445.204

Introduction to Mechanics of Materials

(재료역학개론)

Chapter 10: Deflection of beam

(Ch. 12 in Shames)

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Contents

- Differential equations for deflection of symmetric beams
- Statically indeterminate beams
- Superposition methods
- Energy method

Differential equations

- Recall pure bending

$$R = \frac{EI_{zz}}{M_z}$$

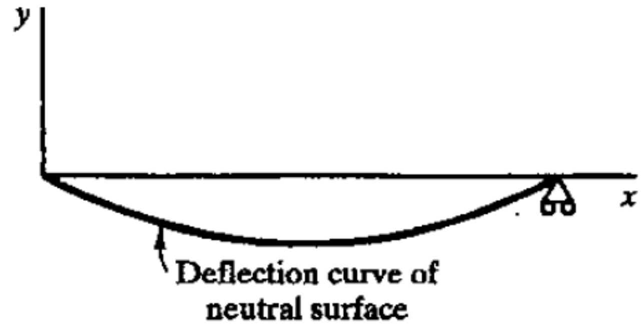


Figure 12.1. Deflection curve of neutral surface.

- For general loading, the above equation gives local radius of curvature of the neutral surface
- Normal stress (or normal strain) results in deflection of the beam

- Radius of curvature as a function of deflection, from analytic geometry

Simplification/small deformation

$$\frac{1}{R} = \kappa = \frac{d^2v / dx^2}{\left[1 + \left(dv / dx\right)^2\right]^{3/2}} \quad \Rightarrow \quad \frac{1}{R} = \frac{d^2v}{dx^2}$$

Differential equations

- Basic differential equations

$$R = \frac{EI_{zz}}{M_z} \quad \& \quad \frac{1}{R} = \frac{d^2v}{dx^2} \quad \Rightarrow \quad \frac{d^2v}{dx^2} = \frac{M_z}{EI_{zz}} \quad \text{or} \quad \boxed{EI_{zz}\frac{d^2v}{dx^2} = M_z}$$

- From $\frac{dM_z}{dx} = V_y$ and $\frac{dV_y}{dx} = -w_y$

$$\frac{d^2M_z}{dx^2} = -w_y$$

- Therefore,

$$\boxed{\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = \frac{d^2M_z}{dx^2} = -w_y} \quad \text{or} \quad \boxed{EI_{zz}\frac{d^4v}{dx^4} = -w_y}$$

Differential equations: example 12.1

A cantilever beam is uniformly loaded over its span as shown in Fig. 12.2. Find the deflection curve $v(x)$.

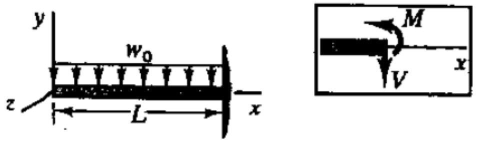


Figure 12.2. Cantilever beam.

The bending moment $M(x)$ is clearly seen by inspection to be $-w_0x^2/2$. Hence, for the region $0 \le x < L$ we can say,[2] using Eq. (12.4),

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left(-\frac{w_0x^2}{2}\right) \tag{a}$$

Integrating twice, we get

$$\frac{dv}{dx} = \frac{1}{EI}\left(-\frac{w_0x^3}{6} + C_1\right) \tag{b}$$

$$v = \frac{1}{EI}\left(-\frac{w_0x^4}{24} + C_1x + C_2\right) \tag{c}$$

where C_1 and C_2 are constants of integration. When $x = L$ we have $v = dv/dx = 0$. Accordingly, we may determine the constants of integration as follows:

$$C_1 = \frac{w_0L^3}{6}$$

$$C_2 = \frac{w_0L^4}{24} - \frac{w_0L^4}{6} = -\frac{1}{8}w_0L^4$$

We thus have as a final result

$$v = \frac{1}{EI}\left(-\frac{w_0x^4}{24} + \frac{w_0L^3}{6}x - \frac{1}{8}w_0L^4\right) \tag{d}$$

$$\boxed{\frac{d^2v}{dx^2} = \frac{M_z}{EI_{zz}}}$$

- Boundary conditions

$$v = \frac{dv}{dx} = 0 \quad at \quad x = L$$

Differential equations: example 12.2

$$\int P\left(x-a\right)^{n} dx = \frac{P\left(x-a\right)^{n+1}}{n+1} + C_{n}$$

Shown in Fig. 12.3 is a simply supported beam with a variety of loadings. We will determine the deflection curve.

