Lecture 23 B.L. (Boundary Layer)

- Exact solutions of N.S are not usually available, except for few ideal cases:
  - ex. parallel channel / pipe flow
  - Irrotational flow w/ viscosity
  - etc.

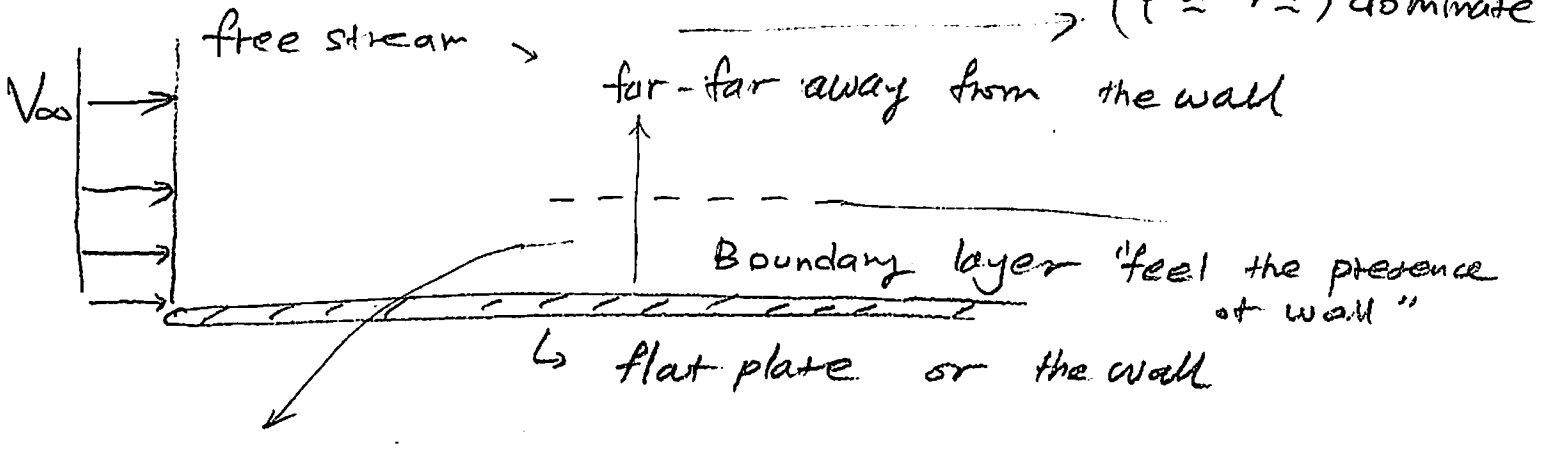
- Like lubrication approximation, one has to simplify N.S. to solve the problem

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \underbrace{\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}} = \underbrace{\mu \nabla^2 \underset{\sim}{u}} + \rho \underset{\sim}{g}$$

when steady state

$\simeq 0$	$\simeq 0$
① ρ or $\lvert \underset{\sim}{u} \rvert \downarrow$	① $\rho \uparrow$ (relatively negligible)
② $\underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \to 0$	② $\mu \downarrow$

- B.L. layer appears near the wall where "no-slip" condition applies.



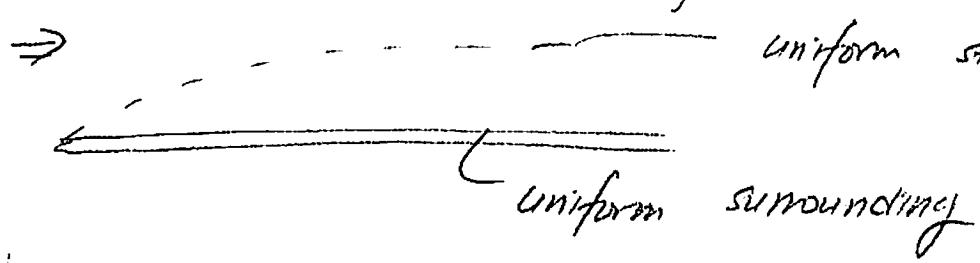
$(\mu \nabla^2 \underset{\sim}{u})$ is comparable to $(\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u})$.

