Lecture 20 — 28 OCT 11

Constitute equation (c more)

Irreducible representation of a dyadic.

The general second rank tensor or dyadic

$$\underset{\approx}{D} = \underbrace{\tfrac{1}{3}\, \underset{\approx}{I}\underset{\approx}{I} : \underset{\approx}{D}}_{\substack{\text{isotropic} \\ \to \text{expansion}}} + \underbrace{\tfrac{1}{2}\left(\underset{\approx}{D} + \underset{\approx}{D}^T - \tfrac{2}{3}\, \underset{\approx}{I}\underset{\approx}{I} : \underset{\approx}{D}\right)}_{\substack{\text{deviatoric} \\ \to \text{distortion}}} + \underbrace{\tfrac{1}{2}\left(\underset{\approx}{D} - \underset{\approx}{D}^T\right)}_{\substack{\text{antisymmetric} \\ \to \text{rotation}}}$$

$\underset{\approx}{D}^T$ → transpose $\quad \underset{\sim}{e}_i \underset{\sim}{e}_j D_{ij} \quad (\underset{\sim}{e}_i \underset{\sim}{e}_j D_{ij})^T = \underset{\sim}{e}_j \underset{\sim}{e}_i D_{ij}^T$

$$\nabla \underset{\sim}{u} = \tfrac{1}{3}\, \underset{\approx}{I}\underset{\approx}{I} : \nabla \underset{\sim}{u} + \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T - \tfrac{2}{3}\, \underset{\approx}{I}\underset{\approx}{I} : \nabla \underset{\sim}{u}\right) + \tfrac{1}{2}\left(\nabla \underset{\sim}{u} - \nabla \underset{\sim}{u}^T\right)$$

$$= \tfrac{1}{3}\, \underset{\approx}{I}\, \nabla \cdot \underset{\sim}{u} + \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T - \tfrac{2}{3}\, \underset{\approx}{I}\, \nabla \cdot \underset{\sim}{u}\right) + \tfrac{1}{2}\left(\nabla \underset{\sim}{u} - \nabla \underset{\sim}{u}^T\right)$$

No. affects

Constitute eqn for Newtonian fluid

$$\underset{\approx}{\tau} = -\tfrac{1}{3}(3K_b)\, \underset{\approx}{I}\, \nabla \cdot \underset{\sim}{v} - \tfrac{1}{2}(2\mu)\left[(\nabla \underset{\sim}{u}) + (\nabla \underset{\sim}{v})^T - \tfrac{2}{3}\, \underset{\approx}{I}\, \nabla \cdot \underset{\sim}{v}\right]$$

bulk viscosity

Coeff of shear viscosity
→ usually "viscosity"

$$\underset{\approx}{\tau} = -K_b\, \underset{\approx}{I}\, \nabla \cdot \underset{\sim}{v} - \mu\left[\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T - \tfrac{2}{3}\, \underset{\approx}{I}(\nabla \cdot \underset{\sim}{v})\right]$$

for incompressible flow $\nabla \cdot \underset{\sim}{v} = 0$ from continuity

$\lambda = 3K_b + 2\mu$

* Cause of flow (excerpts from Skip's lecture note)

when "some conditions" are impressed on a fluid system which make stable equilibrium impossible, flow ensues.

[Definition] flow : relative motion within the fluid. and between the fluid and its surroundings.

△ "Conditions" (What are such conditions?)

1. IMPRESSED RELATIVE MOTION

   a boundary wall moving relative to the fluid
   a moving solid object injected into the system (falling sphere)

   → violation of the condition of mechanical Eq.

$$-\nabla P + \rho \underline{g} = -\nabla P - \nabla \phi_g = -\nabla(P+\phi_g)$$

2 IMPRESSED MECHANICAL POTENTIAL DIFFERENCE

   Pressure difference uncompensated by gravitational potential and acceleration potential diff.
   (under barotropic and conservative conditions)

   → Violation of Newton's 2nd law at mech eq.

3. IMPRESSED SHEAR STRESS

   tangential traction applied to a boundary,
   as by a second fluid phase flowing tangentially along an interface between two,
   ↳ ocean

   or by a surface tension gradient developed in
   ↳ soap free surface

   → violation of the definition of fluid at mech eq.

