Independence Test

Independence Test for Random Numbers

- In testing for independence, the hypotheses are as follows:

$$H_0: R_i \sim \text{ independent}$$
$$H_1: R_i \not\sim \text{ independent}$$

- The null hypothesis, H_0, reads that the numbers are independent. Failure to reject the null hypothesis means that no evidence of dependence has been detected on the basis of this test. This does not imply that further testing of the generator for independence is unnecessary.

Independence Test

1. Gap test

   Counts the number of digits that appear between repetitions of a particular digit and then uses the Kolmogorov–Smirnov test to compare with the expected number of gaps.

2. Run test

   Tests the runs up and down or the runs above and below the mean by comparing the actual values with expected values. The statistic for comparison is the Z–score.

Independence Test (cont.)

3. Poker test

   Treats numbers grouped together as a poker hand. Then the hands obtained are compared with what is expected using the chi-squared test.

4. Autocorrelation test

   Tests the correlation between numbers and compares the sample correlation with the expected correlation of zero.

Gap Test

- The Gap Test measures the number of digits between successive occurrences of the same digit.

(Example) length of gaps associated with the digit 3.

4, 1, 3, 5, 1, 7, 2, 8, 2, 0, 7, 9, 1, 3, 5, 2, 7, 9, 4, 1, 6, 3
3, 9, 6, 3, 4, 8, 2, 3, 1, 9, 4, 4, 6, 8, 4, 1, 3, 8, 9, 5, 5, 7
3, 9, 5, 9, 8, 5, 3, 2, 2, 3, 7, 4, 7, 0, 3, 6, 3, 5, 9, 9, 5, 5
5, 0, 4, 6, 8, 0, 4, 7, 0, 3, 3, 0, 9, 5, 7, 9, 5, 1, 6, 6, 3, 8
8, 8, 9, 2, 9, 1, 8, 5, 4, 4, 5, 0, 2, 3, 9, 7, 1, 2, 0, 3, 6, 3

(22 x 5 = 110 digits)

Note: eighteen 3's in list

==> 17 gaps, the first gap is of length 10

Gap Test (cont.)

- We are interested in the frequency of gaps.

  P(gap of 10) = P(not 3) x . . x P(not 3) x P(3)

  note: there are 10 terms of the type P(not 3) = 0.9

  P(gap size or gap length = 10) = $(0.9)^{10}$ (0.1)

- The theoretical cumulative frequency distribution of the gap size x for randomly ordered digit is given by

  $$f(x) = 0.1\sum_{n=0}^{x}(0.9)^n = 1 - 0.9^{x+1}$$

  for x = 0, 1, 2, . . (the max. gap size or 108=110−2)

Note: The observed frequencies for all digits are compared to the theoretical frequency.

Gap Test (cont.)

Example.

- Based on the frequency with which gaps occur for the digit <u>3</u>, analyze the 110 digits above to test whether they are independent. Use α= 0.05. The total number of gaps (or the number of samples) is given by the number of digits minus 10, or 100.

  * Total number of digits = $\sum_{i=0}^{9}(the \text{ number of ith } digit)$

  $$= \sum_{i=0}^{9}(n_i + 1)$$

  where n_i is the number of gaps for the i-th digit.
  Thus, total number of gaps $\sum_{i=0}^{9} n_i$
  = total number of digits −10.

- The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
# of Gaps	7	8	8	17	10	13	7	8	9	13

(total 100 gaps)

Gap Test (cont.)

Example (cont.)

Gap Length	Frequency	Relative Frequency	Cum. Rel. Frequency, $F(x)$	$S_N(x)$	$\lvert F(x)-S_N(x)\rvert$
0–3	35	0.35	0.35	0.3439	0.0061
4–7	22	0.22	0.57	0.5695	0.0005
8–11	17	0.17	0.74	0.7176	0.0224 ← D
12–15	9	0.09	0.83	0.8147	0.0153
16–19	5	0.05	0.88	0.8784	0.0016
20–23	6	0.06	0.94	0.9202	0.0198
24–27	3	0.03	0.97	0.9497	0.0223
28–31	0	0.00	0.97	0.9657	0.0043
32–35	0	0.00	0.97	0.9775	0.0075
36–39	2	0.02	0.99	0.9852	0.0043
40–43	0	0.00	0.99	0.9903	0.0003
44–47	1	0.01	1.00	0.9936	0.0064

(total 100 gaps)

Table 1 Critical values, $CV(\alpha, n)$, of the KS test with sample size n at the different levels of α.

