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소성재료역학
(Metal Plasticity)

Chapter 13: Normality Rule for Plastic Deformation

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Yield function and plastic strain

What we learn from this chapter

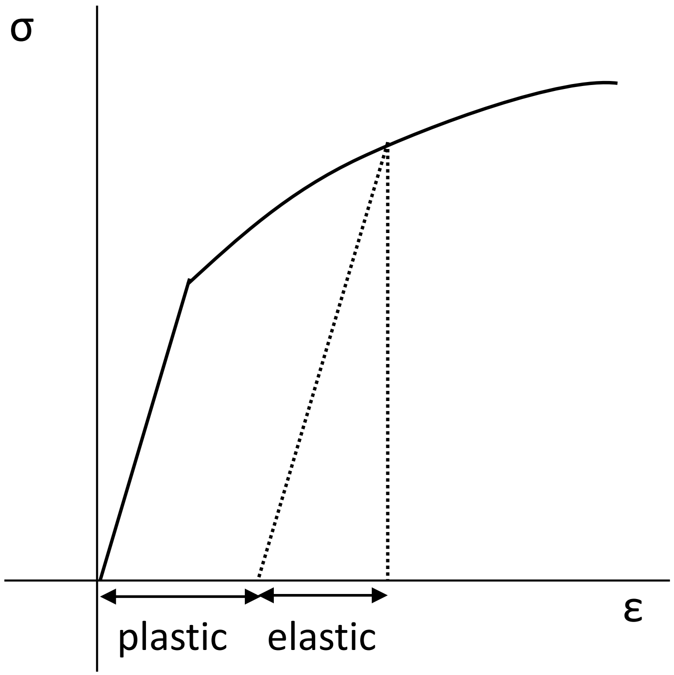
- In chapter 12, yield function states the boundary between elastic and elasto-plastic region and size of equivalent stress

- How can we define the direction of plastic flow?
  → normality rule

- Could we define scalar strain value like a equivalent stress?
  → work equivalence principle & equivalent strain

Additive decomposition of strain increment

1-D stress-strain curve



Additive decomposition
in general state

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{e} + d\boldsymbol{\varepsilon}^{p}$$

Rigid-plasticity
(for large plastic deformation)

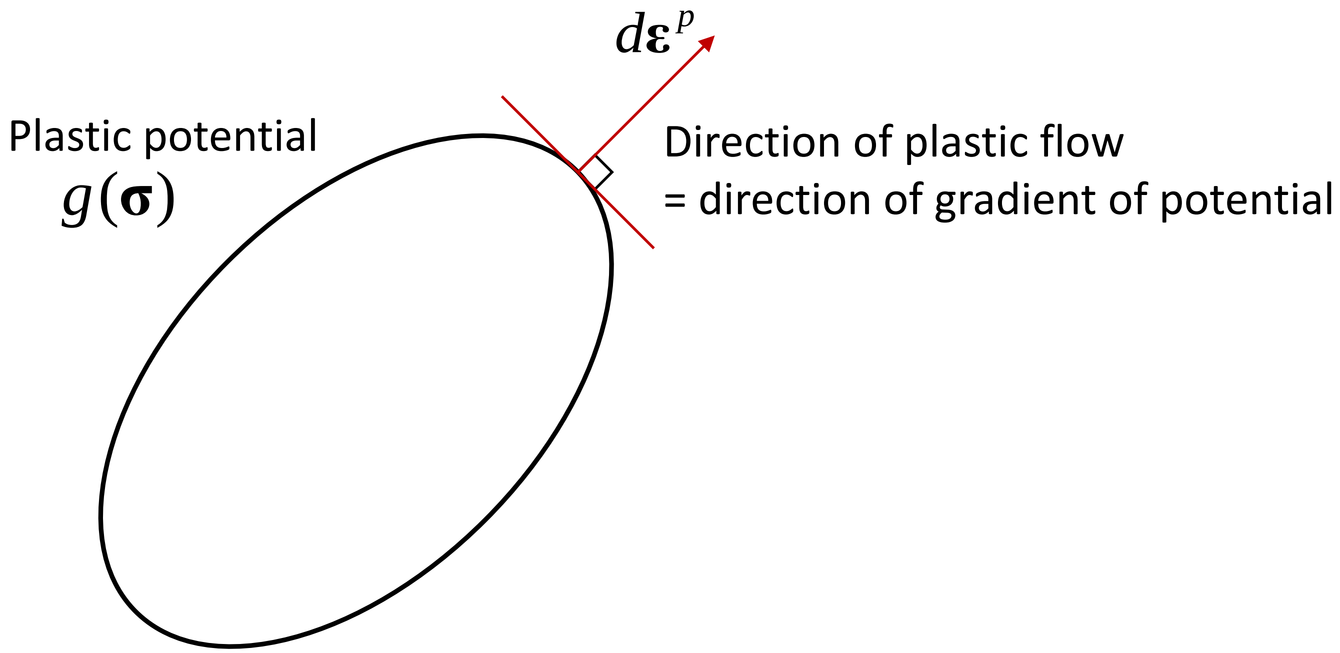
$$d\boldsymbol{\varepsilon} = \cancel{d\boldsymbol{\varepsilon}^{e}} + d\boldsymbol{\varepsilon}^{p}$$

$$\approx d\boldsymbol{\varepsilon}^{p}$$

Normality rule

Surface of plastic potential ⊥ direction plastic flow

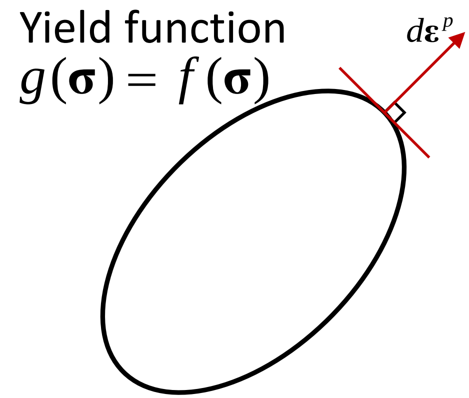
$$d\boldsymbol{\varepsilon}^{p} \left(= \mathbf{D}^{p} dt\right) \sim \frac{\partial g(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$$



Associated/non-associated flow rule

Associated flow rule

Use yield function as plastic potential

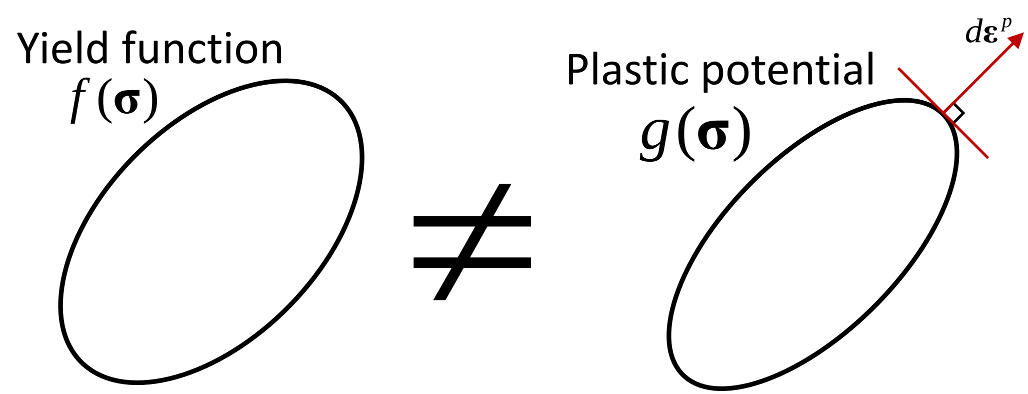


$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = d\lambda \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \left(= d\bar{\varepsilon} \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right)$$

Non-associated flow rule

Introduce another function to point out plastic flow

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial g(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \left(\neq \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \right)$$



Plastic work equivalence principle

- Both yield surface and fixed scalar value, or so called equivalent stress for the given stress state is described in $\bar{\sigma}(\boldsymbol{\sigma}) = const$
- Equivalent stress plays a role in **defining a generalized magnitude of the stress**
- Similarly, **equivalent (or effective) plastic strain increment**, which is a conjugate quantity of equivalent stress, can be defined based on **plastic work equivalence principle**

$$dw^p = tr(\boldsymbol{\sigma} d\boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^p = \sigma_{ij} d\varepsilon$$