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Boundary conditions
Mathmatical point of view
Navier stokes equation

→ Elliptic partial differential equation

$$\rho \frac{D\underset{\sim}{v}}{Dt} = \nabla \cdot \underset{\approx}{T} + \rho \underset{\sim}{g} = -\nabla p + \nabla \cdot \underset{\approx}{\tau} + \rho \underset{\sim}{g}$$

viscous stress

$$\underset{\approx}{\tau} = \mu (\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T)$$

$$\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \mu \nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{g}$$

$$\rho \left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \right) = -\nabla p + \mu \nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{g}$$

Hyperbolic

Non linear

$$(\underset{\sim}{u} + \underset{\sim}{v}) \cdot \nabla (\underset{\sim}{u} + \underset{\sim}{v})$$
$$\neq \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v}$$

B.C propagate
of inlet

Elliptic

linear → requires

$$\mu \nabla^2 (\underset{\sim}{u} + \underset{\sim}{v})$$
$$= \mu \nabla^2 \underset{\sim}{u} + \mu \nabla^2 \underset{\sim}{v}$$

⇒ requires B C for
every boundaries

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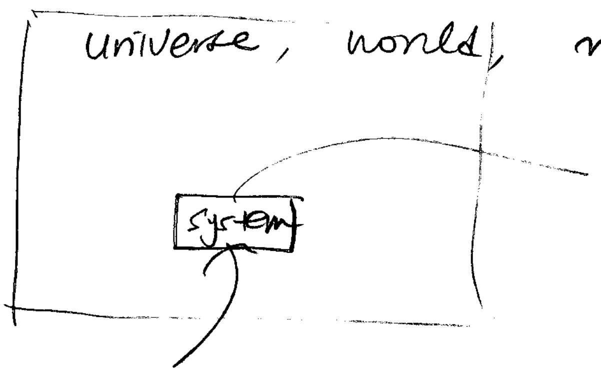
(you may want to skip this part)

When you pick Control volume



interfaces btw "universe" & your system = Boundary.

e.) { inflow boundary, outflow boundary }

Control volume

Since you want to analyze your "system" only, you do not care about "universe".

Boundaries are the places that your system interact w/ "universe".

So proper boundary conditions are essential in analysis of the target system.

So is the momentum balance (i.e. solving Navier Stokes equation).

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In side system, material interfaces also requires boundary conditions as well

ex) solid / fluid boundary — oil / pipe wall

fluid / fluid boundary — air / water

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* Typical Boundary Conditions for momentum balance equation (all comes from nature)

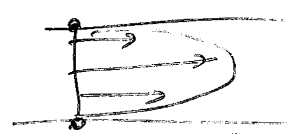
① specify or put restriction on velocity

② describe stress (or pressure)

A) fluid- Solid boundary:

$\underset{\sim}{v} = \underset{\sim}{v}_{wall}$ (vector) → No-slip Condition

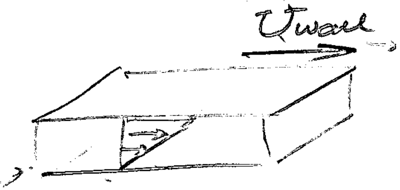
ex) pipe wall



↳ pipe does not move.

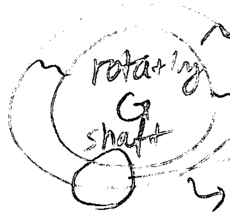
moving wall

U_{wall}



↳ lubrication flow

Bearing



lubrication liq (oil)

rotating shaft

↳ Housing

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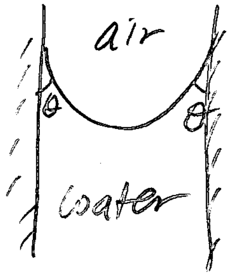
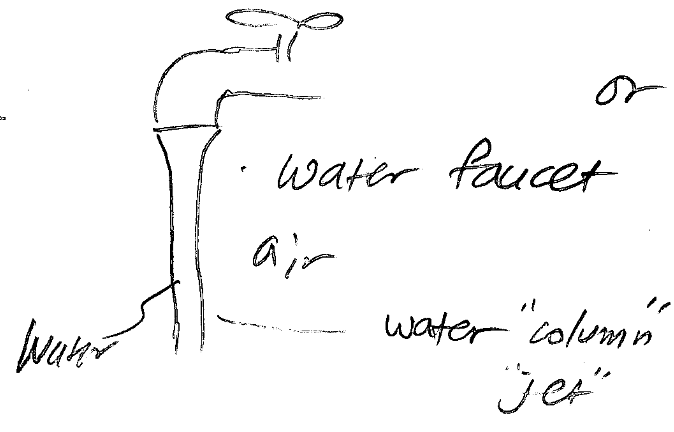
B) Fluid-fluid boundary



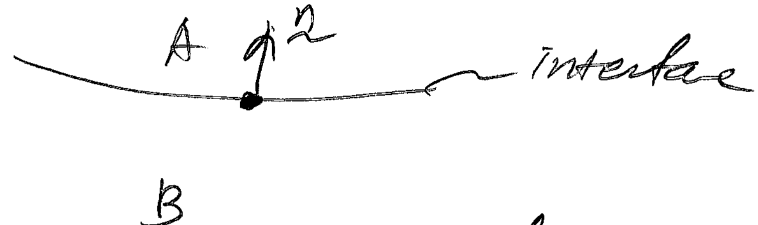
or



or

capillary rising

<u>in general</u>



We only consider <u>constant surface tension</u>.

i) $\underset{\sim}{v}_A = \underset{\sim}{v}_B$ (fluid motion of A = motion of B)
(vector condn)