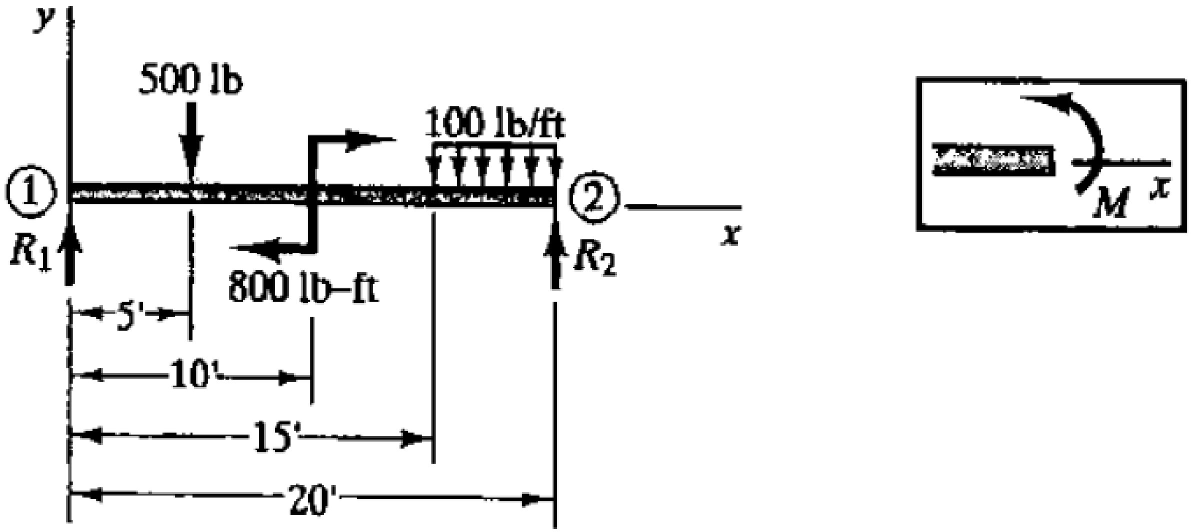


Figure 12.3. Simply supported beam.

Differential equations: example 12.2

As a first step, we compute the supporting forces by rigid-body mechanics.

$\underline{\sum M_2 = 0}$:

$$-R_1(20) + (500)(15) - 800 + (500)(2.5) = 0$$
$$\therefore R_1 = 397.5 \text{ lb}$$

$\underline{\sum M_1 = 0}$:

$$20R_2 - (500)(5) - 800 - (500)(17.5) = 0$$
$$\therefore R_2 = 602.5 \text{ lb}$$

As a check, we can sum forces in the vertical direction. Thus

$$\sum F_y = 0 = 397.5 + 602.5 - 500 - (5)(100)$$
$$0 = 0$$

We now consider a series of domains.

$\underline{0 \le x \le 5}$:

$$M = 397.5x$$
$$\therefore \frac{d^2v}{dx^2} = \frac{1}{EI}(397.5x)$$
$$\frac{dv}{dx} = \frac{1}{EI}\left(397.5\frac{x^2}{2} + C_1\right) \quad \text{(a)}$$
$$v = \frac{1}{EI}\left(397.5\frac{x^3}{6} + C_1x + C_2\right) \quad \text{(b)}$$

$\underline{5 \le x < 10}$:

$$M = 397.5x - 500(x-5)$$
$$\therefore \frac{d^2v}{dx^2} = \frac{1}{EI}[397.5x - 500(x-5)]$$

Integrating and using Eq. (12.9), we get

$$\frac{dv}{dx} = \frac{1}{EI}\left[397.5\frac{x^2}{2} - 500\frac{(x-5)^2}{2} + C_3\right] \quad \text{(c)}$$
$$v = \frac{1}{EI}\left[397.5\frac{x^3}{6} - 500\frac{(x-5)^3}{6} + C_3x + C_4\right] \quad \text{(d)}$$

$\underline{10 < x \le 15}$:

$$M = 397.5x - 500(x-5) + 800$$
$$\therefore \frac{d^2v}{dx^2} = \frac{1}{EI}[397.5x - 500(x-5) + 800]$$
$$\frac{dv}{dx} = \frac{1}{EI}\left[397.5\frac{x^2}{2} - 500\frac{(x-5)^2}{2} + 800x + C_5\right] \quad \text{(e)}$$
$$v = \frac{1}{EI}\left[397.5\frac{x^3}{6} - 500\frac{(x-5)^3}{6} + 800\frac{x^2}{2} + C_5x + C_6\right] \quad \text{(f)}$$

$\underline{15 \le x \le 20}$:

$$M = 397.5x - 500(x-5) + 800 - 100\frac{(x-15)^2}{2}$$
$$\therefore \frac{d^2v}{dx^2} = \frac{1}{EI}\left[397.5x - 500(x-5) + 800 - 100\frac{(x-15)^2}{2}\right]$$
$$\frac{dv}{dx} = \frac{1}{EI}\left[397.5\frac{x^2}{2} - 500\frac{(x-5)^2}{2} + 800x - 100\frac{(x-15)^3}{6} + C_7\right] \quad \text{(g)}$$
$$v = \frac{1}{EI}\left[397.5\frac{x^3}{6} - 500\frac{(x-5)^3}{6} + 800\frac{x^2}{2} - 100\frac{(x-15)^4}{24} + C_7x + C_8\right] \quad \text{(h)}$$

Differential equations: example 12.2

Boundary conditions

1. When $x = 0$, $v = 0$.

From Eq. (b), we can conclude that $C_2 = 0$. Also:

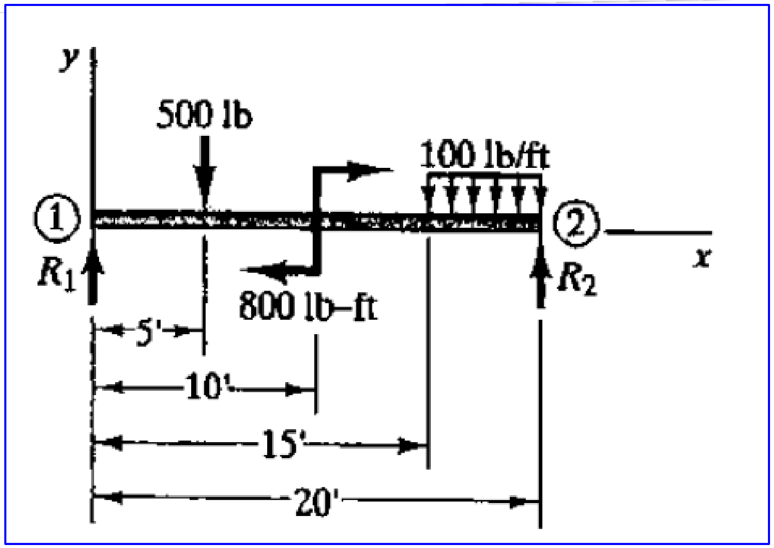
2. When $x = 20$, $v = 0$.

