac{\sigma_{\max} - \sigma_{\min}}{2} = \bar{\sigma}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad f = \left(A \times \frac{\sigma_{\max} - \sigma_{\min}}{2} \right) = \bar{\sigma} = B$$

$$f = \left(A \times \frac{B - 0}{2} \right) = B$$

$$\therefore A = 2$$

$$\left(\sigma_{\max} - \sigma_{\min} \right) = B = \bar{\sigma}$$

Drucker isotropic yield function

- Incompressible, isotropic, and symmetric for tension and compression,

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = f(J_2,\ J_3) = \bar{\sigma}^6(J_2,\ J_3) = J_2^3(1-\xi\frac{J_3^2}{J_2^3}) = K^6$$

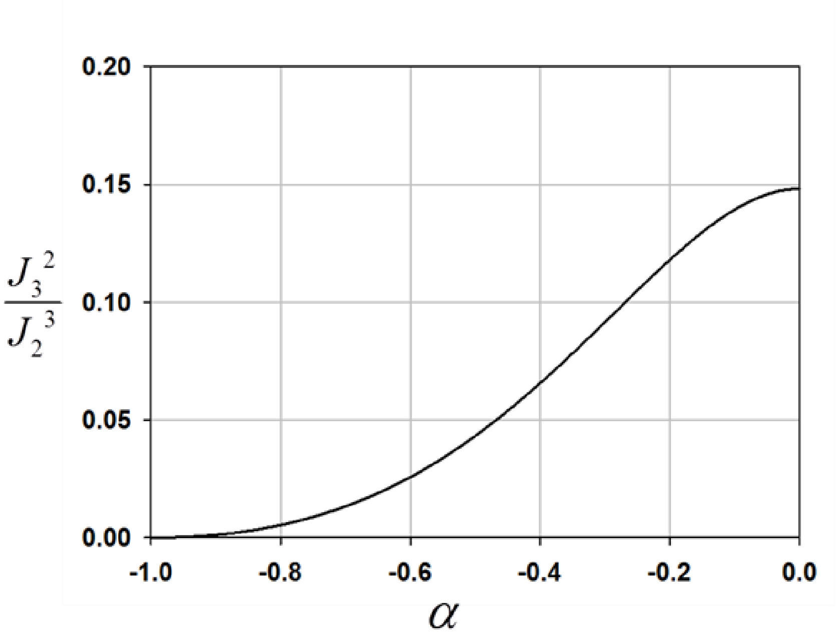
- If ξ=0 → von Mises yield criterion
- So, Drucker yield criterion is extension of von Mises yield criterion, which modifies the shape of yield surface from the von Mises yield surface by material constant ξ

Drucker isotropic yield function

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = f(J_2,\ J_3) = \bar{\sigma}^6(J_2,\ J_3) = J_2^3\left(1-\xi\frac{J_3^2}{J_2^3}\right) = K^6$$

- Let $(\sigma_1, \sigma_2, 0) \to \sigma_1(1, \alpha, 0)$
  where $\alpha = \sigma_2/\sigma_1$ $(-1 \le \alpha \le 0)$

- J_3^2/J_2^2 from $\alpha = -1$(pure shear) to $\alpha = 0$(simple tension)



$$\begin{cases} J_2 = \dfrac{\sigma_I^2}{3}\left(1-\alpha+\alpha^2\right) \\ J_3 = \dfrac{\sigma_I^3}{27}(\alpha-2)(2\alpha-1)(1+\alpha) \\ \dfrac{J_3^2}{J_2^3} = \dfrac{\left((\alpha-2)(2\alpha-1)(1+\alpha)\right)^2}{27\left(1-\alpha+\alpha^2\right)^3} \end{cases}$$