◎ Meaning of B.L.

① A representative example for developing internal structure dynamically

(Laminar B.L. in the flat plate eventually turn to turbulent B.L.)

② Systems in contact w/ uniform surroundings tend to approach a uniform state of Eq.



◎ Why B.L. is important

* In chem. Eng.

Controls the rate of heat & mass transfer operation

* In aerospace Eng.

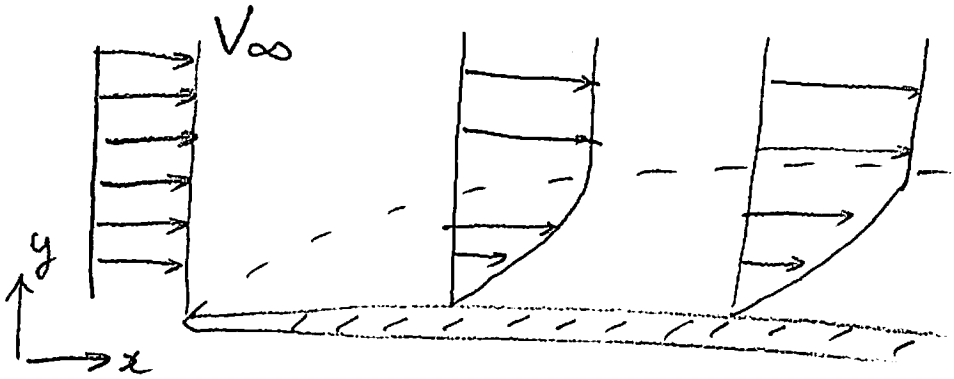
essential in determing the drag on

{ air plane
{ ship

- Key feature of B. L analysis
  - $V_x \gg V_y$

    "nearly unidirectional flow"
  - No paressure variations in transverse direction
    (Consequence of $V_x \gg V_y$)

B. L. FLOW OF INCOMPRESSIBLE FLOW



free stream (Main stream)

Boundary layer

Using order-of-magnitude analysis
(check p4 ~ 9 of the B.L note)

Prandtl's B. L. eqn (1904) → revolutionalize fluid mech.

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 V_x}{\partial y}$$

$\nu \leftarrow \frac{\mu}{\rho}$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

for B.L.

for free stream

$$V_\infty \frac{dV_\infty}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$0 = \frac{\partial p}{\partial y}$$

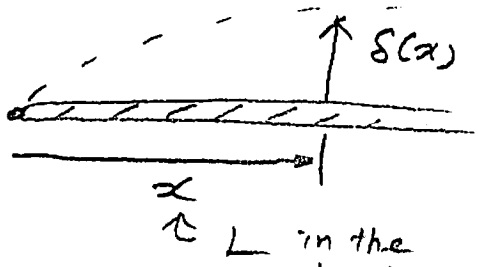
← can be derived from Bernoulli Eq

$\rho \frac{V_\infty^2}{2} + p = \text{Const}$ (along streamline)

→ differentiate w.r.t. x

Important results from B. L. analysis

* ① $$\frac{\delta(x)}{x} \approx \frac{1}{\sqrt{\frac{V_\infty x}{\nu}}} = \frac{1}{\sqrt{N_{Re_x}}}$$



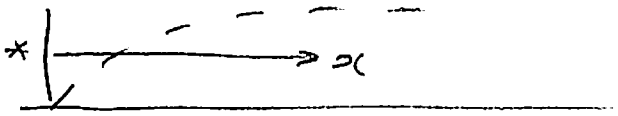
in the detailed lecture note

Consequence of balancing $(\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u})$ & $(\mu \nabla^2 \underset{\sim}{u})$.