From Eq. (h), we get

$$0 = \frac{(397.5)(20^3)}{6} - \frac{(500)(15^3)}{6} + \frac{(800)(20^2)}{2} - \frac{(100)(5^4)}{24} + 20C_7 + C_8$$

$$\therefore 20C_7 + C_8 = -406{,}145 \tag{i}$$

We must next properly *patch* the equations between the domains. That is, the slope and the deflection at the end of one



domain must, respectively, equal the slope and deflection at the beginning of the next domain. Thus between the first and second domains we require **(compatibility)** that

$$\left[\frac{dv(5)}{dx}\right]_{\text{Eq.(a)}} = \left[\frac{dv(5)}{dx}\right]_{\text{Eq.(c)}}$$

$$\therefore \frac{1}{EI}\left(397.5\frac{5^2}{2} + C_1\right) = \frac{1}{EI}\left(397.5\frac{5^2}{2} + 0 + C_3\right)$$

[It is the appearance of the zero on the right-hand side that is the simplification resulting from the use of Eq. (12.9).] Hence,

$$C_1 = C_3 \tag{j}$$

Also,

$$[v(5)]_{\text{Eq.(b)}} = [v(5)]_{\text{Eq.(d)}}$$

$$\frac{1}{EI}\left(397.5\frac{5^3}{6} + 5C_1\right) = \frac{1}{EI}\left(397.5\frac{5^3}{6} - 0 + 5C_3 + C_4\right)$$

$$\therefore 5C_1 - 5C_3 - C_4 = 0 \tag{k}$$

For the next two domains, we have just to the left of $x = 10$ (i.e., $x = 10^-$) and just to the right of $x = 10$ (i.e., $x = 10^+$).

$$\left[\frac{dv(10^-)}{dx}\right]_{\text{Eq.(c)}} = \left[\frac{dv(10^+)}{dx}\right]_{\text{Eq.(e)}}$$

$$\frac{1}{EI}\left[397.5\frac{(10^2)}{2} - \frac{(500)(5^2)}{2} + C_3\right] = \frac{1}{EI}\left[397.5\frac{(10^2)}{2} - \frac{(500)(5^2)}{2} + (800)(10) + C_5\right]$$

$$\therefore C_3 - C_5 = 8000 \tag{l}$$

$$[v(10^-)]_{\text{Eq.(d)}} = [v(10^+)]_{\text{Eq.(f)}}$$

$$\frac{1}{EI}\left[397.5\frac{(10^3)}{6} - 500\frac{(5^3)}{6} + 10C_3 + C_4\right] = \frac{1}{EI}\left[397.5\frac{(10^3)}{6} - 500\frac{(5^3)}{6} + 800\frac{(10^2)}{2} + 10C_5 + C_6\right]$$

$$\therefore 10C_3 + C_4 - 10C_5 - C_6 = 40{,}000 \tag{m}$$

Differential equations: example 12.2

Finally, we have

$$\left[\frac{dv(15)}{dx}\right]_{\text{Eq.(e)}} = \left[\frac{dv(15)}{dx}\right]_{\text{Eq.(g)}}$$

$$\frac{1}{EI}\left[397.5\frac{(15^2)}{2} - 500\frac{(10^2)}{2} + (800)(15) + C_5\right] =$$

$$\frac{1}{EI}\left[397.5\frac{(15^2)}{2} - 500\frac{(10^2)}{2} + (800)(15) + 0 + C_7\right.$$

$$\therefore C_5 = C_7 \qquad \text{(n)}$$

$$[v(15)]_{\text{Eq.(f)}} = [v(15)]_{\text{Eq.(h)}}$$

$$\frac{1}{EI}\left[397.5\frac{(15^3)}{6} - 500\frac{(10^3)}{6} + 800\frac{(15^2)}{2} + 15C_5 + C_6\right] =$$

$$\frac{1}{EI}\left[397.5\frac{(15^3)}{6} - 500\frac{(10^3)}{6} + 800\frac{(15^2)}{2} + 0 + 15C_7 + C_8\right]$$

$$\therefore 15C_5 + C_6 - 15C_7 - C_8 = 0 \qquad \text{(o)}$$

In forming the patching equations you need not write everything down as we have done. By inspection you can arrive directly at the proper equations such as Eq. (o), since many of the terms are either zero or cancel, as can be easily observed from the domain equations. We now rewrite the equations for the constants.