	Level of significance (α)					
n	0.40	0.20	0.10	0.05	0.04	0.01
5	0.369	0.447	0.509	0.562	0.580	0.667
10	0.268	0.322	0.368	0.409	0.422	0.487
20	0.192	0.232	0.264	0.294	0.304	0.352
30	0.158	0.190	0.217	0.242	0.250	0.290
50	0.123	0.149	0.169	0.189	0.194	0.225
>50	$\frac{0.87}{\sqrt{n}}$	$\frac{1.07}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.37}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$

Gap Test (cont.)

- The critical value of D in Kolmogorov–Smirnov test is given by

$$D_{0.05} = 1.36 / \sqrt{100} = 0.136$$

Since $D = \max |F(x) - S_N(x)| = 0.0224$ is less than $D_{0.05}$, do not reject the hypothesis of independence on the basis of this test.

(total number of gaps = the number of samples = 100)

Run Test

➢ Up and Down

- Consider the 40 numbers; both the Kolmogorov–Smirnov and Chi-square would indicate that the numbers are uniformly distributed. But, not so.

0.08	0.09	0.23	0.29	0.42	0.55	0.58	0.72	0.89	0.91
0.11	0.16	0.18	0.31	0.41	0.53	0.71	0.73	0.74	0.84
0.02	0.09	0.30	0.32	0.45	0.47	0.69	0.74	0.91	0.95
0.12	0.13	0.29	0.36	0.38	0.54	0.68	0.86	0.88	0.91

→ Increasing

Run Test (cont.)

- Now, rearrange and there is less reason to doubt independence.

0.41	0.68	0.89	0.84	0.74	0.91	0.55	0.71	0.36	0.30
0.09	0.72	0.86	0.08	0.54	0.02	0.11	0.29	0.16	0.18
0.88	0.91	0.95	0.69	0.09	0.38	0.23	0.32	0.91	0.53
0.31	0.42	0.73	0.12	0.74	0.45	0.13	0.47	0.58	0.29

Run Test (cont.)

- Concerns:
  - ✓ Number of runs
  - ✓ Length of runs

- Note.

  If N is the number of numbers (or elements) in a sequence, the maximum number of runs is $N-1$, and the minimum number of runs is one.

  If "a" is the total number of runs in a sequence, the mean and variance of "a" is given by

Run Test (cont.)

$$\mu_a = (2N - 1) / 3$$
$$\sigma_a^2 = (16N - 29) / 90$$

- For $N > 20$, the distribution of "a" is approximated by a normal distribution, $N(\mu_a, \sigma_a^2)$.
- This approximation can be used to test the independence of numbers from a generator.

$$Z_0 = (a - \mu_a) / \sigma_a$$: the Z score

Run Test (cont.)