② as $N_{Re_x} = \frac{V_\infty x}{\nu} = \frac{\rho V_\infty x}{\mu} \uparrow$ @ fixed x

- Magnitude of viscous stress $\left(\nu \frac{\partial^2 U_x}{\partial y^2}\right) \downarrow$
- $\delta(x) \downarrow$ — size of B.L. $\downarrow$

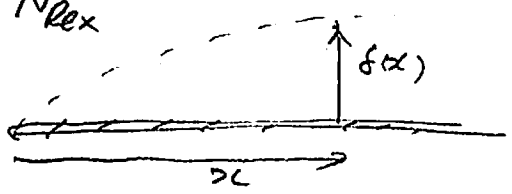
③ Normal stress for x $\frac{\partial^2 U_x}{\partial x^2}$ disappears :
parabolic in x-dir (only one b.c. req. not two)

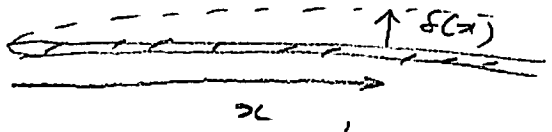


up stream event affects downstream only

④ Solution of Prandtl B.L. eqn depend on N_{Re_x} @ fixed x

$N_{Re_x}^1$ << $N_{Re_x}^2$





shape is dynamically similar

"Stretching & Compression"

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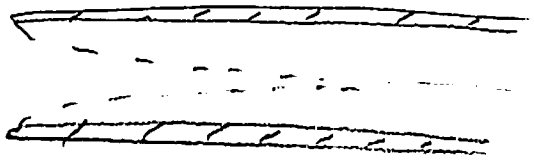
Example of B.L.) P11. of the note

ex) developing flow in channel or pipe



← eventually "flooded" with B.L.

Two approaches to handle B.L. eqn.

① Find exact solution (w/ many restrictions, very ideal case)
→ Blasius B.L. solution

② Find approximated solution (can applied to various cases)
→ von Kármán / Pohlhausen approach

Blasius B.L. Solution P13 ~ 17 of the detailed lecture note

1) Introduce stream function $\psi(x, y)$

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

2) Simplify eqn further by New variable

$$\zeta = y\sqrt{\frac{V_\infty}{\nu x}}$$

← based on $\frac{\delta}{x} = 4/\sqrt{N_{Re_x}}$ from Prandtl's analysis

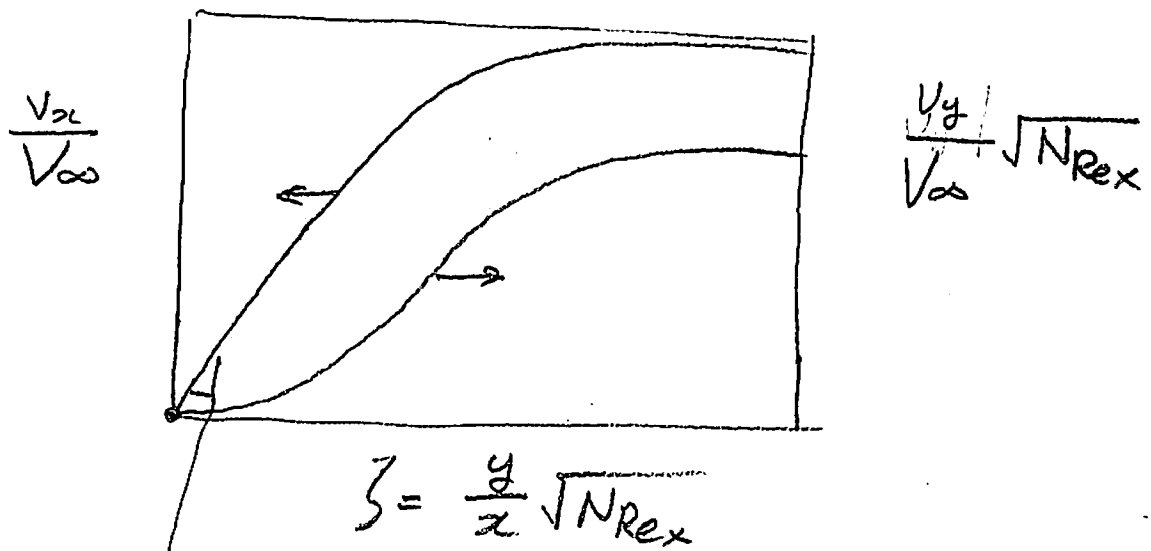
3) Using separation of variable

$$\psi = g(x) f(\zeta)$$

4) solving resulting 3rd order PDE

$$f f'' + 2 f''' = 0 \qquad \text{w/ B.Cs} \begin{cases} f = f' = 0 & @\ \zeta = 0 \\ f' = 1 & @\ \zeta \to \infty \end{cases}$$