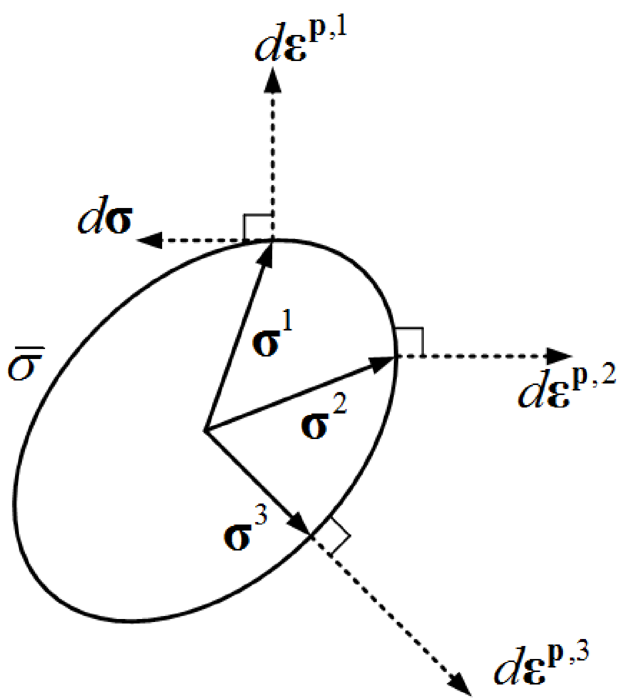
$$= \bar{\sigma}(\boldsymbol{\sigma}) d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p) = constant$$

Plastic work equivalence principle

- Effective plastic strain increment surface gives two information:

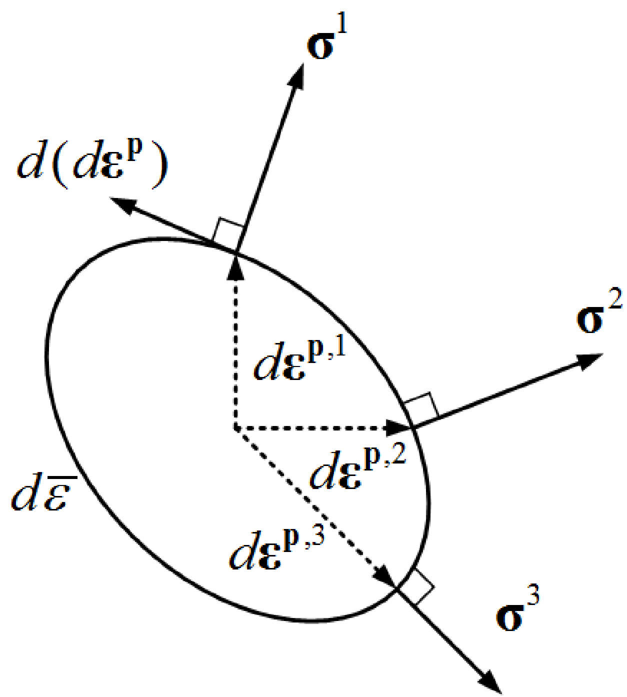
  generalized magnitude of plastic strain increment and its surface

Yield surface $\bar{\sigma}(\boldsymbol{\sigma}) = const$



Effective plastic strain increment surface

$$d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p) = \frac{dw^p}{\bar{\sigma}(\boldsymbol{\sigma})(= const)} = const$$



Dual normality rules

$$dw^p = \sigma_{ij} d\varepsilon_{ij}^p = \bar{\sigma} d\bar{\varepsilon} = constant$$

$$\rightarrow d\left(dw^p\right) = d\left(\sigma_{ij}\right) d\varepsilon_{ij}^p + \sigma_{ij} d\left(d\varepsilon_{ij}^p\right) = 0$$

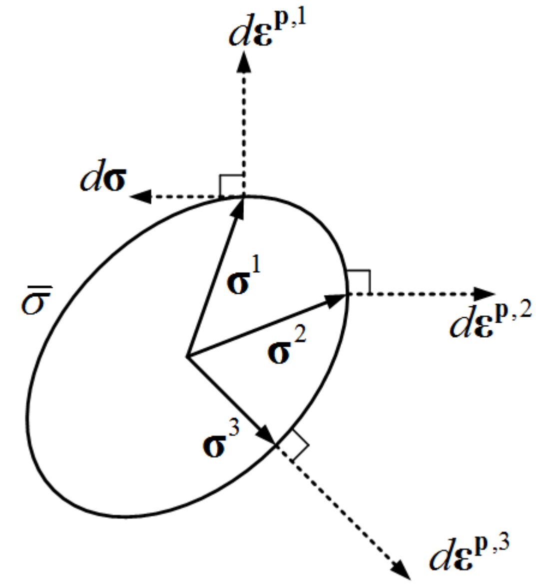
By the normality rule, $\sigma_{ij} d\left(d\varepsilon_{ij}^p\right) = 0$

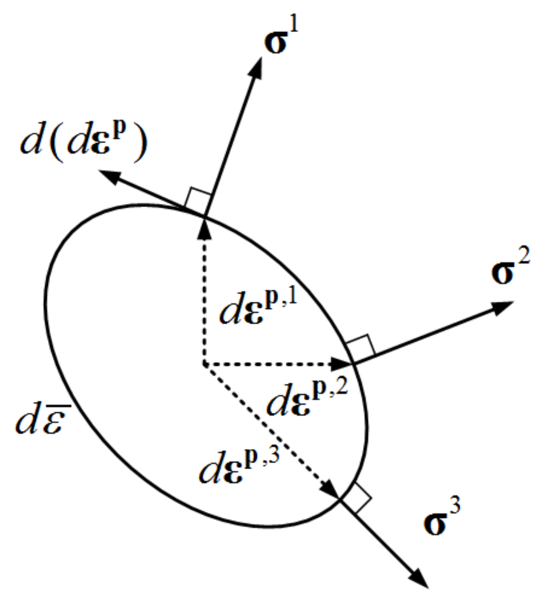
Therefore, $\sigma_{ij} d\left(d\varepsilon_{ij}^p\right) = 0$

also,

$$\boldsymbol{\sigma} = A\frac{\partial dg(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)} = A\frac{\partial d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)} (= \bar{\sigma}\frac{\partial d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)})$$

$$d\boldsymbol{\varepsilon}^p = d\lambda\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} = d\lambda\frac{\partial\bar{\sigma}(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} (= d\bar{\varepsilon}\frac{\partial\bar{\sigma}(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}})$$





Effective plastic strain increment as a first order homogeneous function

<u>**Proof**</u>

$$\left(\alpha\boldsymbol{\sigma}\right)\cdot\left(\beta d\boldsymbol{\varepsilon}^{p}\right)=\alpha\beta dw^{p}$$

$$\begin{aligned}\alpha\beta dw^{p} &=\alpha\beta\bar{\sigma}\left(\boldsymbol{\sigma}\right)d\bar{\varepsilon}\left(d\boldsymbol{\varepsilon}^{p}\right)\\ &=\bar{\sigma}\left(\alpha\boldsymbol{\sigma}\right)d\bar{\varepsilon}\left(\beta d\boldsymbol{\varepsilon}^{p}\right)\\ &=\alpha\bar{\sigma}(\boldsymbol{\sigma})\beta^{n}d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^{p})\end{aligned}$$

$$\therefore n=1$$

Effective plastic strain increment as a first order homogeneous function

Since $\bar{\sigma}(\boldsymbol{\sigma})$ and $\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p)$ are first homogeneous function,

$$\boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} = d\lambda \bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{\sigma}) d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p)$$

$$\therefore d\lambda = d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p)$$

From normality rule,

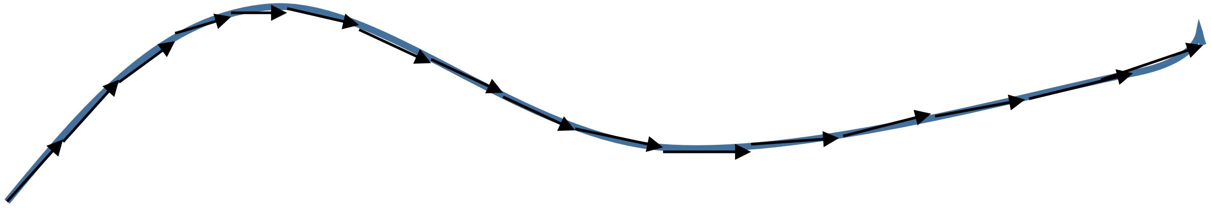
$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = d\lambda \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} (= d\bar{\varepsilon} \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}) \longrightarrow \boldsymbol{\sigma} = \boldsymbol{\sigma}(d\boldsymbol{\varepsilon}^p)$$

Then, equivalent strain can be calculated like in below

$$d\bar{\varepsilon} = \frac{\boldsymbol{\sigma}(d\boldsymbol{\varepsilon}^p) \cdot d\boldsymbol{\varepsilon}^p}{\bar{\sigma}(\boldsymbol{\sigma}(d\boldsymbol{\varepsilon}^p))}$$

For the simple cases like von Mises or Hill 1948,
analytical expressions can be obtained.

