Chapther 2 Lagrangian Dynamics

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Introduction

Formulation of eqns of motion ~ Calculus of Variation

(Variational method)

: T and V expression in terms of $d^2x/dt^2, dx/dt, x$

Start ~ Energy expressions (T and V) : Scalar quantity ! : ***Vector Mechanics***

Focus : Particle – single or milky way
Rigid body : 6 d.o.f.
Flexible body : Helicopter, Spacecraft..

T,V : invariant under coordinate transformation :

- **Newton's Law : g : Experimental Result : (Rotation of Earth ignored w.r.t inertial coordinate system)**
  ; Intuitive idea for free body diagram ! Vector quantity

- **Lagrangian Expression for T and V ~ Limitation and Priority !**

  : Systematic approach can be set up for n - d.o.f. systems
  : Scalar quantity

*** Important Things to Remember**

~ Mutual Limitation and Priority ! have to be remembered.

Vector mechanics – linear and angular momentum for isolated body : (Inertial force)

~ Pendulum.: g and internal force

~ Elevator : lope : cut at first and derive differential equation of motion

Leibniz : Change of energy due to the force (Work & potential)~ Variational method

– Lagrangian
Hamiltonian..

Analysis of Energy: Scalar quantity : Easier than *v*, *a*

- **Differential calculus – Variatinal calculus ~ Optimization : Comfortable but careful !**

  Vector mechanics: Applied and reacting force should be considered.

How about to deal with constraint if we choose a coordinate system ?

Ex: Pendulum : curvilinear coordinate system ?

: Easy to extend to n-d.o.f system & Continuous system? ~Mission impossible ??

Objective of this Chapter

-Formulation of Eqn of motion which is independent of a system of eqn.

-Energy or variational methods provide a very elegant way to develop the eqns of motion in a systematic way ~ Formulation is independent of any particular coordinate system.

<u>Advantage of Lagrangian Dynamics</u> : Relative to Vector Dynamics(Free- body ~)

1. **<u>System as a whole</u> rather than being separated into it's individual component :**
   advantages and drawbacks ?
2. **Based on scalar quantities such as K.E. and Potential Energy, if we can find the quantities.**

3. **Forces of constraints without doing a work may not be included.**

4. **Use of generalized coordinates - Ex: Angles, Fourier's coefficient etc.**

5. **In some cases for stress analysis and design, <u>constrained forces</u> can be determined by Lagrange's Multiplier method.**

Generalized coordinates

Coordinate systems : Descartes : 17th century-one to one correspondence between physical locations ~ coordinates (In space)

-One reference frame is chosen –well defined coordinates

The locations of particle in space can be uniquely defined by a set of three numbers in some specified Frame of Reference

Ex: (x,y,z) to transform any other 3D system

Usually, the choice is based on kinematics

Degrees of freedom

: Particle - 3 numbers are necessary and sufficient condition to uniquely specify the physical locations of particles

Fig.1 A Point in space (3) : (q_1, q_2, q_3) translations

Fig.2 Generalized coordinate (6) : (q_1, q_2, q_3) translations + rotations

Fig.3 Two dimensional body (3) : (q_1, q_2) translations + rotation(q_3)

$$m\ddot{x} + \varepsilon\dot{x} + kx = 0$$

- **<u>Boundary layer theory</u>:** $$\varepsilon\ddot{x} + c\dot{x} + kx = 0$$

$$\varepsilon = 0$$

$$\varepsilon \neq 0$$

Wing model

Re

- **Actual motion of a particle is independent of the variables used to observe the motion – Since no one coordinate system is preferred, we can 'label' the variables as**

3 independent quantities as shown in Fig.1. - Rigid body in space; 6 coordinates.

***x* < - > *q*   : Invertible**

$$q_i = q_i(x_1(\mathrm{t}),\dots x_{3N}(\mathrm{t}),t) \;\dots\; x_i = x_i(\mathrm{q}_1(\mathrm{t}),\dots \mathrm{q}_{3N}(\mathrm{t}),t) \quad \textbf{(2.1),(2.2)}$$

(for i=1..,3N)

Generalized coordinate(*q*) – Physical coordinate(*x*)

Ex:Fig.2.4 Rolling cart with a pendulum

- **Free-body diagram ? Constraint force ?**

- **Degree of freedom considering the constraint.** $q_1 = x, q_2 = \theta$ **(2.3),(2.4)**
  Configuration space as Fig.2.5

Ex:4-bar linkage as in Fig.2.6 : g ? dof ?

$$(x_1, x_2) <-> (\theta, x_1) <-> (\theta, x_2)$$

Choice of coordinate : Ex) Pendulum model for 2 dof.