Drucker isotropic yield function

- Principal stress σ_I can be derived as follows

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = f(J_2,\ J_3) = \bar{\sigma}^6(J_2,\ J_3) = J_2^3(1-\xi\frac{J_3^2}{J_2^3}) = K^6$$

$$\begin{cases} J_2 = \dfrac{\sigma_I^2}{3}\left(1-\alpha+\alpha^2\right) \\ J_3 = \dfrac{\sigma_I^3}{27}\left(\alpha-2\right)\left(2\alpha-1\right)\left(1+\alpha\right) \\ \dfrac{J_3^2}{J_2^3} = \dfrac{\left(\left(\alpha-2\right)\left(2\alpha-1\right)\left(1+\alpha\right)\right)^2}{27\left(1-\alpha+\alpha^2\right)^3} \end{cases}$$

$$\sigma_I(\alpha,\xi) = \sqrt{3}K\left(\left(1-\alpha+\alpha^2\right)^3\left(1-\xi\frac{\left(\left(\alpha-2\right)\left(2\alpha-1\right)\left(1+\alpha\right)\right)^2}{27\left(1-\alpha+\alpha^2\right)^3}\right)\right)^{-\frac{1}{6}}$$

- Since $\sigma_I(\alpha=0,\xi) = Y(=B)$ and $\sigma_I(\alpha=-1,\xi) = K$,

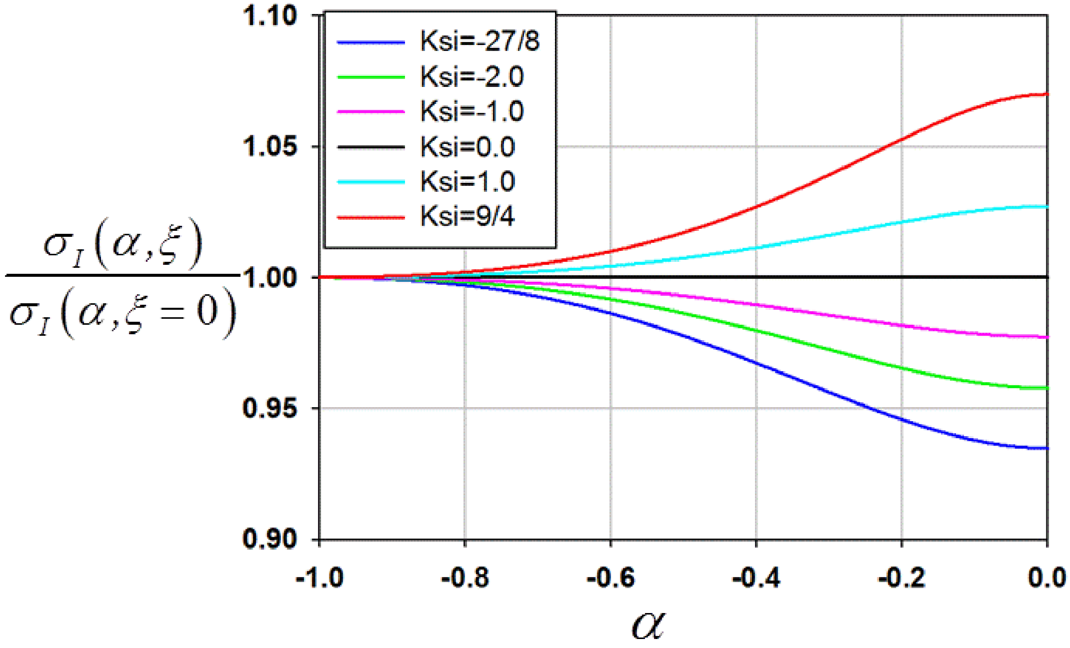
$$K^6 = \frac{\left(27-4\xi\right)}{27^2}Y^6 = \frac{\left(27-4\xi\right)}{27^2}B^6$$

Drucker isotropic yield function

- By normalizing $\sigma_I(\alpha,\xi)$ with $\sigma_I(\alpha,\xi=0)$ which is von Mises yield function case,

$$\frac{\sigma_I(\alpha,\xi)}{\sigma_I(\alpha,\xi=0)} = \left(1-\xi\frac{\left((\alpha-2)(2\alpha-1)(1+\alpha)\right)^2}{27\left(1-\alpha+\alpha^2\right)^3}\right)^{-\frac{1}{6}} = \left(1-\xi\frac{J_3^2}{J_2^3}\right)^{-\frac{1}{6}}$$