Accumulative effective plastic strain



Similar to the concept of travel length in geometry,

$$s = \int ds(d\,\boldsymbol{x}) = \int \left| d\,\boldsymbol{x} \right|$$

Accumulative effective plastic strain

$$\bar{\varepsilon} = \int d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^{p})$$

Remark

Remark #13.3 Note that the standard elasto-plastic constitititutive formation with Eq. (13.1) is based on the normality rule in Eq. (13.3) so that it requires the yield function as well as its diverse development. Meanwhile, the standard rigid-plastic formulation is based on the normality rule in Eq. (13.7) so that it requires the plastic strain increment function as well as its diverse development. However, the effective plastic strain increment is required in order to calculate the accumulative effective plastic strain in Eq. (13.13). Since rigid-plasticity is based on the plastic strain increment function, calculating the accumulative effective plastic strain is straightforward. For elasto-plasticity, a standard formulation to be discussed in Chap. 16 calculates the magnitude of the effective plastic strain increment, without requiring the explicit expression of the effective plastic strain increment as a function of the plastic strain increment tensor. Similarly, rigid-plasticity does not require the explicit expression of the effective stress as a function of the stress tensor in order to calculate the size of the yield stress. Consequently, elasto-plasticity requires only the effective stress as a function of the stress tensor, while rigid-plasticity requires only the effective plastic strain increment as a function of the plastic strain increment tensor. These will be discussed in Chap. 16.

Incompressibility

For the crystalline metals,

$$tr(d\boldsymbol{\varepsilon}^{p}) = d\boldsymbol{\varepsilon}_{ii}^{p} = \tilde{I}_{1} = 0$$

$\tilde{\mathrm{I}}_{1}$: 1st inveriant of plastic strain increment

For the normality rule,

$$d\varepsilon_{ii}^{p} \sim \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ii}} = \frac{\partial f(S_{ij}, \cancel{\sigma_{kk}})}{\partial \sigma_{ii}} = 0$$

← Hydrostatic stress independence

$$or\ d\varepsilon_{ij}^{p} = de_{ij}^{p} \sim \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f(S_{ij}, \cancel{\sigma_{kk}})}{\partial S_{kl}} \frac{\partial S_{kl}}{\partial \sigma_{ij}} + \frac{\partial f(S_{ij}, \cancel{\sigma_{kk}})}{\partial \sigma_{kk}} \frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \frac{\partial f(S_{ij}, \cancel{\sigma_{kk}})}{\partial S_{kl}} \frac{\partial S_{kl}}{\partial \sigma_{ij}}$$

de_{ij}^{p} : deviatoric plastic strain increment

Incompressibility

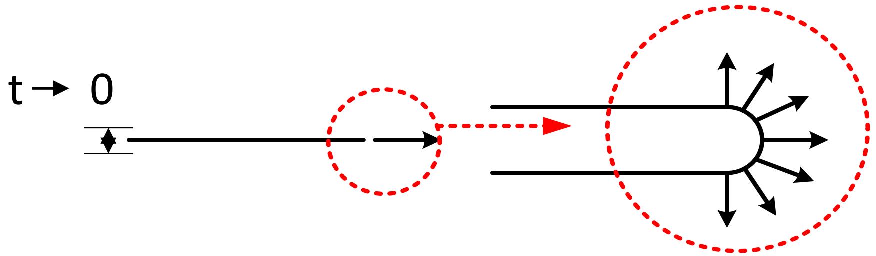
$$\begin{pmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{21} \\ S_{23} \\ S_{32} \\ S_{13} \\ S_{31} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} \\ -\frac{1}{3}\sigma_{11} + \frac{2}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} \\ -\frac{1}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} + \frac{2}{3}\sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \end{pmatrix}, \qquad \begin{pmatrix} d\varepsilon_{11}^{p} \\ d\varepsilon_{22}^{p} \\ d\varepsilon_{33}^{p} \\ d\varepsilon_{12}^{p} \\ d\varepsilon_{21}^{p} \\ d\varepsilon_{23}^{p} \\ d\varepsilon_{32}^{p} \\ d\varepsilon_{13}^{p} \\ d\varepsilon_{31}^{p} \end{pmatrix} \sim \begin{pmatrix} \frac{2}{3}\frac{\partial f}{\partial S_{11}} - \frac{1}{3}\frac{\partial f}{\partial S_{22}} - \frac{1}{3}\frac{\partial f}{\partial S_{33}} \\ -\frac{1}{3}\frac{\partial f}{\partial S_{11}} + \frac{2}{3}\frac{\partial f}{\partial S_{22}} - \frac{1}{3}\frac{\partial f}{\partial S_{33}} \\ -\frac{1}{3}\frac{\partial f}{\partial S_{11}} - \frac{1}{3}\frac{\partial f}{\partial S_{22}} + \frac{2}{3}\frac{\partial f}{\partial S_{33}} \\ \frac{\partial f}{\partial S_{12}} \\ \frac{\partial f}{\partial S_{21}} \\ \frac{\partial f}{\partial S_{23}} \\ \frac{\partial f}{\partial S_{32}} \\ \frac{\partial f}{\partial S_{13}} \\ \frac{\partial f}{\partial S_{31}} \end{pmatrix}$$

Incompressible effective plastic strain increment surface in normal component space

- Normality rule leads to **2D membrane of the plastic strain increment surface** in the 3D normal stress component space, while the conjugate yield surface is a 3D cylinder
- **Constructing a 3D cylindrical yield surface from the 2D plastic strain increment surface** based on the dual normality rule would be **troublesome algebraically**
- A figurative justification would be possible by **considering the two-dimensional membrane** as the limit of a 2D closed structure having a smooth curved side wall (with a half circular cross-section)

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = d\bar{\varepsilon} \frac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \rightarrow d\bar{\varepsilon} = \frac{d\boldsymbol{\varepsilon}^p}{\dfrac{\partial \bar{\sigma}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}}$$

Normal direction of side wall cover all possible hydrostatic stress



Remark #13.6 Applying the normality rule in Eq. (13.3) figuratively and algebraically leads to the two-dimensional membrane of the plastic strain increment surface in the three-dimensional normal stress component space, while the conjugate yield surface is a three-dimensional cylinder. However, constructing a three-dimensional cylindrical yield surface from the two-dimensional plastic strain increment surface based on the dual normality rule in Eq. (13.7) would be troublesome algebraically. A figurative justification would be possible by considering the two-dimensional membrane as the limit of a thin three-dimensional closed structure having a smooth curved side wall (with a half circular cross-section), when its thickness converges to zero. Then, the normal direction of its side wall would cover all possible hydrostatic stress, which is vertical to the deviatoric plane as shown in Fig. 13.2.