$$C_2 = 0$$

$$20C_7 + C_8 = -406{,}145 \qquad \text{(i)}$$

$$C_1 = C_3 \qquad \text{(j)}$$

$$5C_1 - 5C_3 - C_4 = 0 \qquad \text{(k)}$$

$$C_3 - C_5 = 8000 \qquad \text{(l)}$$

$$10C_3 + C_4 - 10C_5 - C_6 = 40{,}000 \qquad \text{(m)}$$

$$C_5 = C_7 \qquad \text{(n)}$$

$$15C_5 + C_6 - 15C_7 - C_8 = 0 \qquad \text{(o)}$$

The equations are rather simple to handle. By substituting for C_3 in Eq. (k), using Eq. (j), we obtain

$$5C_1 - 5C_1 - C_4 = 0$$

$$\therefore C_4 = 0 \qquad \text{(p)}$$

Next going to Eq. (m), we get

$$10(C_3 - C_5) - C_6 = 40{,}000 \qquad \text{(q)}$$

But $(C_3 - C_5) = 8000$, from Eq. (l). Hence, we may solve for C_6.

$$C_6 = 40{,}000 \qquad \text{(r)}$$

Now we go to Eq. (o). Replacing C_5, using Eq. (n), we get

$$15C_7 + C_6 - 15C_7 - C_8 = 0$$

$$\therefore C_8 = C_6 = 40{,}000 \qquad \text{(s)}$$

Going to Eq. (i), we may determine C_7:

$$20C_7 + 40{,}000 = -406{,}145$$

$$\therefore C_7 = -22{,}307 \qquad \text{(t)}$$

From Eq. (n),

$$C_5 = -22{,}307 \qquad \text{(u)}$$

From Eq. (l),

$$C_3 = 8000 - 22{,}307 = -14{,}307$$

From Eq. (j),

$$C_1 = -14{,}307 \qquad \text{(v)}$$

The example is thus complete.

Differential equations: example 12.4

Find the maximum deflection of the pin-connected beams shown in Fig. 12.5. The weight of the beam has been included in the 180 N/m uniform loading. Take $E = 2 \times 10^{11}$ Pa.

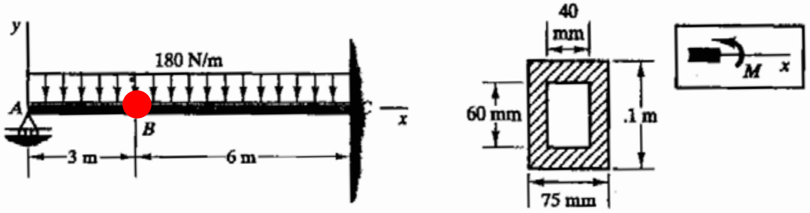


Figure 12.5. Pin-connected beams.

It would at first seem that we have a statically indeterminate support system here, but this is not the case. We can take $\overline{AB}$ as free body with the bending moment zero at the pin at B and solve for the supporting force at A. Thus observing Fig. 12.6, we can say:

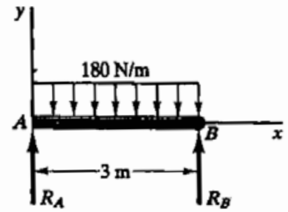


Figure 12.6. Free body of AB.

$\Sigma M_B = 0$:

$$-R_A(3) + (180)(3)\left(\frac{3}{2}\right) = 0$$

$$\therefore R_A = 270 \text{ N}$$

We can now proceed with the deflection curve analysis.

$0 \le x < 3$:

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left[270(x) - 180\left(\frac{x^2}{2}\right)\right]$$

$$\frac{dv}{dx} = \frac{1}{EI}\left[270\left(\frac{x^2}{2}\right) - 180\left(\frac{x^3}{6}\right) + C_1\right] \qquad \text{(a)}$$

$$v = \frac{1}{EI}\left[270\left(\frac{x^3}{6}\right) - 180\left(\frac{x^4}{24}\right) + C_1x + C_2\right] \qquad \text{(b)}$$

$3 < x < 9$:

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left[270(x) - 180\left(\frac{x^2}{2}\right)\right]$$

$$\frac{dv}{dx} = \frac{1}{EI}\left[270\left(\frac{x^2}{2}\right) - 180\left(\frac{x^3}{6}\right) + C_3\right] \qquad \text{(c)}$$

Pin connected beam

Q: Find max. deflection

Solution step

1. Apply moment equilibrium at the position of pin
And determine the reaction force at A
2. Obtain bending moment along the beam axis
Note: divide the region at the pin position
3. Apply boundary conditions
BCs. 0 deflection at x=0 & x=9, zero slope at x=9
Apply patch condition.

Differential equations: example 12.4

$$v = \frac{1}{EI}\left[270\left(\frac{x^3}{6}\right) - 180\left(\frac{x^4}{24}\right) + C_3 x + C_4\right] \qquad \text{(d)}$$

You will note that except for the constants of integration the deflection equations are identical for this simple problem for both domains.

Boundary conditions:

1. When $x = 0$, $v = 0$.

$$\therefore C_2 = 0$$

2. When $x = 9$, $dv/dx = 0$.

$$270\left(\frac{9^2}{2}\right) - 180\left(\frac{9^3}{6}\right) + C_3 = 0$$

$$\therefore C_3 = 1.094 \times 10^4$$

3. When $x = 9$, $v = 0$.

$$270\left(\frac{9^3}{6}\right) - 180\left(\frac{9^4}{24}\right) + (1.094 \times 10^4)(9) + C_4 = 0$$

$$\therefore C_4 = -8.206 \times 10^4$$

Patch condition:

$$[v(3)]_{\text{Eq.(b)}} = [v(3)]_{\text{Eq.(d)}}$$

$$\therefore C_1(3) + C_2 = C_3(3) + C_4$$

Noting that $C_2 = 0$, $C_3 = 1.094 \times 10^4$, and $C_4 = -8.206 \times 10^4$, we can solve for the remaining unknown constant C_1. That is,

$$3C_1 + 0 = (3)(1.094 \times 10^4) - 8.206 \times 10^4$$

$$\therefore C_1 = -1.641 \times 10^4$$

We now look for *zero slopes* of v in the two domains. Thus for the left domain we have

$$\frac{dv}{dx} = 0 = \frac{1}{EI}\left(270\frac{x^2}{2} - 180\frac{x^3}{6} - 1.641 \times 10^4\right) \qquad \text{(e)}$$