(can solve w/ computer software)



※ Some important results from Blasius analysis

$$\left(\frac{dV_x}{dy}\right)_y = 0.332\, V_\infty \sqrt{\frac{V_\infty}{\nu x}} \qquad \text{slope @ } y=0$$

- τ_w (wall shear stress) $= 0.332 \dfrac{\rho V_\infty^2}{\sqrt{N_{Re_x}}}$ → singular @ $x=0$

$$= \frac{1}{\sqrt{N_{Re_x}}} = \sqrt{\frac{\nu}{V_\infty x}}$$

- C_f (drag coeff) $= \dfrac{\tau_w}{\frac{1}{2}\rho V_\infty^2} = \dfrac{0.664}{\sqrt{N_{Re,x}}}$

- Drag force on a finite length

$$F_D = \int_0^{\ell} \tau_w \, dx = 0.664 \sqrt{V_\infty^3 \rho \mu \ell}$$

↳ stress is integrable

- Boundary layer thickness

$$\delta_{0.99} = 4.86 \frac{x}{\sqrt{N_{Re,x}}} \quad \left(@ \ \frac{V_x}{V_\infty} = 0.99 \right)$$

this dotted line is $\delta_{0.99}$ for Blasius B.L.

◎ von Kármán / Pohlhausen approach

- von Kármán integral momentum eqn
  - ↳ he recognized that the details of Velocity are not crucial
- Pohlhausen velocity field approximation
  - ↳ he proposed 'piecewise polynomial' to approximate velocity profile.

Similar to modern numerical methods

⊙ von Kármán integral momentum eqn

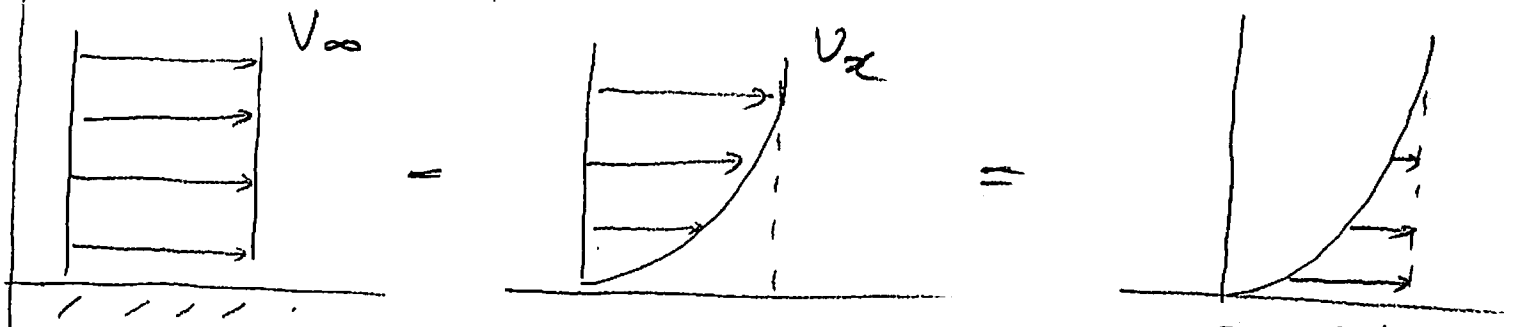
(P. 18 ~ 22 of the detailed note)

for $V_\infty = \text{Const}$ case:

$$\frac{d}{dx}\int_0^\infty \Big\{ \underbrace{\rho u_x}_{\text{mass flux}} \underbrace{(V_\infty - u_x)}_{\text{momentum per unit mass}} \Big\} dy = \underbrace{\mu \frac{\partial u_x}{\partial y}\Big|_{y=0}}_{\tau_w \text{ (wall shear stress)}} \quad \cdots \circledast$$

This eqn is the same as Eq (8.4) of Wilkes. see p22-1 of the note.