Relations for incompressible plasticity

Work equivalence for the incompressible plasticity

$$dw^p = \sigma_{ij} d\varepsilon_{ij}^p = (\mathrm{S}_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij})(de_{ij}^p + \frac{1}{3} d\varepsilon_{mm}^p \delta_{ij}) = \mathrm{S}_{ij} de_{ij}^p = \mathrm{S}_{ij} d\varepsilon_{ij}^p = \bar{\sigma}(\boldsymbol{S}) d\bar{\varepsilon}(d\,\boldsymbol{e}^p) = constant$$

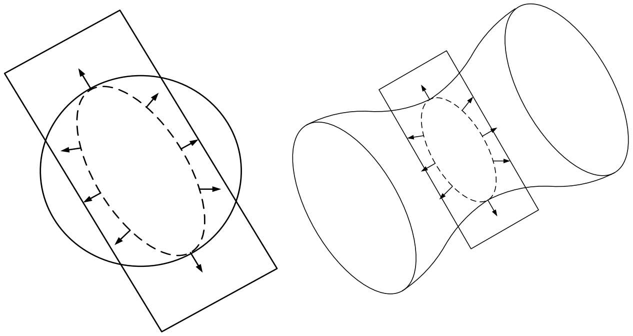
Normality rule for the incompressible plasticity

$$d\boldsymbol{\varepsilon}^p (= d\boldsymbol{e}^p) = d\lambda \frac{\partial f(\boldsymbol{S})}{\partial \boldsymbol{S}} = d\lambda \frac{\partial \bar{\sigma}(\boldsymbol{S})}{\partial \boldsymbol{S}} (= d\bar{\varepsilon}(d\boldsymbol{\varepsilon}^p) \frac{\partial \bar{\sigma}(\boldsymbol{S})}{\partial \boldsymbol{S}})$$

Dual normality rule for the incompressible plasticity

$$\boldsymbol{S} = A \frac{\partial dg(d\boldsymbol{e}^p)}{\partial\left(d\boldsymbol{e}^p\right)} = A \frac{\partial d\bar{\varepsilon}^p(d\boldsymbol{e}^p)}{\partial\left(d\boldsymbol{e}^p\right)} = A \frac{\partial d\bar{\varepsilon}^p(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)} (= \bar{\sigma}(\boldsymbol{S}) \frac{\partial d\bar{\varepsilon}^p(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)})$$

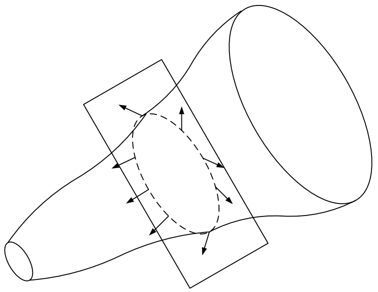
Yield surfaces are symmetric with respect to deviatoric plane



From dual normality rule,

$$\therefore \boldsymbol{\sigma} = \bar{\sigma}(\boldsymbol{S}) \frac{\partial d\bar{\varepsilon}^p(d\boldsymbol{\varepsilon}^p)}{\partial\left(d\boldsymbol{\varepsilon}^p\right)} + B\boldsymbol{I}$$

Yield surfaces are asymmetric with respect to deviatoric plane



Incompressible & Isotropic case

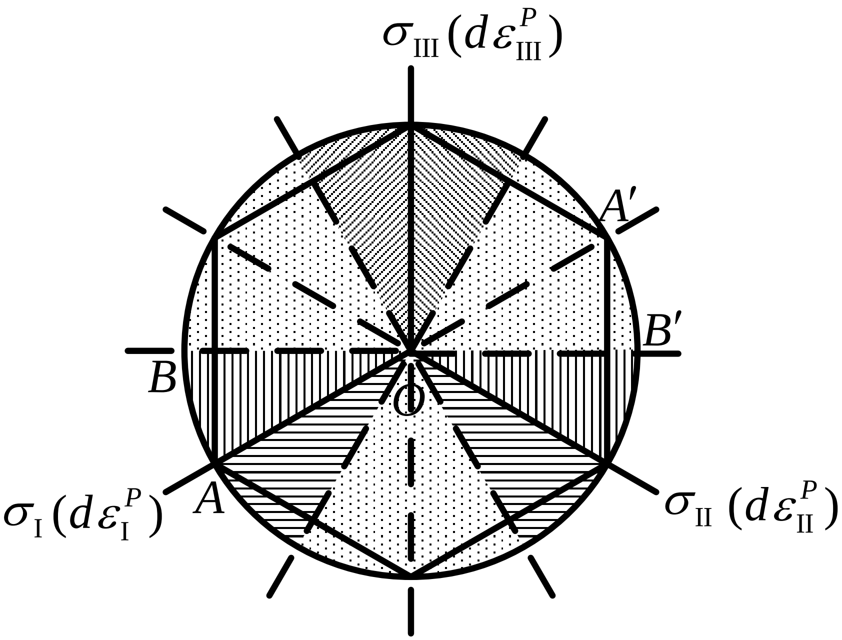
$$dg(d\boldsymbol{\varepsilon}^p)$$
$$= dg(d\varepsilon_I^p, d\varepsilon_{II}^p, d\varepsilon_{III}^p, n_I, n_{II}, n_{III})$$
$$= dg(d\varepsilon_I^p, d\varepsilon_{II}^p, d\varepsilon_{III}^p) \longleftarrow \text{Isotropic case}$$
$$= dg(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3) \longleftarrow \text{Denote with invariants}$$
$$= dg(\tilde{I}_1, \tilde{J}_2, \tilde{I}_3)$$
$$= dg(\tilde{J}_2, \tilde{J}_3) \longleftarrow \text{Incompressible case, } \tilde{J}_1 \text{ or } \tilde{I}_1 = 0$$

where

$\tilde{I}_1, \tilde{I}_2, \tilde{I}_2$:invariants of plastic strain increment

$\tilde{J}_1, \tilde{J}_2, \tilde{J}_2$:invariants of deviatoric plastic strain increment

π-diagram



Reference plastic strain increment

- As a conjugate of the reference yield stress for the yield surface (and the effective stress), there is the reference plastic strain increment (as a scalar quantity) for the plastic strain increment surface

- Reference state for the incompressible, isotropic and symmetric plasticity shown in below

Direction of plastic flow for the different reference state

$$\begin{cases} (d\varepsilon_I^{pST}, d\varepsilon_{II}^{pST}, d\varepsilon_{III}^{pST}) = -(d\varepsilon_I^{pSC}, d\varepsilon_{II}^{pSC}, d\varepsilon_{III}^{pSC}) = d\varepsilon_I^{pST}(1, \frac{-1}{2}, \frac{-1}{2}) \\ (d\varepsilon_I^{pBBT}, d\varepsilon_{II}^{pBBT}, d\varepsilon_{III}^{pBBT}) = -(d\varepsilon_I^{pBBC}, d\varepsilon_{II}^{pBBC}, d\varepsilon_{III}^{pBBC}) = \left| d\varepsilon_{III}^{pBBT} \right| (\frac{1}{2}, \frac{1}{2}, -1) \\ (d\varepsilon_I^{pPS2}, d\varepsilon_{II}^{pPS2}, d\varepsilon_{III}^{pPS2}) = d\varepsilon_I^{pPS2}(1, 0, -1) \\ (d\varepsilon_I^{pPS3}, d\varepsilon_{II}^{pPS3}, d\varepsilon_{III}^{pPS3}) = d\varepsilon_I^{pPS3}(1, -1, 0) \end{cases}$$

Simple tension/compression-ST/SC
balanced biaxial tension/Compression: BBT/BBC
Pure shear : PS

Reference plastic strain increment

Plastic work equivalence principle for the different reference state

$$\bar{\sigma} d\bar{\varepsilon} = \begin{cases} \sigma_I^{ST} d\varepsilon_I^{pST} = \sigma_I^{SC} d\varepsilon_I^{pSC} = YdY \\ \sigma_I^{BBT} \left| d\varepsilon_{III}^{pBBT} \right| = \left| \sigma_I^{BBC} \right| d\varepsilon_{III}^{pBBT} = BdB \\ 2\sigma_I^{PS2} d\varepsilon_I^{pPS2} = 2\sigma_I^{PS3} d\varepsilon_I^{pPS3} = KdK \end{cases}$$

Simple tension/compression-ST/SC
balanced biaxial tension/Compression: BBT/BBC
Pure shear : PS

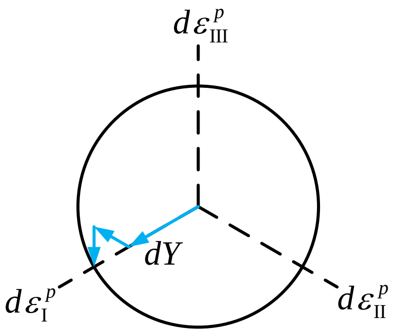
where

$$d\bar{\varepsilon} = \begin{cases} dY = d\varepsilon_I^{pST} = \left| d\varepsilon_I^{pSC} \right| \\ dB = \left| d\varepsilon_{III}^{pBBT} \right| = d\varepsilon_{III}^{pBBC} \\ dK = 2d\varepsilon_I^{pPS2} = 2d\varepsilon_I^{pPS3} \end{cases} \qquad \bar{\sigma} = \begin{cases} Y = \sigma_I^{ST} = \left| \sigma_I^{SC} \right| \\ B = \sigma_I^{BBT} = \left| \sigma_I^{BBC} \right| \\ K = \sigma_I^{PS2} = \sigma_I^{PS3} \end{cases}$$