We find as a real root for this equation,

$$x = -6.92 \text{ m}$$

Pin connected beam

Q: Find max. deflection

Solution step

1. Apply moment equilibrium at the position of pin And determine the reaction force at A
2. Obtain bending moment along the beam axis Note: divide the region at the pin position
3. Apply boundary conditions
 BCs. 0 deflection at x=0 & x=9, zero slope at x=9
 Apply patch condition.

Differential equations: example 12.4

Clearly, we discard this result, coming as it does outside the domain of Eq. (e). Look next at the remaining domain.

$$\frac{dv}{dx} = 0 = \frac{1}{EI}\left[270\left(\frac{x^2}{2}\right) - 180\left(\frac{x^3}{6}\right) + 1.094 \times 10^4\right]$$

We get as the only zero-slope position,

$$x = 9.00 \text{ m}$$

This corresponds to the base of the cantilever and represents the trivial condition of a minimum deflection of zero.

We should check the pin. Thus, from Eq. (b) we have

$$v(3) = \frac{1}{EI}\left[270\left(\frac{3^3}{6}\right) - 180\left(\frac{3^4}{24}\right) - (1.641 \times 10^4)(3)\right]$$

$$= -\frac{4.862 \times 10^4}{EI} \text{ m}$$

It should now be clear that the maximum deflection must occur at the pin.

The value of EI is next computed.

$$EI = (2 \times 10^{11})\left[\left(\frac{1}{12}\right)(.075)(.1)^3 - \left(\frac{1}{12}\right)(.040)(.060)^3\right]$$

$$= 1.106 \times 10^6 \text{ N-m}^2$$

The maximum deflection then is

$$v(3) = -\frac{4.862 \times 10^4}{1.106 \times 10^6} = -.0440 \text{ m}$$

Pin connected beam

Q: Find max. deflection

Statically indeterminate problem: example 12.5

The cantilever beam shown in Fig. 12.8 supports a uniform loading w_0 of 10 kN/m and a concentrated couple-moment M_0 having the value of 100 kN-m. Find the supporting forces and the deflection curve in terms of EI. The beam is 10 m long.

The free-body diagram for the entire beam is shown in Fig. 12.9. We shall consider the supporting force R_1 as the redundant constraint in the ensuing computations. We can here compute the bending moment M in terms of R_1 without the necessity of determining other supporting forces or torques in terms of R_1. Accordingly, we shall employ Eq. (12.4) for two spans of the beam as follows:

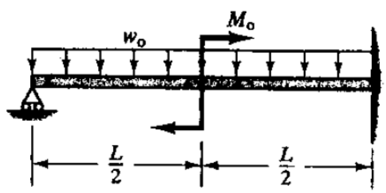


Figure 12.8. Cantilever beam.

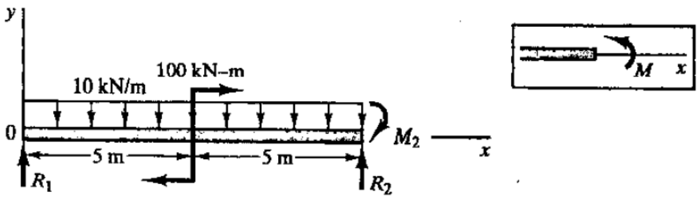


Figure 12.9. Free-body diagram of cantilever beam.

$0 \le x < 5$:

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left(R_1x - \frac{10x^2}{2}\right) \qquad \text{(a)}$$

$5 < x < 10$:

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left(R_1x - 10\frac{x^2}{2} + 100\right) \qquad \text{(b)}$$

Integrating twice, for the spans we get

$0 \le x < 5$:

$$\frac{dv}{dx} = \frac{1}{EI}\left(R_1\frac{x^2}{2} - \frac{10x^3}{6} + C_1\right) \qquad \text{(c)}$$

$$v = \frac{1}{EI}\left(R_1\frac{x^3}{6} - \frac{10x^4}{24} + C_1x + C_2\right) \qquad \text{(d)}$$

$5 < x < 10$:

$$\frac{dv}{dx} = \frac{1}{EI}\left(R_1\frac{x^2}{2} - \frac{10x^3}{6} + 100x + C_3\right) \qquad \text{(e)}$$

Solution step

1. Calculate bending moment as a function of x with unknown R1 (considered as redundant constant)
2. Obtain deflection and slope for each region between 0~5, and 5~10
3. Apply proper boundary condition at x=0, x=10, and patch conditions at x=5
4. Obtain additional unknowns for R2 and M2 at the wall (or x=10)

Statically indeterminate problem: example 12.5

$$v = \frac{1}{EI}\left(R_1\frac{x^3}{6} - \frac{10x^4}{24} + 100\frac{x^2}{2} + C_3x + C_4\right) \qquad \text{(f)}$$