Refer to Page 132

Constraint

: **Kinematic reactions on the motion-limit the motion of the systems**

: **plane motion ?, tube ?…**

- **If 'R' constraint : 3 N-R degrees of freedom for number of <u>independent</u> generalized coordinates**

<u>Possibility of the elimination of R constraints ?</u>

***Elimination is possible ? , Retaining of R constraint ?**

Ex: ~<u>Massless</u> spring + rigid body with mass :

~Infinite d.o.f – continuous system

~Rigid bar with a tip mass : d.o.f = 1

- **Constraint in general form- Pfaffian form : $a_{jt}\,dt + a_{jk}dx_k = 0$ (j=1…R)** o Catastatic system : all $a_{jt} = 0$ for j=1,…R < > Acatastatic system : at least one a_{jt} not equal to 0

o Holonomic system : all Pfaffian forms are integrable $df_j = a_{jt}\,dt + a_{jk}dx_k \Rightarrow f_j$

< > Non-Holonomic system : At least one Pfaffian form is not integrable ->

Impossible to eliminate any variable -> Must use excessive

Coordinates

o Sclenonomic system : Holonomic and t does <u>not appear explicitly</u> in $f_j(x_1,x_2,\ldots x_M)$: Eqn(2.5)

< > Rheonomic system : Holonomic and t appear explicitly in $f_j(x_1,x_2,\ldots x_M)$: Eqn(2.6)

(Diagram ?)

o Unilateral system : (Contact problem)

: Constraint is expressed as an inequality ~ r > a

Ex) **A bead is free to slide along a rod rotating in the *x-y* plane with a constant angular velocity about z axis.**

a) **Draw a model**
b) **Classify this system**

Ex) **2-dimensional motion of a boat**

- **Constraint is that any translation of the center of mass of the boat must be in the direction of its heading -> Check the integrability**

Ex) **Fig.2.7, Rolling wheel without slipping:** $\dot{x} = r\dot{\theta} \quad dx = rd\theta$ **:** $x - x_0 = r(\theta - \theta_0)$
: Holonimic constraint !

Ex) **Free to change direction: Non-holonomic constraint — (2.9)**

Kinetic Energy and Generalized Momenta

$q_i = q_i(x_1,\ldots\ldots x_{3N},t)$, dq_i/dt

Ex) Car pendulum: (q_1)distance+(q_2)angular rotation angle.

One particle <-> N particles

➔ Generalization: Generalized velocity are <u>not necessarily</u> absolute velocity.

Single particle : $x_i=x_i(\underline{q_1,q_2,q_3},t)$

| **: generalized**

|

physical

Among the *N* particles, choose *i*-th particle, and then dq_i/dt can be expressed in terms of generalized coordinates ! -> Just through a coordinate transformation by using a Chain

rule !!

wrt $x -> q$

: Rate of change of physical coordinate x_i dependens on the rate of change of the

generalized coordinates. It may also depend on time t if the change of coordinates contain t

explicitly. (* Moving frame)

Kinetic energy of N-particles : Sum of $(1/2m_i dx_i/dt*dx_i/dt)$ -> Using a Chain rule and Applying Einstein summation convention):

→ In terms of generalized coordinate : T=T(q, dq/dt ,t)

Actually depend on generalized coordinates ~ determined by the nature of transformation :

$T(q,q_1,q_2) = T_2 + T_1 + T_0$ ~ Homogeneous Quadratic + Linear + Constant

: Coordinate transformation does not dependent on t → $T_1=T_0=0$

Generalized momentum = Partial differantiation of *T* wrt dq_i/dt :

Def !

Kinetic energy/ Generalized velocity

Physical interpretation of a particular component of a generalized momentum p_i depends

on the nature of the corresponding generalized coordinates.

In 3–dimensional space : $1/2m(v_x^2+v_y^2+v_z^2)$ ~ quardratic function !

→ linear momentum

Ex) Earth surface!

Generalized coordinates may be actual *x*-,*y*- and *z*-components of position

Using the definition, $p_x = mdx/dt$, p_y=.. , p_z=..

In spherical coordinates, the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\cos^2\phi\,\dot{\theta}^2\right)$$

Generalized coordinates : distance, two angles

Generalized momenta conjugate to these coordinates :

$p_r = m\dot{r}$ (linear..momentum) $\quad p_\theta = mr^2\cos^2\phi\,\dot{\theta}$ (angular..momentum) $\quad p_\phi = mr^2\dot{\phi}$ (angular..momentum)

*Based on geometric configuration ~ Vector mechanics ?
→ independent of the type of generalized coordinates !

~ Vector mechanics ?

Generalized Force

Vector mechanics : Time rate of change of the momenta of a system ~ force, moment
Analytical mechanics : Geometric relationships between generalized coordinates obscure
the distinction between the two momentum !

***Energy* concept !**

Virtual Work due to the actual forces is defined as

$$\delta W = \Sigma F_i \bullet \delta r_i \, (i = 1 \ldots N)$$

Applying the chain rule,

$$\delta x_i = \Sigma \frac{\partial x_i}{\partial q_j} \delta q_j \, (j = 1 \ldots n)$$

Virtual displacement is defined for *time is fixed*

$$\delta t = 0$$

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