Reference plastic strain increment

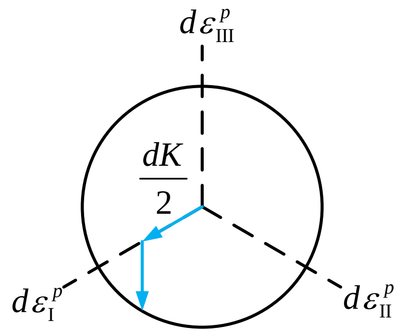
Four reference states in the π diagram in terms of plastic strain increment

Simple tension



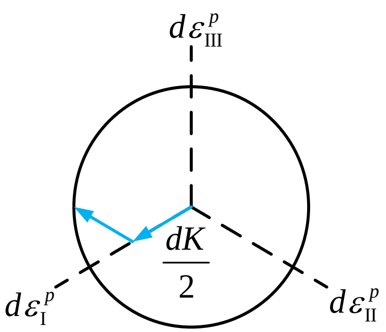
$$d\boldsymbol{\varepsilon}^{p} : (dY, -\frac{dY}{2}, -\frac{dY}{2})$$

Balanced biaxial



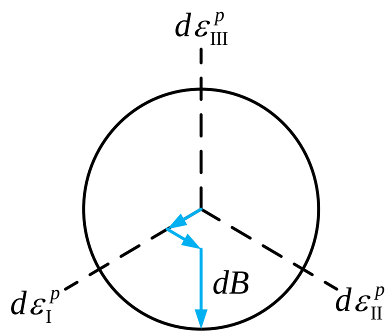
$$d\boldsymbol{\varepsilon}^{p} : (\frac{dK}{2}, 0, -\frac{dK}{2})$$

Pure shear(PS3)
for plane strain(PLS3)



$$d\boldsymbol{\varepsilon}^{p} : (\frac{dK}{2}, -\frac{dK}{2}, 0)$$

Pure shear(PS32
for plane strain(PLS2)



$$d\boldsymbol{\varepsilon}^{p} : (\frac{dB}{2}, \frac{dB}{2}, -dB)$$

Von Mises isotropic yield criterion

From normality rule,

$$d\varepsilon_{ij}^{p} = AS_{ij}$$

From work equivalence principle,

$$AS_{ij}S_{ij} = S_{ij}d\varepsilon_{ij}^{p} = \bar{\sigma}d\bar{\varepsilon} = \frac{A}{\alpha}\bar{\sigma}^{2} \quad \rightarrow \quad A = \frac{\alpha d\bar{\varepsilon}}{\bar{\sigma}}$$

From above equations,

$$d\varepsilon_{ij}^{p} = AS_{ij} = \alpha d\bar{\varepsilon}\frac{S_{ij}}{\bar{\sigma}}$$

In addition,

$$d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p} = AS_{ij}d\varepsilon_{ij}^{p} = A\bar{\sigma}d\bar{\varepsilon} = \alpha d\bar{\varepsilon}^{2}$$

$$\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha}d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p}}$$

Von Mises isotropic criterion

Reference state : simple tension

$$\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}} \qquad \alpha = \frac{3}{2}$$

$$\sigma_{ij} = \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow S_{ij} = \begin{pmatrix} \frac{2}{3}Y & 0 & 0 \\ 0 & -\frac{1}{3}Y & 0 \\ 0 & 0 & -\frac{1}{3}Y \end{pmatrix} \sim d\varepsilon_{ij}^{p} \sim \begin{pmatrix} d\varepsilon_{11}^{ST,p} & 0 & 0 \\ 0 & -\frac{1}{2}d\varepsilon_{11}^{ST,p} & 0 \\ 0 & 0 & -\frac{1}{2}d\varepsilon_{11}^{ST,p} \end{pmatrix}$$

$$d\bar{\varepsilon} = \sqrt{\frac{2}{3}(1+\frac{1}{4}+\frac{1}{4})} d\varepsilon_{11}^{ST,p} = d\varepsilon_{11}^{ST,p}$$

Von Mises isotropic criterion

Ref: Balanced biaxial tension case

$$\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}} \qquad \alpha = \frac{3}{2}$$

$$\sigma_{ij} = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow S_{ij} = \begin{pmatrix} \frac{1}{3}B & 0 & 0 \\ 0 & \frac{1}{3}B & 0 \\ 0 & 0 & -\frac{2}{3}B \end{pmatrix} \sim d\varepsilon_{ij}^{p} \sim \begin{pmatrix} d\varepsilon_{11}^{BB,p} & 0 & 0 \\ 0 & d\varepsilon_{11}^{BB,p} & 0 \\ 0 & 0 & -2d\varepsilon_{11}^{BB,p} \end{pmatrix}$$

$$d\bar{\varepsilon} = \sqrt{\frac{2}{3}(1+1+4)} d\varepsilon_{11}^{BB,p} = 2d\varepsilon_{11}^{BB,p} = \left| d\varepsilon_{thickness}^{BB,p} \right| = d\varepsilon_{11}^{ST,p}$$

Von Mises isotropic criterion

Pure shear case

$$\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}} \qquad \alpha = \frac{3}{2}$$

$$\sigma_{ij} = \begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow S_{ij} = \begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim d\varepsilon_{ij}^{p} \sim \begin{pmatrix} 0 & d\varepsilon_{12}^{PS,p} & 0 \\ d\varepsilon_{12}^{PS,p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d\bar{\varepsilon} = \sqrt{\frac{2}{3}(1+1)} d\varepsilon_{12}^{PS,p} = \frac{2}{\sqrt{3}} d\varepsilon_{12}^{PS,p} = d\varepsilon_{11}^{ST,p} = \left| d\varepsilon_{thickness}^{BB,p} \right|$$

while

$$\bar{\sigma} = Y = B = \sqrt{3}K$$

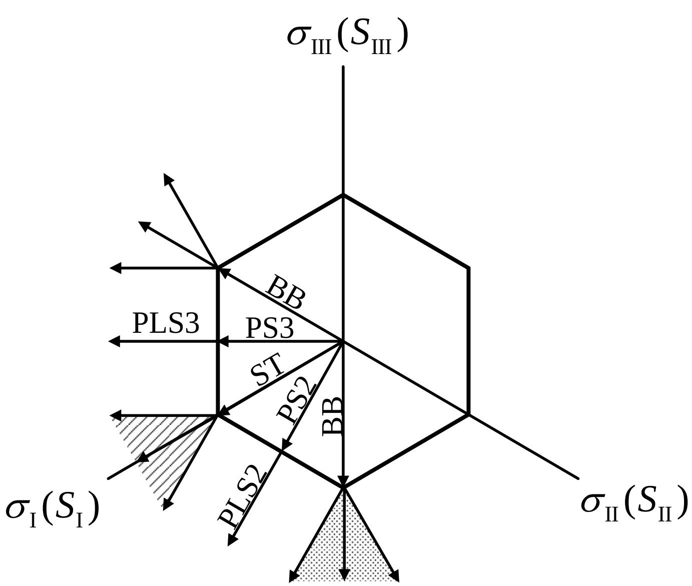
Tresca isotropic yield criterion

Tresca yield criterion

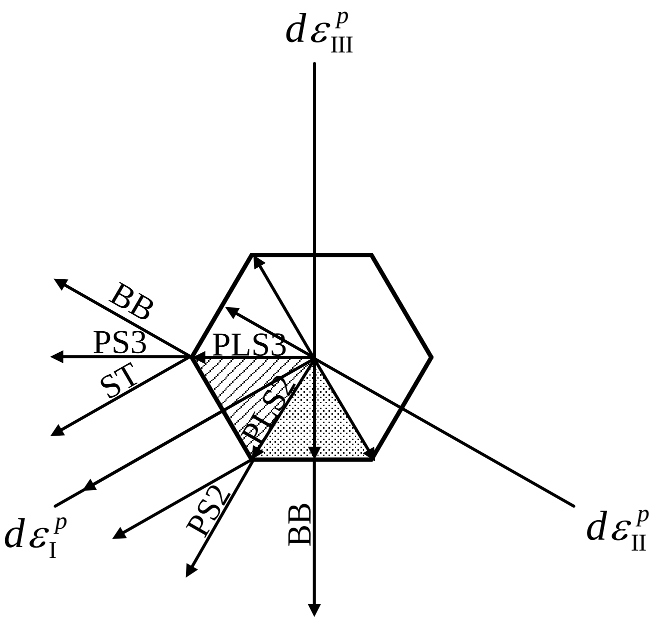
$$\bar{\sigma} = \frac{S_{max} - S_{min}}{2} \quad \text{equivalent} \quad \bar{\sigma} = \left\{ \alpha \left(\left| S_{\mathrm{I}} - S_{\mathrm{II}} \right| + \left| S_{\mathrm{II}} - S_{\mathrm{III}} \right| + \left| S_{\mathrm{III}} - S_{\mathrm{I}} \right| \right) \right\}$$