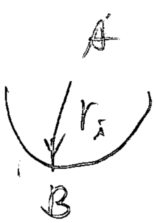
ii) stress

a) shear stress is continuous along the interface

$$\underset{\sim}{t}\,\underset{\sim}{n} : \underset{\approx}{T}^B = \underset{\sim}{t}\,\underset{\sim}{n} : \underset{\approx}{T}^A$$

b) normal stress has "jump" due to interfacial tension

"mean" curvature

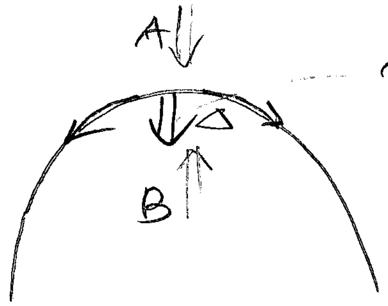


$$\underset{\sim}{n}\,\underset{\sim}{n} : \underset{\approx}{T}^B = \underset{\sim}{n}\,\underset{\sim}{n} : \underset{\approx}{T}^A - \sigma(\nabla\cdot\underset{\sim}{n})$$

$$= \underset{\sim}{n}\,\underset{\sim}{n} : \underset{\approx}{T}^A - \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$r_i > 0$ if side B is convex

Quick way to determine sign of r_i

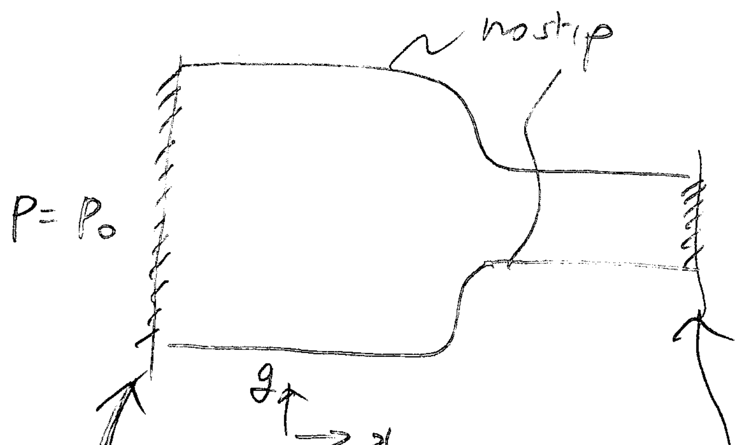


net surface force

$$\underset{\sim}{t}\underset{\sim}{n} : \underline{\underline{T}}^{B} = \underset{\sim}{t}\underset{\sim}{n} : \underline{\underline{T}}^{A} + \Delta$$

$$\Delta > 0 \rightarrow -\sigma\left(\frac{1}{r_1}\right) \qquad r_1 < 0$$

c) Open flow boundaries or in/out boundaries



i) pressure specification $P = P_0$ usually

ii) $\underset{\sim}{n}\underset{\sim}{n} : \underline{\underline{T}} = \underline{\underline{T}}_0$ sometimes

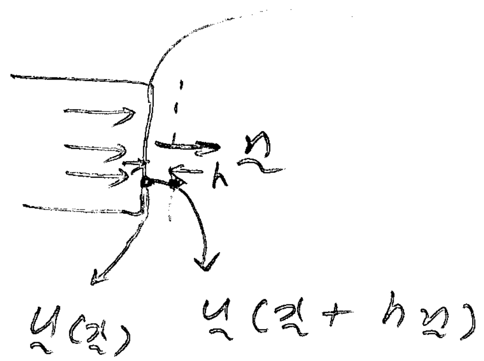
iii) set velocity profile ex) parabolic profile like pipe flow

iv) $\underset{\sim}{n} \cdot \nabla \underset{\sim}{u} = 0$ sometimes

or

$v_y = 0$ → some component vanishes or is specified fully developed flow condn

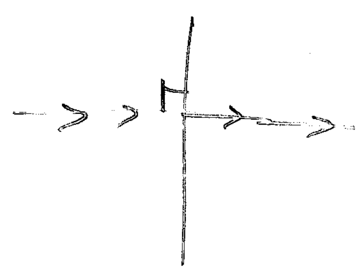
$$\underset{\sim}{n} \cdot \nabla \underset{\sim}{u} = \lim_{h \to 0} \frac{\underset{\sim}{u}(\underset{\sim}{x} + h\underset{\sim}{n}) - \underset{\sim}{u}(\underset{\sim}{x})}{h} = 0$$



$$\underset{\sim}{u}(\underset{\sim}{x} + h\underset{\sim}{n}) = \underset{\sim}{u}(\underset{\sim}{x})$$

no change along normal direction

requirement: ① out flow boundary need to be perpendicular to flow direction



or ② $\underset{\sim}{n}$ is the same as flow direction

ex) $\dfrac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \pm \underset{\sim}{n}$