VIRTUAL WORK

Method of Virtual Work : Earliest tool for analytical mechanics

Mechanics:

- Vector Mechanics : Free body diagram for isolated body
  - Physical constraints ~> Appropriate reactions using forces or moments ! – Vector !

o Analytical Mechanics – Energy of a system

(~ motion, deformation) + External force: work done

~ System as a whole !

They yield the same results : Which one is more efficient ?

- Kinematic constraint :

Fig. 1.30

Freeze the time !!

$$\delta W = \vec{F} \bullet \delta \vec{r} \qquad (1.78)$$

or

$$\delta W = M \bullet \delta\theta \qquad (1.79)$$

: Virtual displacement ~ Principle of Virtual Work

~ static problem – D'Alembert principle ~ dynamics

At any time

$$\delta W = \vec{F}_1 \bullet \delta \vec{r}_1 + \ldots\ldots + \vec{F}_n \bullet \delta \vec{r}_n \qquad (1.80)$$

(1,81)

iff Kinematically admissible :

Virtual displacement must be compatible with the special constraint

Constraint free ?

~> Virtual displacements are independent

For static equilibrium:

$$\delta W = \vec{F} \bullet \delta \vec{r} \qquad (1.82)$$

- Orthogonality !

δ :

d :

Variation : (Total) Derivative

$$df(x_1,.....x_n,\mathrm{t}) = \frac{\partial f}{\partial x_1}dx_1 + ... + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial t}$$

$$\delta f(x_1,.....x_n,\mathrm{t}) = \frac{\partial f}{\partial x_1}\delta x_1 + ... + \frac{\partial f}{\partial x_n}\delta x_n + neglect\,!$$

Variation : (Total) Derivative

Chapther 2 Lagrangian Dynamics

Formulation of eqns of motion ~ Calculus of Variation

(Variational method)

: T and V expression in terms of $d^2x/dt^2, dx/dt, x$

Start ~ Energy expressions (T and V) : Scalar quantity ! : *Vector Mechanics*

<u>Focus</u> : Particle – single or milky way
Rigid body : 6 d.o.f.
Flexible body : Helicopter, Spacecraft..

T,V: invariant under coordinate transformation

- Newton’s Law : g : Experimental Result : (Rotation of Earth ignored
w.r.t inertial coordinate system)

; Intuitive idea for free body diagram ! Vector quantity

- Lagrangian Expression for T and V ~ Limitation and Priority !

  : Systematic approach can be set up for n - d.o.f. systems
  : Scalar quantity

<u>Important Things to Remember</u>

~ Mutual Limitation and Priority ! have to be remembered.

Vector mechanics – linear and angular momentum for isolated body : (Inertial force)

-Pendulum..g and internal force
Elevator : lope : cut at first and derive differential equation of motion

Leibniz : Change of energy due to the force (Work & potential)~ Variational method

—Lagrangian, Hamiltonian..

Analysis of Energy : Scalar quantity : Easier than *v, a*

- Differential calculus – Variatinal calculus~Optimization : Comfortable but careful !

  Vector mechanics: Applied and reacting force should be considered.

How about to deal with constraint if we choose a coordinate system ?

Ex: Pendulum : curvilinear coordinate system ?

: Easy to extend to n-d.o.f system & Continuous system? ~Mission impossible ??

Objective of this Chapter

-Formulation of Eqn of motion which is independent of a system of eqn.

-Energy or variational methods provide a very elegant way to develop the eqns of motion in a systematic way ~ Formulation is independent of any particular coordinate system.

Advantage of Lagrangian Dynamics : Relative to Vector Dynamics(Free- body ~)

1. System as a whole rather than being separated into it’s individual component :

advantages and drawbacks ?

2. Based on scalar quantities such as K.E. and Potential Energy, if we can find the quantities.

3. Forces of constraints without doing a work may not be included.
4. Use of generalized coordinates - Ex: Angles, Fourier's coefficient etc.

5. In some cases for stress analysis and design, constrained forces can be determined by

   Lagrange's Multiplier method.

   Generalized coordinates

   Coordinate systems : Descartes : 17^{th} century-one to one correspondence between physical locations ~ coordinates (In space)

   -one reference frame is chosen –well defined coordinates

   The locations of particle in space can be uniquelydefined by a set of three

   numbers in some specified Frame of Reference

Ex: (x,y,z) to transform any other 3D system

Usually, the choice is based on kinematics

Degrees of freedom

: Particle - 3 numbers are necessary and sufficient condition to uniquely specify

the physical locations of particles

- Actual motion of a particle is independent of the used to observe the motion
  – Since no one coordinate system is preferred, we can 'label' the variables as

  3 independent quantities as shown in Fig.1. - Rigid body in space; 6 coordinates.

  x <-> *q* : Invertible

- Constraint : Kinematic reactions on the motion-limit the motion of the systems

  : plane motion ?, tube ?…

- If 'R' constraint : 3 N-R degrees of freedom for number of independent generalized coordinates

Possibility of the elimination of R constraints ?