Gradient of stress or strain increment surface is **not unique** for the sharp corner

Yield surface of Tresca

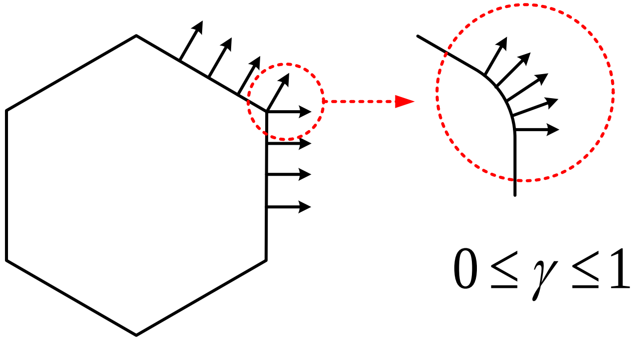


Plastic strain increment surface of Tresca



Tresca isotropic yield criterion

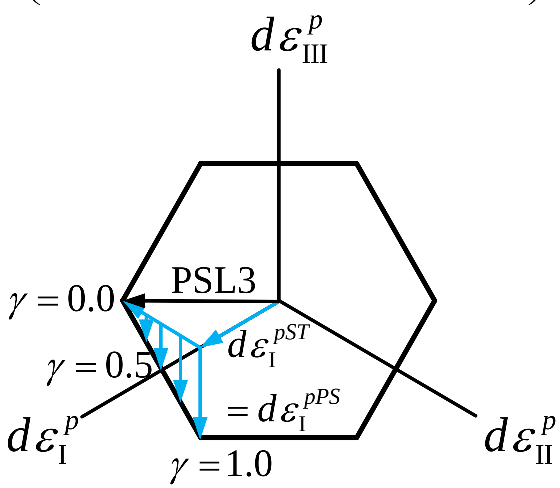
Introducing smoothing at the sharp corner of the Tresca



$$0 \le \gamma \le 1$$

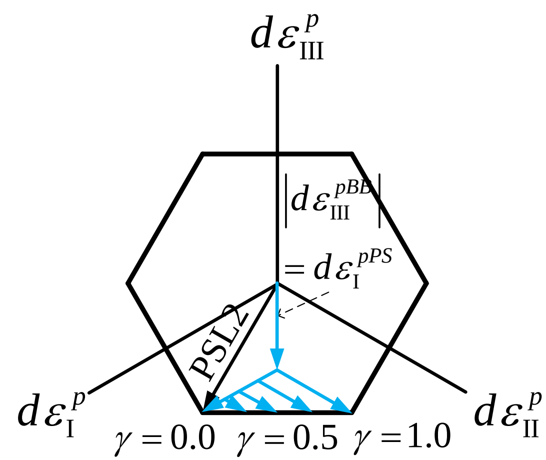
For the simple tension

$$d\boldsymbol{\varepsilon}^{pST} = \begin{pmatrix} d\varepsilon_I^{pST} & 0 & 0 \\ 0 & (-1+\gamma)d\varepsilon_I^{pST} & 0 \\ 0 & 0 & -\gamma d\varepsilon_I^{pST} \end{pmatrix}$$



For the balanced biaxial

$$d\boldsymbol{\varepsilon}^{pBB} = \begin{pmatrix} (1-\gamma)\left|d\varepsilon_{III}^{pBB}\right| & 0 & 0 \\ 0 & \gamma\left|d\varepsilon_{III}^{pBB}\right| & 0 \\ 0 & 0 & -\left|d\varepsilon_{III}^{pBB}\right| \end{pmatrix}$$

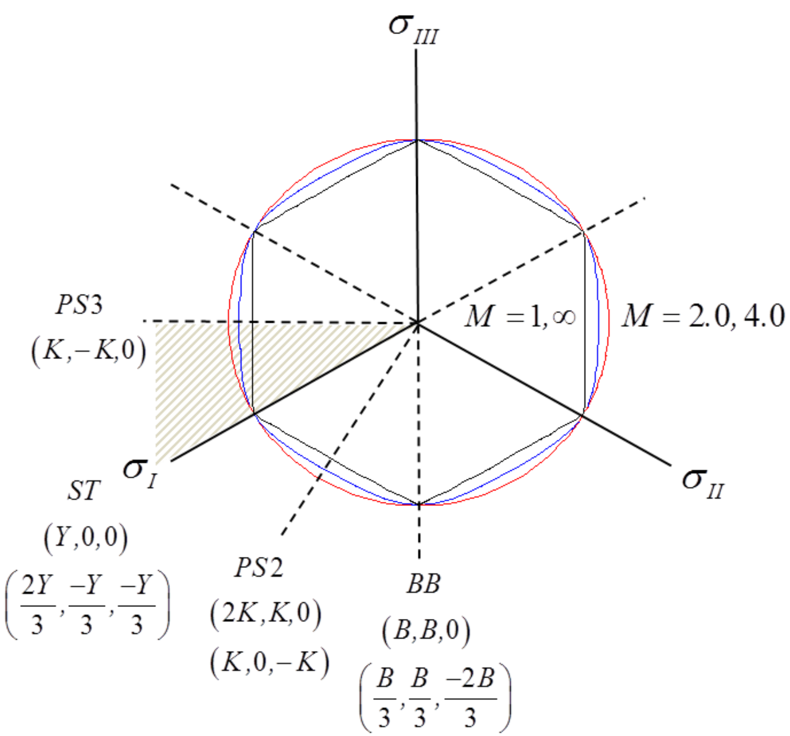


Non-quadratic isotropic incompressible yield criterion

Hosford set

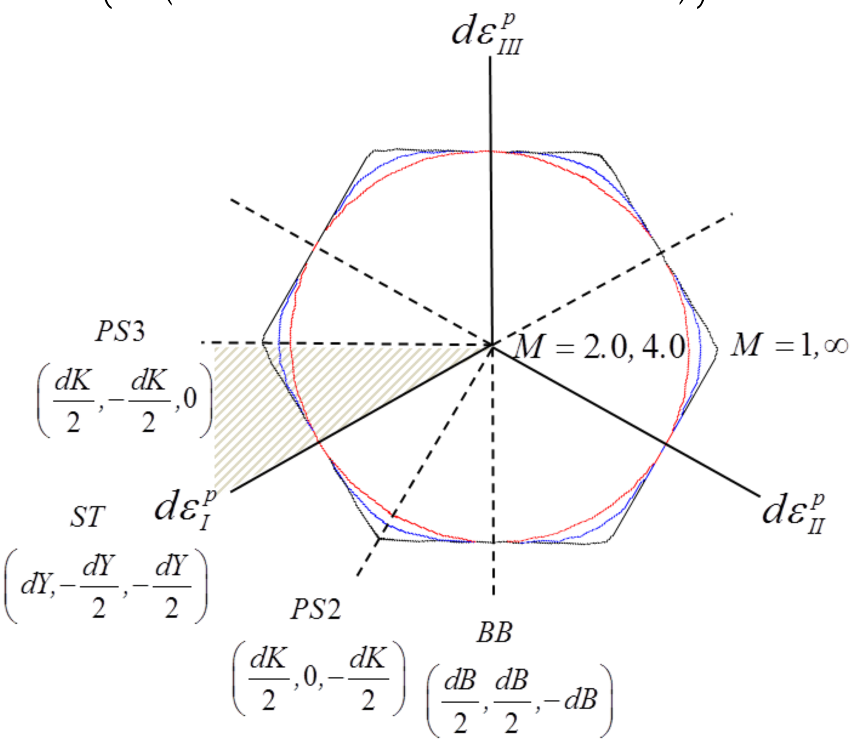
Yield surface

$$\bar{\sigma} = \left\{ \alpha \left(|S_{\mathrm{I}}|^M + |S_{\mathrm{II}}|^M + |S_{\mathrm{III}}|^M \right) \right\}^{\frac{1}{M}}$$