- $V_\infty - V_x$?



$$V_\infty - u_x = (V_\infty - u_x)$$

→ meaning: Wall shear stress is the rate of removal of momentum by viscous drag.

⊙ Pohlhausen approximation :

- let define dimensionless velocity $\Phi \equiv \frac{V_x}{V_\infty}$
  (piece wise polynomial)

- introduce new variable $\eta = \frac{y}{\delta(x)} = \frac{y}{(a x/\sqrt{N_{Re,x}})}$

  (Note that $\eta \neq \zeta = y x/\sqrt{N_{Re,x}}$)
  from Blasius

Plugging Φ into von Kármán eqn ⊛

$$a^2 = \frac{2 \left.\frac{d\Phi}{d\eta}\right|_{\eta=0}}{\int_0^\infty \Phi(1-\Phi)\, d\eta}$$

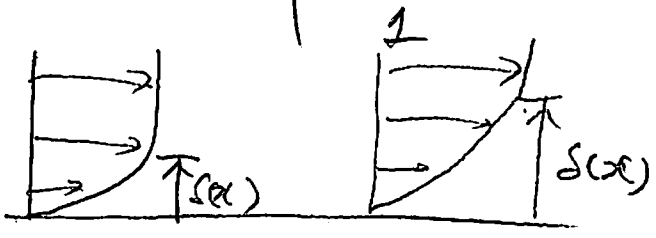
meaning : $\delta(x)$ depends on the choice of Φ

- Choice of Φ

  requirements) $\Phi(\eta)$ rise monotonically from 0 (@ $\eta=0$) to 1 (@ $\eta \to \infty$)
  $\therefore V_\infty$

ex) Pohlhausen's choice

$$\Phi = \frac{V_x}{V_\infty} = \begin{cases} \Phi(\eta) = 2\eta - 2\eta^3 + \eta^4 & : 0 \le \eta \le 1 \\ & : \eta > 1 \\ & (y > \delta(x)) \end{cases}$$



※ Some important results from von Kármán / Pohlhausen analysis

- $a = 5.84$ B.L. thickness

$$\delta(x) = 5.84\sqrt{\frac{\nu x}{V_\infty}} = 5.84\, x\sqrt{\frac{1}{N_{Rex}}}$$

$$\rightarrow \frac{\delta}{x} = \frac{5.84}{\sqrt{N_{Rex}}}$$

- τ_w (wall shear stress) $= \left| \mu \frac{\partial V_x}{\partial y}\Big|_{y=0} \right|$

$$= 0.343\, \frac{\rho V_\infty^2}{\sqrt{N_{Rex}}}$$

$$C_f \text{ (drag coeff)} = \frac{\tau_w}{\frac{1}{2}\rho V_\infty^2} = \frac{0.686}{\sqrt{N_{Rex}}}$$

(Very close to Blasius's result

$$C_f(\text{Blasius}) = \frac{0.664}{\sqrt{N_{Rex}}}$$

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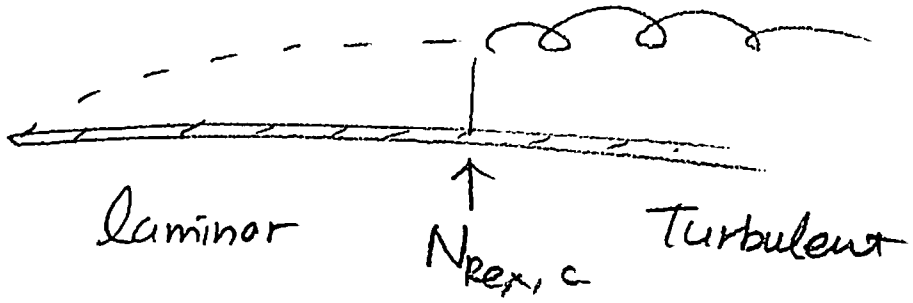
Turbulent B.L

- B.L. on the flat plate (Blasius B.L.) become unstable $\sim N_{Re_x} = N_{Re_{x,c}} \sim 3.5 \times 10^5$



- For simplicity, we follow Wilks argument (Consider the whole B.L. as turbulent B.L.)