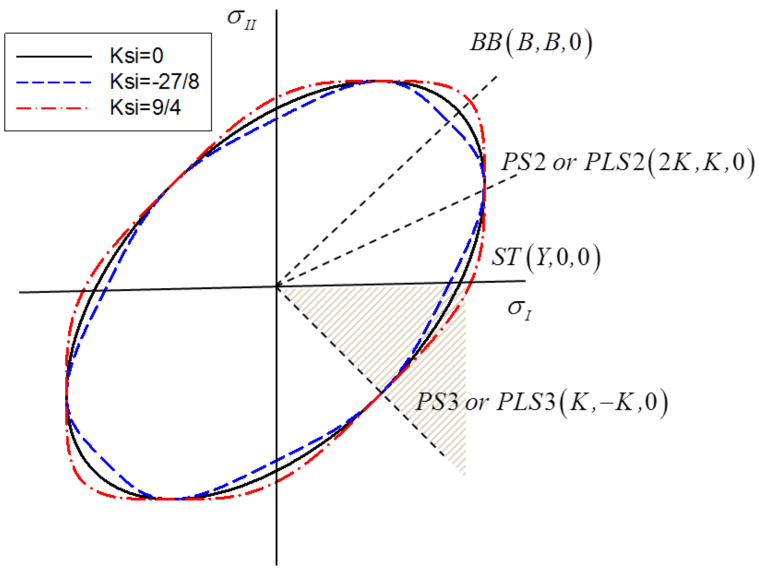
- From convexity condition, $-\frac{27}{8} \le \xi \le \frac{9}{4}$



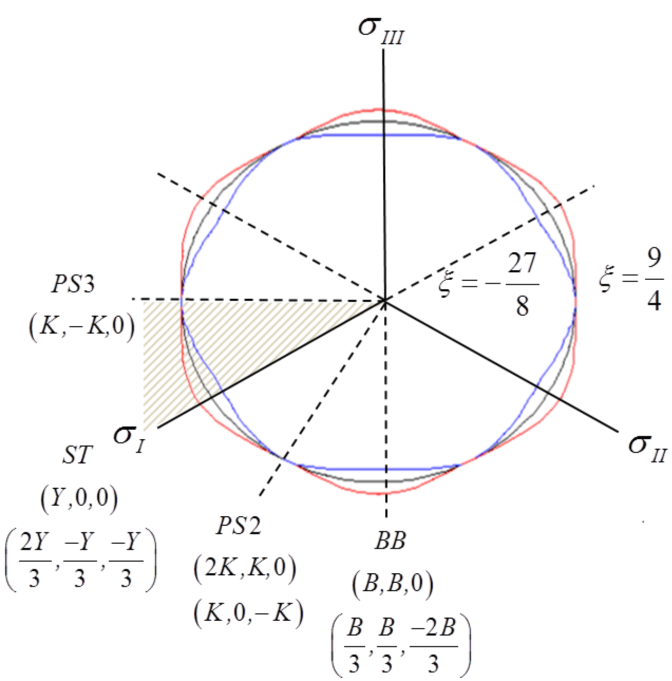
Drucker isotropic yield function

- From previously driven equations, Drucker isotropic yield function can be plotted

In plane stress condition



In π diagram



Non-quadratic isotropic yield function generalized from von Mises yield function

Von Mises yield function

$$\bar{\sigma} = \left\{ \alpha \left(\left| S_{\mathrm{I}} - S_{\mathrm{II}} \right|^2 + \left| S_{\mathrm{II}} - S_{\mathrm{III}} \right|^2 + \left| S_{\mathrm{III}} - S_{\mathrm{I}} \right|^2 \right) \right\}^{\frac{1}{2}}$$