We have four constants of integration plus the unknown R_1 to be determined. We can note that

$$\text{at } x = 0, \qquad v = 0$$
$$\text{at } x = L, \qquad \frac{dv}{dx} = v = 0$$

Applying these conditions, we have

$$C_2 = 0 \qquad \text{(g)}$$

$$C_3 = -\frac{R_1(10)^2}{2} + \frac{(10)(10)^3}{6} - (100)(10) = -50R_1 + 667 \qquad \text{(h)}$$

$$C_4 = -\frac{R_1(10^3)}{6} + \frac{(10)(10^4)}{24} - (100)\frac{(10^2)}{2} - (-50R_1 + 667)(10) \qquad \text{(i)}$$
$$= 333R_1 - 7.51 \times 10^3$$

Next we apply the *patch conditions* (**compatibility**) at $x = 5$. Thus

$$\left[\frac{dv(5^-)}{dx}\right]_{\text{Eq.(c)}} = \left[\frac{dv(5^+)}{dx}\right]_{\text{Eq.(e)}}$$

$$R_1\left(\frac{5^2}{2}\right) - 10\frac{(5^3)}{6} + C_1 = R_1\left(\frac{5^2}{2}\right) - 10\frac{(5^3)}{6} + (100)(5) + C_3$$

$$\therefore C_1 = 500 + C_3 \qquad \text{(j)}$$

Also,

$$[v(5^-)]_{\text{Eq.(d)}} = [v(5^+)]_{\text{Eq.(f)}}$$

$$R_1\left(\frac{5^3}{6}\right) - \frac{(10)(5^4)}{24} + C_1(5) = R_1\left(\frac{5^3}{6}\right) - \frac{(10)(5^4)}{24} + 100\frac{(5^2)}{2} + C_3(5) + C_4$$

$$\therefore 5C_1 = 1250 + 5C_3 + C_4 \qquad \text{(k)}$$

Replacing C_3 and C_4 using Eqs. (h) and (i) in Eqs. (j) and (k), we get the following simultaneous equations for C_1 and R_1:

$$C_1 = 1167 - 50R_1$$
$$5C_1 = 83R_1 - 2.92 \times 10^3$$

Solving for R_1, we get

$$R_1 = 26.3 \text{ kN}$$

The other supporting forces are now readily available from rigid-body mechanics. Thus,

$\underline{\sum F_y = 0}$:

$$R_1 - (10)(10) + R_2 = 0$$
$$\therefore R_2 = 73.7 \text{ kN}$$

$\underline{\sum M_0 = 0}$:

$$-(100)(5) - 100 + R_2(10) - M_2 = 0$$
$$\therefore M_2 = 137 \text{ kN-m}$$

We have accordingly determined both the deflection equation and the supporting forces simultaneously.

Superposition method

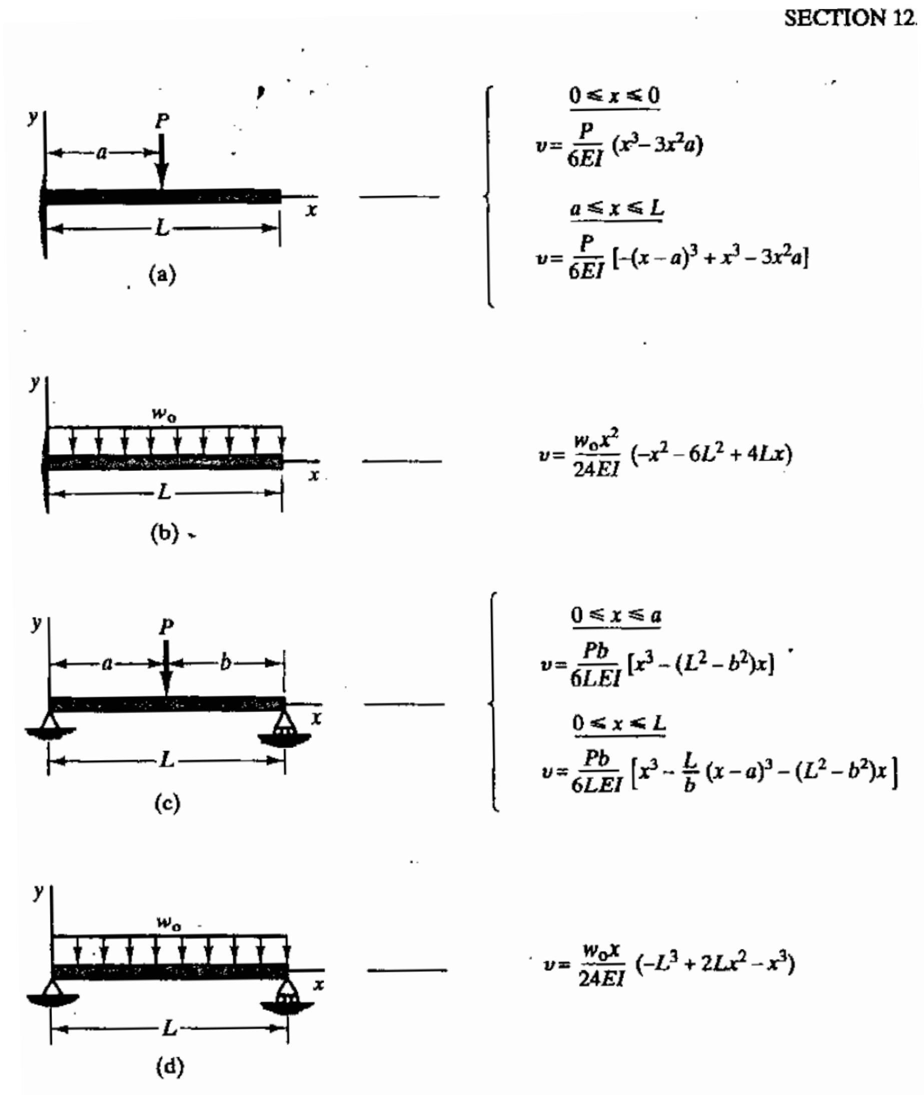


Figure 12.11. Deflection formulas for simple beam loadings.