*Elimination is possible ? ,Retaining of R constraint ?

Ex: Massless spring + rigid body with mass : Infinite d.o.f – continuous system

Rigid bar with a tip mass : d.o.f =1

- Costraint in general form- Pfaffian form : $a_{jt}\,dt + a_{jk}dx_k = 0$ (j=1…R)

o Catastatic system : all $a_{jt} = 0$ for j=1,…R <>Acatastatic system : at least one a_{jt} not equal to 0

o Holonomic system : all Pfaffian forms are integrable $df_j = a_{jt}\,dt + a_{jk}dx_k$ => f_j

<> Non-Holonomic system : At least one Pfaffian form is not integrable ->

Impossible to eliminate any variable -> Must use excessive

Coordinates

o Sclenonomic system : Holonomic and t does not appear explicitly in $f_j(x_1,x_2,\ldots x_M)$: Eqn(2.5)

<> Rheonomic system : Holonomic and t appear explicitly in $f_j(x_1,x_2,\ldots x_M)$: Eqn(2.6)

o Unilateral system : (Contact problem)

: Constraint is expressed as an inequality ~ $r > a$

Ex) A bead is free to slide along a rod rotating in the *x-y* plane with a constant angular velocity about z axis.

a) Draw a model
b) Classify this system

Ex) 2-dimensional motion of a boat

- Constraint is that any translation of the center of mass of the boat must be in the direction of its heading -> Check the integrability

Ex) Non-holonomic constraint — (2.9)

Kinetic Energy and Generalized Momenta

$q_i = q_i(x_1, \ldots, x_{3N}, t)$, dq_i/dt

Ex) Car pendulum: (q_1)distance+(q_2)angular rotation angle.

One particle <-> N particles

➔ Generalization: Generalized velocity are <u>not necessarily</u> absolute velocity.

Single particle : $x_i = x_i(\underline{q_1, q_2, q_3}, t)$

$$\begin{array}{ll} | & \quad : \text{generalized} \\ | & \\ \text{physical} & \end{array}$$

Among the *N* particles, choose *i*-th particle, and then dq_i/dt can be expressed in terms of

generalized coordinates ! -> Just through a coordinate transformation by using a Chain rule !!

wrt x - > q

: Rate of change of physical coordinate x_i dependens on the rate of change of the generalized coordinates. It may also depend on time t if the change of coordinates contain t explicitly. (* Moving frame)

Kinetic energy of N-particles : Sum of $(1/2m_i dx_i/dt*dx_i/dt)$ -> Using a Chain rule and Applying Einstein summation convention):

➔ In terms of generalized coordinate : T=T(q, dq/dt ,t)

Actually depend on generalized coordinates ~ determined by the nature of transformation :

$T(q,q_1,q_2) = T_2 + T_1 + T_0$ ~ Homogeneous Quadratic + Linear + Constant

: Coordinate transformation does not dependent on t → $T_1=T_0=0$

Generalized momentum = Partial differantiation of T wrt dq_i/dt **:**

Def !

Kinetic energy/ Generalized velocity

Physical interpretation of a particular component of a generalized momentum p_i depends

on the nature of the corresponding generalized coordinates.

In 3–dimensional space : $1/2m(v_x^2+v_y^2+v_z^2)$ ~ quardratic function !

→ linear momentum

Ex) Earth surface!

Generalized coordinates may be actual x-,y- and z-components of position

Using the definition, $p_x = mdx/dt$, p_y=.. , p_z=..

In spherical coordinates, the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\cos^2\phi\,\dot{\theta}^2\right)$$

Generalized coordinates : distance, two angles

Generalized momenta conjugate to these coordinates :

$p_r = m\dot{r}$ $(linear..momentum)$ $p_\theta = mr^2\cos^2\phi\dot{\theta}$ $(angular..momentum)$ $p_\phi = mr^2\dot{\phi}$ $(angular..momentum)$

*Based on geometric configuration ~ Vector mechanics ?
→ independent of the type of generalized coordinates !

~ Vector mechanics ?
Generalized Force

Vector mechanics : Time rate of change of the momenta of a system ~ force, moment
Analytical mechanics : Geometric relationships between generalized coordinates obscure the distinction between the two momentum !

Energy concept !

Virtual Work due to the actual forces is defined as

$$\delta W = \Sigma F_i \bullet \delta r_i (i = 1...N)$$

Applying the chain rule,

$$\delta x_i = \Sigma \frac{\partial x_i}{\partial q_j}\delta q_j (j = 1...n)$$

Virtual displacement is defined for *time is fixed*

$$\delta t = 0$$

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