generalization

Hosford non-quadratic isotropic yield function

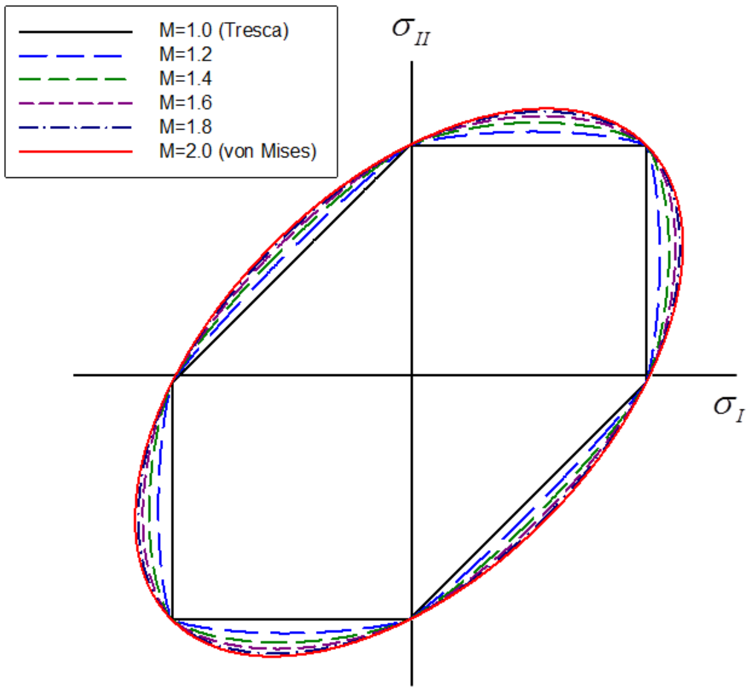
$$\bar{\sigma} = \left\{ \alpha \left(\left| S_{\mathrm{I}} - S_{\mathrm{II}} \right|^M + \left| S_{\mathrm{II}} - S_{\mathrm{III}} \right|^M + \left| S_{\mathrm{III}} - S_{\mathrm{I}} \right|^M \right) \right\}^{\frac{1}{M}}$$

Non-quadratic isotropic yield function generalized from von Mises yield function

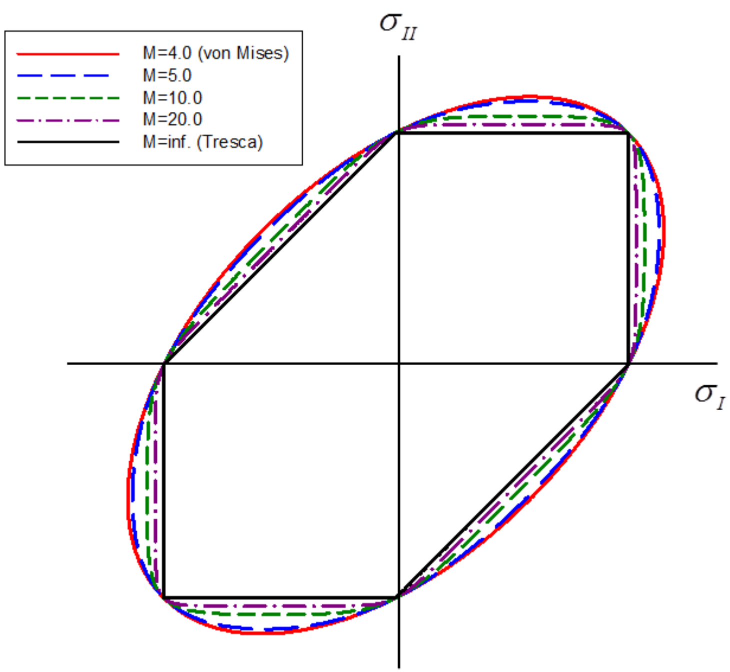
Hosford non-quadratic isotropic yield function in plane stress condition

$$\bar{\sigma} = \left\{ \alpha \left(\left| S_{\mathrm{I}} - S_{\mathrm{II}} \right|^{M} + \left| S_{\mathrm{II}} - S_{\mathrm{III}} \right|^{M} + \left| S_{\mathrm{III}} - S_{\mathrm{I}} \right|^{M} \right) \right\}^{\frac{1}{M}}$$