Homework!!!!
Derive the left equations!

Superposition method

Shown in Fig. 12.12 is a simply-supported beam carrying a uniform loading of 50 N/m and a 5000-N concentrated load. What is the deflection at the midpoint of the beam?

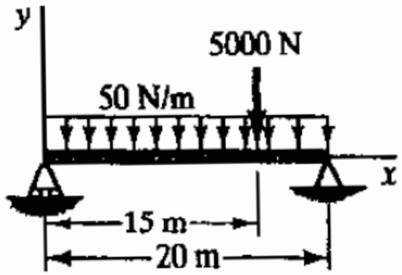


Figure 12.12. Simply-supported beam.

Noting that the value of x to be used is less than $a = 15$, from Fig. 12.11(c) and 12.11(d) we have, on using the proper domain for the 5000-lb force

$$v = \frac{(5000)(5)}{(6)(20)(EI)}[x^3 - (20^2 - 5^2)x] + \frac{50x}{24EI}[-20^3 + (2)(20)x^2 - x^3]$$

At $x = 10$ we have

$$v(10) = \frac{1}{EI}\left\{\frac{(5000)(5)}{(6)(20)}[1000 - (400 - 25)(10)]\right.$$

$$\left. + \frac{(50)(10)}{24}[-8000 + (2)(20)(100) - 1000]\right\}$$

$$v(10) = -\frac{6.77 \times 10^5}{EI}\,\text{m}$$

Energy method

- Strain energy related 1) normal stress, 2) transverse shear stress
- For normal stress

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}}$$

$$U^* = \int_V \frac{\sigma_{xx}^2}{2E} dv = \int_0^L \left[\int_A \frac{M_z^2 y^2}{2EI_{zz}^2} dA \right] dx = \int_0^L \left[\frac{M_z^2}{2EI_{zz}^2} \int_A y^2 dA \right] dx$$

$$U^* = \int_0^L \left[\frac{M_z^2}{2EI_{zz}} \right] dx$$

- For shear stress resulting from V_y

$$U^* = \int_V \frac{\tau_{xy}^2}{2G} dv$$

Energy method

- Shear stress may vary across the cross section as a function of both y and z
- Therefore, instead of determining the exact distribution of shear stress, use "tabulated shape factors"
- The shape factors correlate the energy computed using actual shear stress distribution with that of a cross section with equal area but with a constant distribution of shear stress

$$U^* = \int_V \frac{\tau_{xy}^2}{2G} dv \bigg|_{\tau_{xy->actual}} = \alpha_y \int_V \frac{\tau_{xy}^2}{2G} dv \bigg|_{\tau_{xy->simplified}} \qquad \tau_{xy->simplified} = \frac{V_y}{A}$$

$$U^* = \alpha_y \int_0^L \left[\int_A \frac{V_y^2}{2GA^2} dA \right] dx = \alpha_y \int_0^L \frac{V_y^2}{2GA} dx$$

Energy method – Example 12.11

For a beam with a rectangular cross section, as shown in Fig. 12.21, compute the shape factor associated with shear strain energy. The beam is prismatic and the loading is in the xy plane.

We start by assuming that the shear stress distribution across the face of the section can be adequately described by Jourowski's formula

$$\tau_{xy} = \frac{V_y Q_z}{I_{zz} b}$$

so that Eq. (12.15) becomes[5]

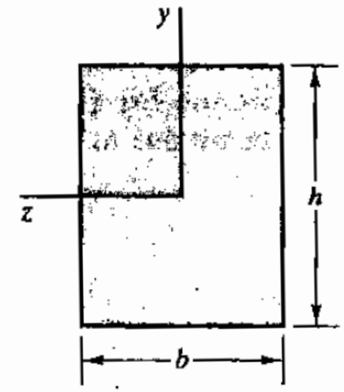


Figure 12.21. Rectangular beam section.

Energy method

$$U^* = \iiint_V \frac{1}{2G}\left[\frac{V_y Q_z}{I_{zz} b}\right]^2 dv = \alpha_y \iiint_V \frac{1}{2G}\left[\frac{V_y}{A}\right]^2 dv \qquad \text{(a)}$$

Canceling terms from both sides of the equation and, recognizing that the beam is prismatic of length L and cross section area A, we integrate with respect to x on both sides and then with respect to A on the right side. Then we get on cancelling L,

$$\frac{1}{I_{zz}^2}\iint_A \left[\frac{Q_z}{b}\right]^2 dA' = \frac{\alpha_y}{A}$$

$$\therefore \alpha_y = \frac{A}{I_{zz}^2}\iint_A \left[\frac{Q_z}{b}\right]^2 dA' \qquad \text{(b)}$$

For the rectangle we have

$$A = bh$$

$$I_{zz} = \tfrac{1}{12}bh^3$$

$$Q_z = \frac{b}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right] \qquad \text{(From Example 11.7)}$$

Substitution of the above parameters into Eq. (b) yields

$$\alpha_y = \frac{bh}{\frac{1}{144}b^2h^6}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{\frac{b^2}{4}\left[\left(\frac{h}{2}\right)^2 - y^2\right]^2}{b^2}\right] dy\,dz$$

$$\therefore \alpha_y = \frac{6}{5}$$

So we see that the shape factor $\alpha_y = 6/5$ for the rectangular section. Note that we get the same shape factor for loading in the xz plane (i.e., $\alpha_z = 6/5$).[6]

Shape factors for other common cross sections are shown in Fig. 12.22.