- Here we use von Kármán / Pohlhausen approach (VKP)

$$\Phi(x,y) \equiv \frac{U_x(x,y)}{V_\infty} = \left(\frac{y}{\delta(x)}\right)^{1/7}$$

← from turbulent velocity profile in pipe flows

Wilke's choice.

- Perform analysis on Turbulent B.L. (P 27 – 29)

$$\frac{\delta}{x} = \frac{0.376}{(N_{Re_x})^{1/5}}, \quad C_f = \frac{\tau_w}{\frac{1}{2}\rho V_\infty^2} = \frac{0.0576}{N_{Re_x}^{1/5}}$$

- This analysis is valid for a "streamlined" object.

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- $\delta_{turbulent} \sim 3 \times \delta_{laminar}$ (P 30)

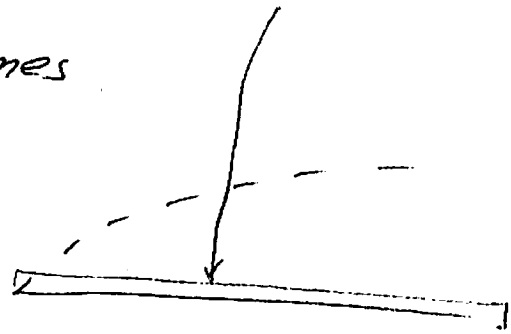
  C_f (turbulent) $\sim 4 \times C_f$ (laminar) (P31)

  ↳ holds for "streamlined" object

Boundary layer separation

- Velocity profile very closed to the wall ($y \sim 0$)

Prandtl's B.L. Eqn becomes



$$\begin{cases} 0 \doteq -\dfrac{\partial P}{\partial x} + \mu \dfrac{\partial^2 U_x}{\partial y^2} \\ \rho V_\infty \dfrac{dV_\infty}{dx} = -\dfrac{dP}{dx} \end{cases}$$

- meaning free stream flow generate pressure gradient and it will change velocity profile near the wall

$$\Rightarrow \quad \frac{\partial P}{\partial x} = \mu \frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y}\right)$$

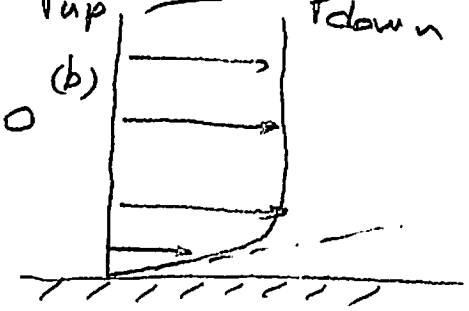
(※ Typically B.L. separation occur
① Laminar B.L.)

- Similar to lubrication approximation

Note)
We are interested in velocity profile near $y=0$.

$P_{up} > P_{down}$

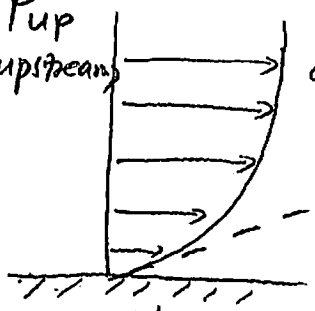
$\frac{\partial P}{\partial x} < 0$ (b)



steeper velocity profile

$\frac{\partial P}{\partial x} = 0$ (a)

P_{up} (upstream) P_{down} (down stream)



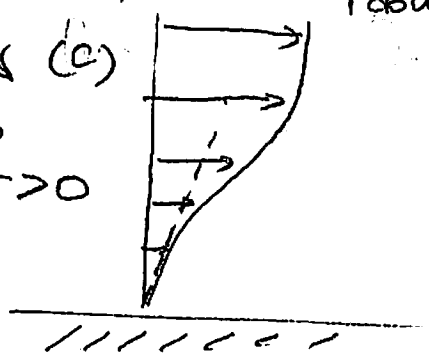
$\frac{\partial U_x}{\partial y} = \text{Const}$