Case 1 : 1 ≤ M ≤ 2



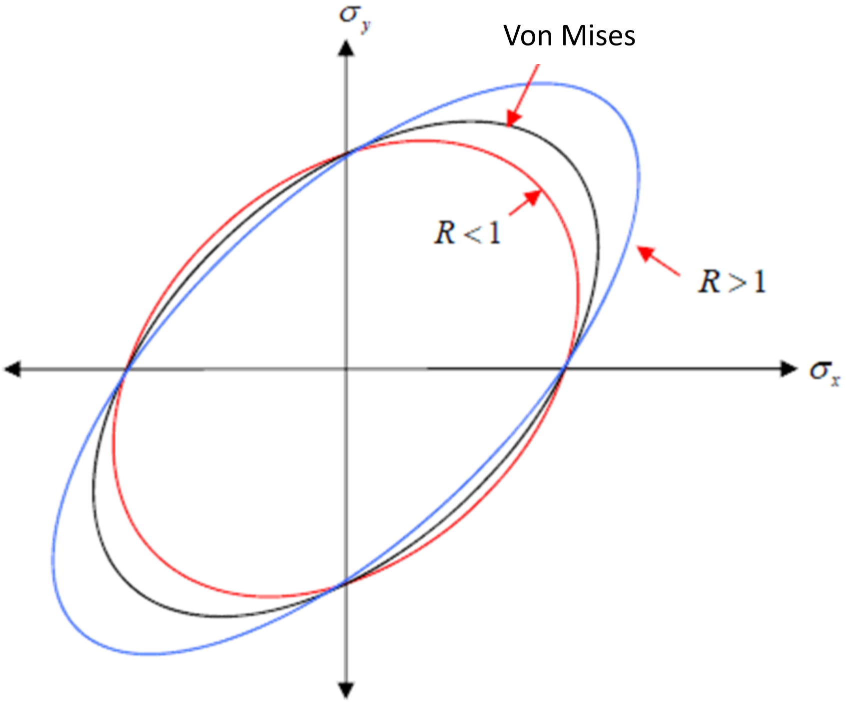
Case 2 : 2 ≤ M ≤ ∞



Hill 1948 quadratic anisotropic yield function

- Anisotropic expansion of von Mises yield criterion
- This will be further discussed in Chapter 14

$$f(\boldsymbol{\sigma}) = \bar{\sigma}^2 = \mathrm{F}(\sigma_{yy} - \sigma_{zz})^2 + \mathrm{G}(\sigma_{zz} - \sigma_{xx})^2 + \mathrm{H}(\sigma_{xx} - \sigma_{yy})^2$$
$$+ 2\mathrm{L}\sigma_{yz}{}^2 + 2\mathrm{M}\sigma_{zx}{}^2 + 2\mathrm{N}\sigma_{xy}{}^2$$