Plastic strain increment surface

$$d\bar{\varepsilon} = \left\{ \beta \left(|d\varepsilon_I^p|^M + |d\varepsilon_{II}^p|^M + |d\varepsilon_{III}^p|^M \right) \right\}^{\frac{1}{M}}$$

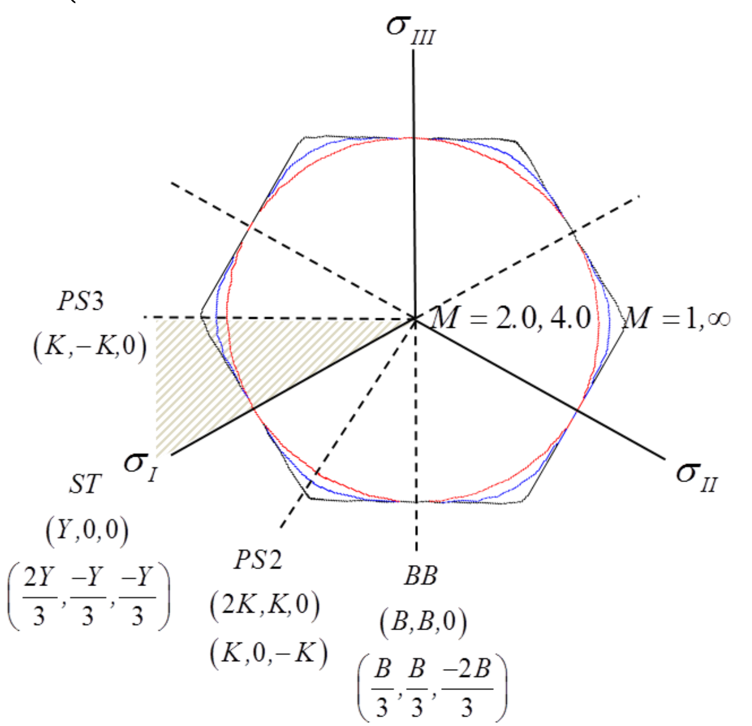


Non-quadratic isotropic incompressible yield criterion

Inverse Hosford set

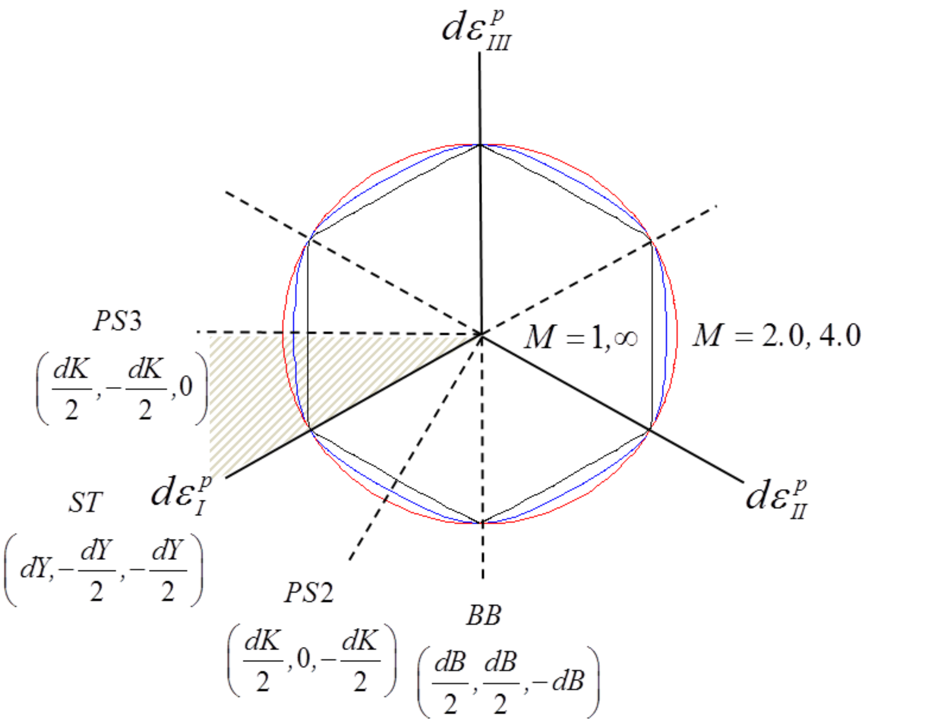
Yield surface

$$\bar{\sigma} = \left\{ \alpha \left(\left| S_{\mathrm{I}} - S_{\mathrm{II}} \right|^{M} + \left| S_{\mathrm{II}} - S_{\mathrm{III}} \right|^{M} + \left| S_{\mathrm{III}} - S_{\mathrm{I}} \right|^{M} \right) \right\}^{\frac{1}{M}}$$



Plastic strain increment surface

$$d\bar{\varepsilon} = \left\{ \beta \left(\left| d\varepsilon_I^p - d\varepsilon_{II}^p \right|^{M} + \left| d\varepsilon_{II}^p - d\varepsilon_{III}^p \right|^{M} + \left| d\varepsilon_{III}^p - d\varepsilon_I^p \right|^{M} \right) \right\}^{\frac{1}{M}}$$



Hill 1948 yield criterion

Yield surface

$$f(\boldsymbol{\sigma}) = \bar{\sigma}^2 = \mathrm{F}(\sigma_{yy} - \sigma_{zz})^2 + \mathrm{G}(\sigma_{zz} - \sigma_{xx})^2 + \mathrm{H}(\sigma_{xx} - \sigma_{yy})^2$$
$$+ 2\mathrm{L}\sigma_{yz}{}^2 + 2\mathrm{M}\sigma_{zx}{}^2 + 2\mathrm{N}\sigma_{xy}{}^2$$

Conjugate plastic strain increment surface

$$d\bar{\varepsilon}^2 = (G+H)\left\{ \frac{\left(F\left(d\varepsilon_{xx}^p\right)^2 + G\left(d\varepsilon_{yy}^p\right)^2 + H\left(d\varepsilon_{zz}^p\right)^2\right)}{(FG+GH+HF)} + \frac{2\left(d\varepsilon_{yz}^p\right)^2}{L} + \frac{2\left(d\varepsilon_{zx}^p\right)^2}{M} + \frac{2\left(d\varepsilon_{xy}^p\right)^2}{N} \right\}$$

Drucker-Prager isotropic compressible yield criterion

Drucker-Prager criterion

$$\bar{\sigma} = \sqrt{2\alpha J_2} + \beta I_1$$

From normality rule and work equivalence principle,

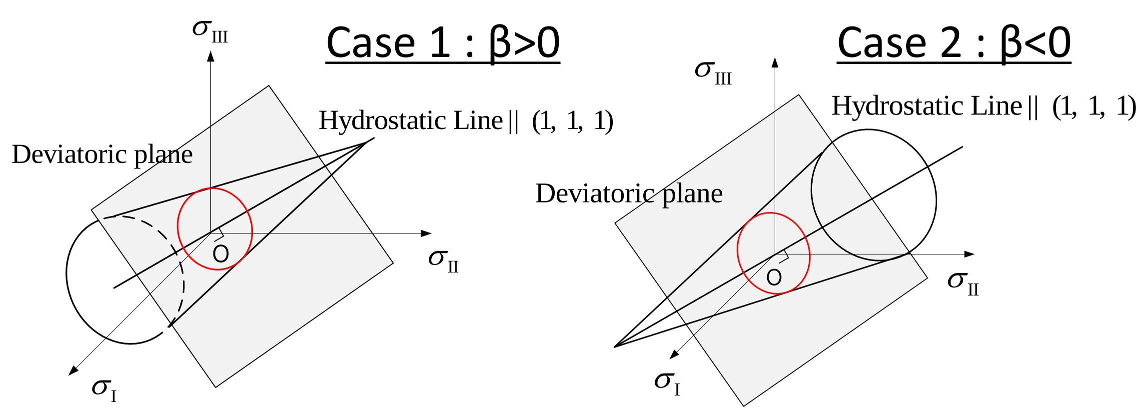
$$dw^p = \sigma_{ij} d\varepsilon_{ij}^p = S_{ij} de_{ij}^p + \frac{1}{3}(\sigma_{ii})(d\varepsilon_{jj}^p) = (\sqrt{2J_2})(\sqrt{2\tilde{J}_2}) + (\frac{I_1}{\sqrt{3}})(\frac{\tilde{I}_1}{\sqrt{3}}) = \boldsymbol{x} \cdot \tilde{\boldsymbol{x}} = \bar{\sigma} d\bar{\varepsilon}$$