B.L. Separation occur here

$P_{up} < P_{down}$

(c)

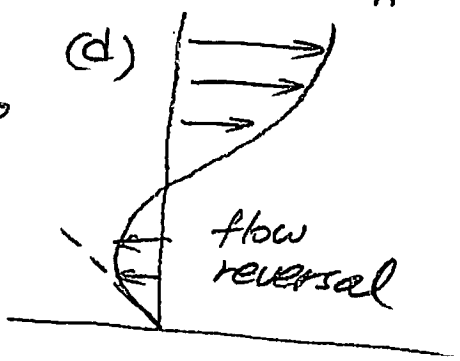
$\frac{\partial P}{\partial x} > 0$



less steep velocity profile

$P_{up} \ll P_{down}$

(d)

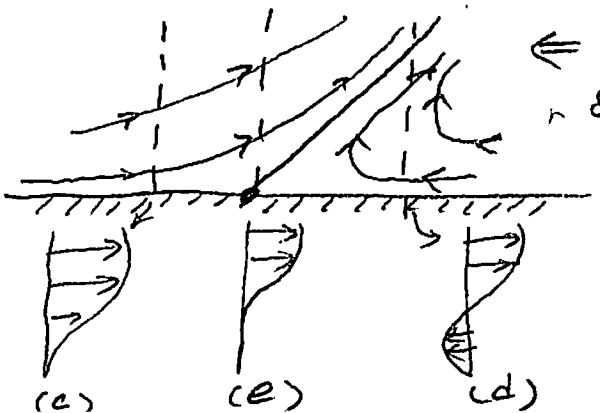


flow reversal

change sign of $\frac{\partial U_x}{\partial y}$

See page 33

* Stream line plot

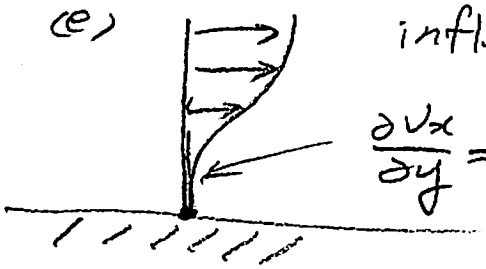


⇐ B.L. separation

(c) (e) (d)

Transition

(e)

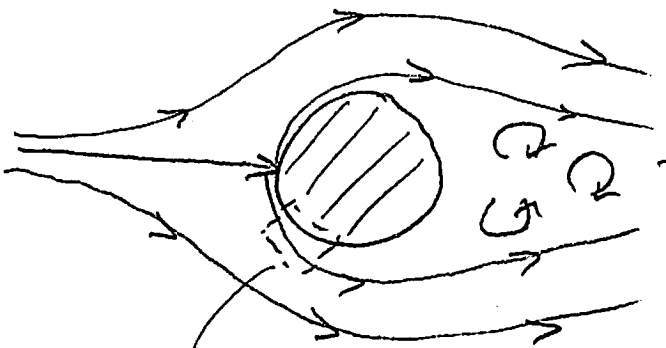


inflection point

$\frac{\partial U_x}{\partial y} \approx 0$

- B.L. separation usually accompanied w/ wake formation that causes a substantial increase in the drag force.

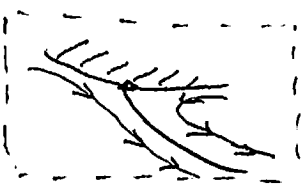
ex) Blunt object "non streamlined object"!



large wake

show van Dyke's picture.

B.L. separate!



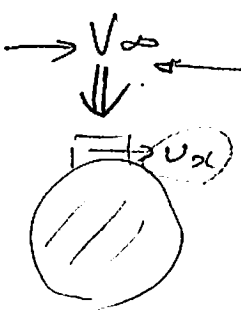
- How to delay B.L. separation?

1) make an object streamlined

ex) air plane wing

2) INDUCE TURBULENT B.L. ex) dimples on golf ball

→ why? Turbulent flow increases momentum transfer from freestream to B.L. delays B.L. separation



Induce turbulent BL

(flows need to be slowed down for the separation)