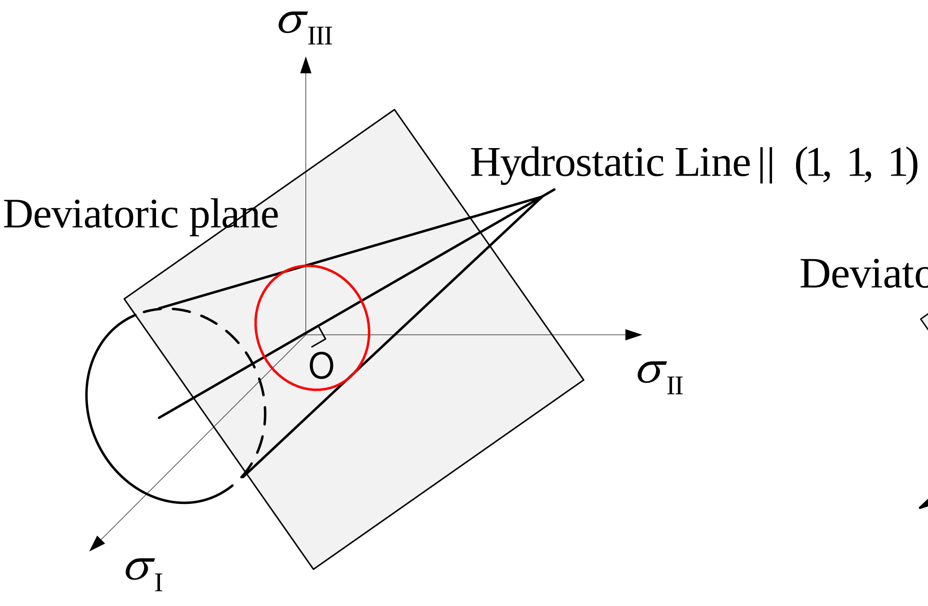


Drucker-Prager compressible isotropic yield function

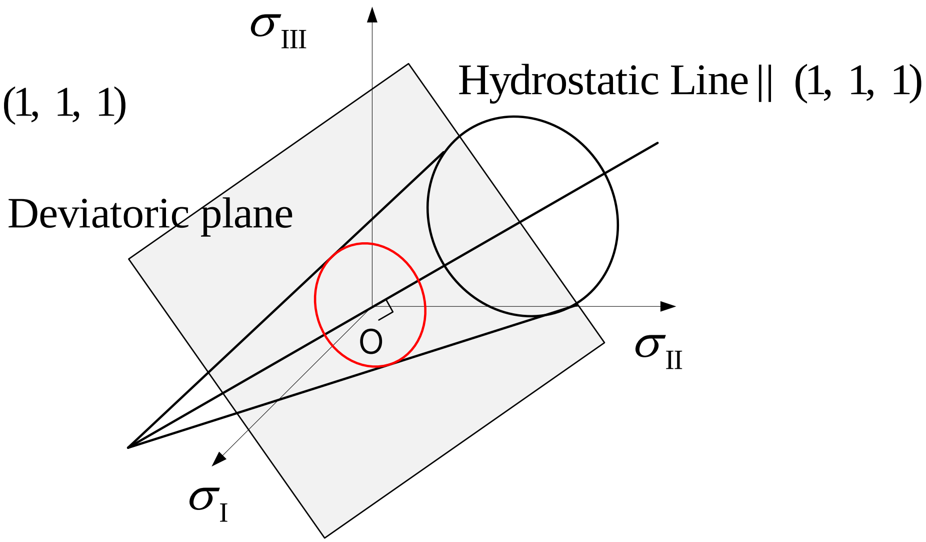
- Drucker-Prager yield function can be utilized for Soil, Concrete, or other hydrostatic stress dependent materials

$$\bar{\sigma}(I_1, I_2, I_3) = \bar{\sigma}(I_1, J_2, I_3) = \sqrt{2\alpha J_2} + \beta I_1$$

<u>Case 1 : β>0</u>



<u>Case 2 : β<0</u>

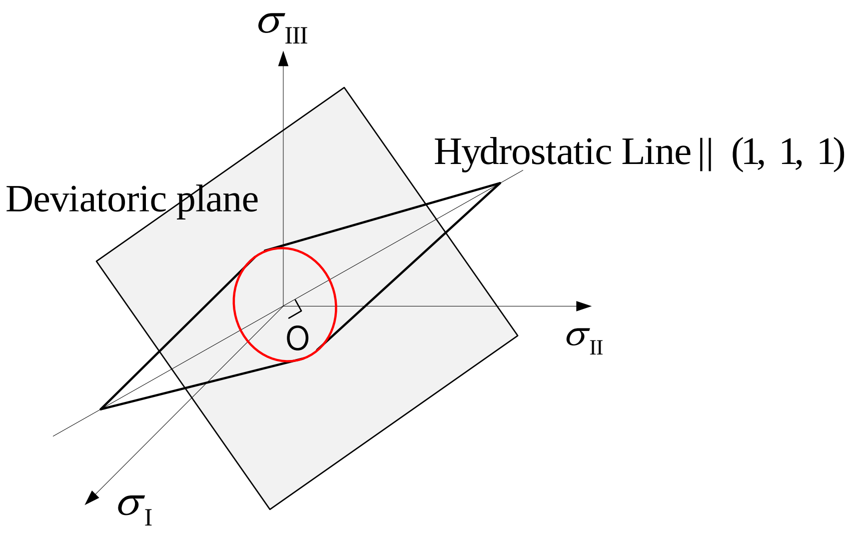


Drucker-Prager compressible isotropic yield function

- Variation of Drucker-Prager yield criterion

<u>Double cone shape</u>

$$\bar{\sigma} = \sqrt{2\alpha J_2} + \beta \left| I_1 \right|$$



<u>Ellipsoid shape</u>

$$\bar{\sigma} = \sqrt{2\alpha J_2 + \beta I_1^2}$$