Drucker-Prager criterion is axisymmetric in the principle stress space
Therefore, $\boldsymbol{S}$ and $d\boldsymbol{e}$ are parallel to each other

$$\because S_{ij} de_{ij}^p = |\boldsymbol{S}||d\,\boldsymbol{e}^p| = (\sqrt{2J_2})(\sqrt{2\tilde{J}_2})$$

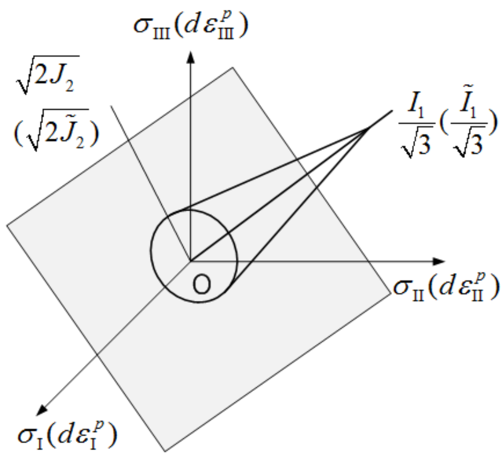
$$\because \boldsymbol{x}^T = (\frac{I_1}{\sqrt{3}}, \sqrt{2J_2})$$

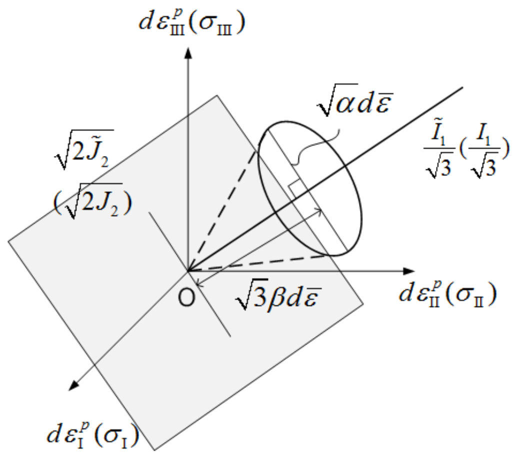
$$\because \tilde{\boldsymbol{x}}^T = (\frac{\tilde{I}_1}{\sqrt{3}}, \sqrt{2\tilde{J}_2})$$



Drucker-Prager yield criterion

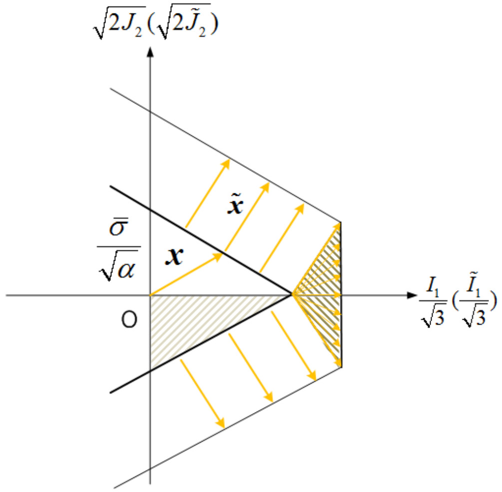
Drucker-Prager yield surface and its conjugate surface when β>0

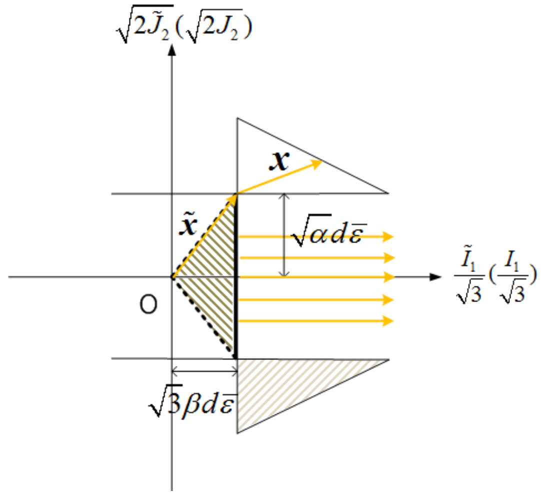




$$\bar{\sigma} = \sqrt{2\alpha J_2} + \beta I_1$$

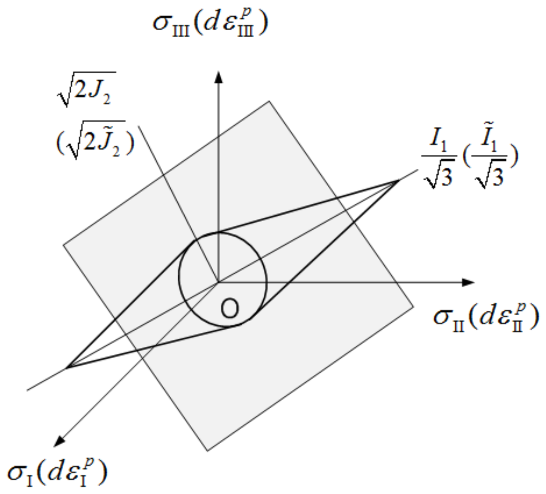
2-D side view

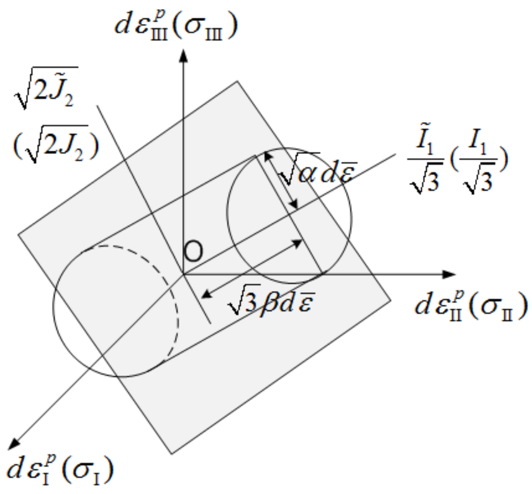




Drucker-Prager yield criterion

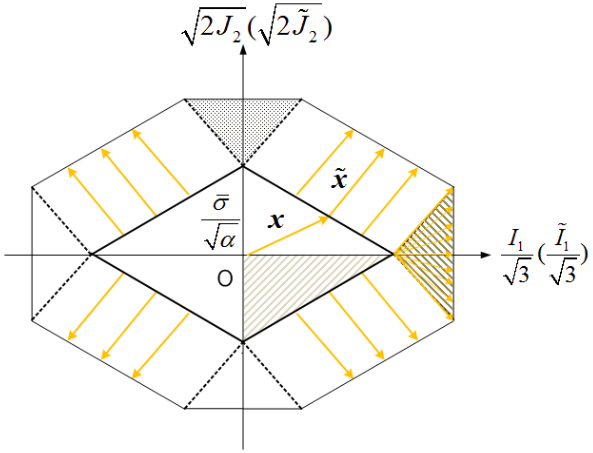
Variation of Drucker –Prager yield criterion(double cone shape case)
Yield surface and its conjugate surface

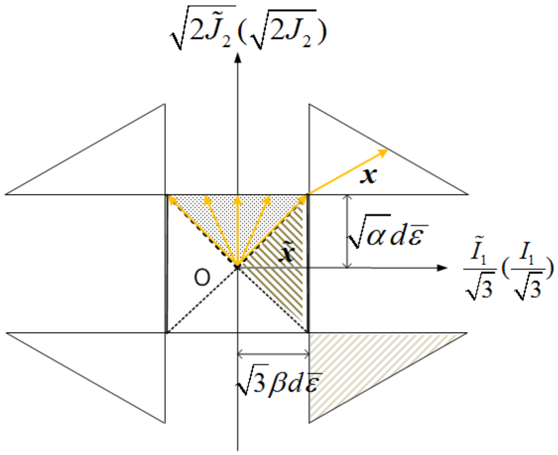




$$\bar{\sigma} = \sqrt{2\alpha J_2} + \beta |I_1|$$

2-D side view





Drucker-Prager yield criterion

Variation of Drucker –Prager yield criterion(ellipsoid shape case)
Yield surface and its conjugate surface

$$\bar{\sigma} = \sqrt{2\alpha J_2 + \beta I_1^2}$$