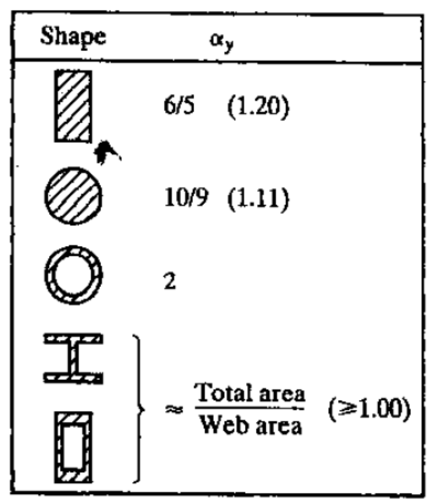


Figure 12.22. Shape factors for common cross sections.

Energy method- Example 12.12

Determine the displacement at the tip ($x = 0$) of the end-loaded cantilevered beam shown in Fig. 12.23 using Castigliano's second theorem. Assume that the beam is prismatic and is made from a linear elastic material.

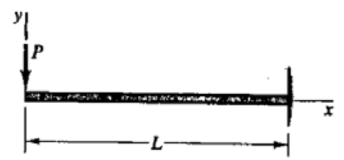


Figure 12.23. Tip loaded cantilevered beam.

We start by assuming that the displacement at the beam tip is due to both bending and shear deformation in the beam. We can write U^* (or U since the material is linear elastic) as the sum of the flexural energy given by Eq. (12.14) and the shear energy given by Eq. (12.17) **(constitutive law)**

$$U^* = \int_0^L \frac{M_z^2}{2EI_{zz}}\,dx + \alpha_y \int_0^L \frac{V_y^2}{2GA}\,dx \qquad \text{(a)}$$

The bending moment at any position is $M_z(x) = -Px$ and the shear is simply $V_y(x) = -P$ **(equilibrium)**. Substitution into Eq. (a) gives

$$U^* = \int_0^L \frac{(-Px)^2}{2EI_{zz}}\,dx + \alpha_y \int_0^L \frac{(-P)^2}{2GA}\,dx \qquad \text{(b)}$$

Evaluating the integrals in Eq. (b) we get

$$U^* = \frac{P^2L^3}{6EI_{zz}} + \alpha_y \frac{P^2L}{2GA} \qquad \text{(c)}$$

Using the second Castigliano theorem given in Eq. (6.58) we can readily determine the displacement Δ at the *location* of and in the *direction* of the load P as **(compatibility)**

$$\frac{\partial U^*}{\partial P} = \Delta = \frac{PL^3}{3EI_{zz}} + \alpha_y \frac{PL}{GA} \qquad \text{(d)}$$

Since the solution for Δ is positive, the beam deflects in the direction of the load P, as was expected. From our work in this chapter, we recognize the first term on the right-hand side of Eq. (d) as the displacement at the tip of a point-loaded cantilevered beam. The second term is the additional displacement due to the shear deformation in the beam.

Carrying the calculations further, we can determine the relative influence of the shear deformation on the tip displacement. Assuming a rectangular cross section of height h and width b we have from Example 12.11 [Fig. 12.22] $\alpha_y = 6/5$. The displacement given by Eq. (d) is now expressed as

$$\Delta = \frac{PL}{bh}\left[\frac{4}{E}\left(\frac{L}{h}\right)^2 + \frac{6}{5G}\right] \qquad \text{(e)}$$

If we assume the beam is made from steel ($\nu = 0.3$) then from Eq. (6.10) we have $E = G/2.6$. Rewriting Eq. (e)

$$\Delta = \frac{PL}{bhE}\left[4\left(\frac{L}{h}\right)^2 + 3.12\right] \qquad \text{(f)}$$

The first term in the bracket of Eq. (f) is the contribution due to bending, the second term is due to shear. The ratio of beam length to height (L/h) is often called the *aspect ratio* of the beam. We can see from Eq. (f) that for a beam with an aspect ratio of unity (i.e., $L = h$), the bending and shear contributions are of the same order of magnitude; the shear deformation in this case is significant. As the aspect ratio increases, the beam becomes *slender* and the relative contribution to the displacement of the bending and shear terms change. We can see in Eq. (f) that the aspect ratio, which appears in the bending term, is to the second power while the shear term is constant. Clearly, as the length of the beam becomes much greater than its height, the affect of shear on the tip displacement becomes negligible.[7] For example, with an aspect ratio $L/h = 10$, the contribution of shear to the deformation is less than 1%, this drops to less than 0.2% for $L/h = 20$. Similar results are obtained with other beam cross sections. We conclude that for slender beams (say, $L/h \geq 10$), the effect of shear on the displacement may be ignored.

Homework by yourself

Solve examples 12.3, 12.6, 12.8, 12.13, 12.14