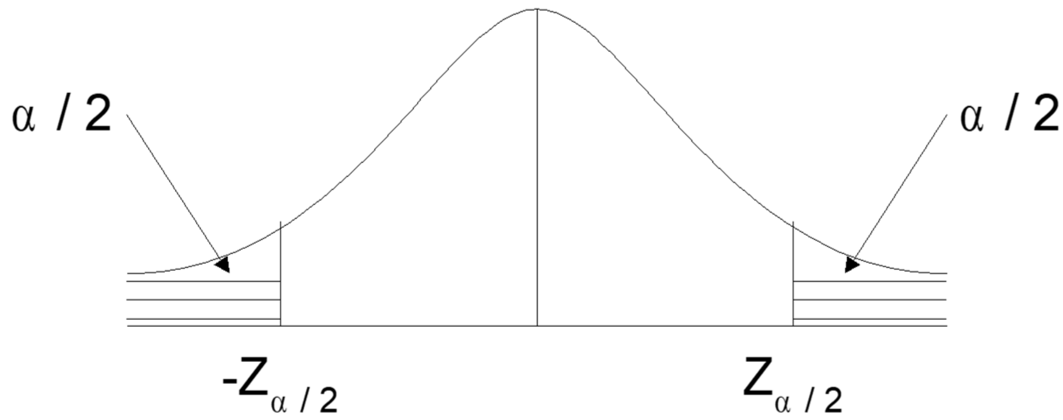
- *Substituting for* μ_a *and* σ_a ==>

$$Z_a = \{a - [(2N-1)/3]\} / \{\sqrt{(16N-29)/90}\},$$

where $Z \sim N(0,1)$

- *Acceptance region for hypothesis of independence*

$$-Z_{\alpha/2} \le Z_0 \le Z_{\alpha/2}$$



* *Too many ("up and then down"s or "down and then up"s) or too small ("monotonously increasing" or "monotonously decreasing") runs make a reason to doubt!*

Run Test (cont.)

- Example
  - Based on runs up and runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$.

0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.57	0.63	0.29

  - The sequence of runs up and down is as follows:

⊕ + + ⊖ ⊕ ⊖ ⊕ ⊖ − − ⊕ + ⊖ ⊕ ⊖ − ⊕ ⊖ ⊕

⊖ − ⊕ ⊖ − ⊕ ⊖ ⊕ + ⊖ − ⊕ + ⊖ ⊕ ⊖ − ⊕ + ⊖

Run Test (cont.)

- The sequence of runs up and down is as follows:

  + + + − + − + − − − + + − + − − + − + − − + − − + − + + − − + + − + − − + + −

  There are 26 runs (13 ups + 13 downs) in this sequence.

  With $N=40$ and $a=26$,

  $$\mu_a = \{2(40) - 1\} / 3 = 26.33 \quad \text{and}$$

  $$\sigma_a^2 = \{16(40) - 29\} / 90 = 6.79$$

  Then, $Z_0 = (26 - 26.33) / \sqrt{(6.79)} = -0.13 < Z_{0.025}$

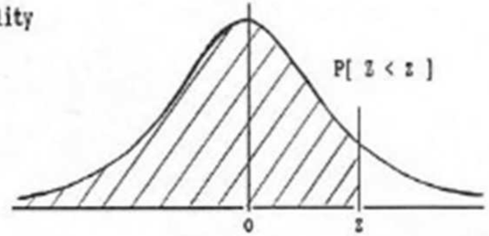
- Now, the critical value is $Z_{0.025} = 1.96$, so the independence of the numbers cannot be rejected on the basis of this test.

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z
i.e.

$$P[\, Z < z \,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

$Z_{0.025} = 1.96$

Poker Test

- based on the frequency with which certain digits are repeated.

Example:

0.255 0.577 0.331 0.414 0.828 0.909

Note: a pair of like digits appear in each number generated.

Poker Test (cont.)

- In 3-digit numbers, there are only 3 possibilities.
  - P(3 different digits)
    = (2nd diff. from 1st) * P(3rd diff. from 1st & 2nd)
    = (0.9) (0.8) = 0.72
  - P(3 like digits)
    = (2nd digit same as 1st) * P(3rd digit same as 1st)
    = (0.1) (0.1) = 0.01
  - P(exactly one pair) = 1 − 0.72 − 0.01 = 0.27 or
    = (2nd digit same as 1st) * (3rd diff. from 1st)
    + (2nd digit diff. from 1st) * (3rd same as 1st or 2nd)
    = (0.1) (0.9) + (0.9) (0.2) = 0.27

Poker Test (cont.)

- Example

  A sequence of 1000 three-digit numbers has been generated and an analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits, and 31 contain three like digits. Based on the poker test, are these numbers independent?

  Let $\alpha = 0.05$.

  The test is summarized in the next table.

Poker Test (cont.)

Example (cont.)

Combination, i	Observed Frequency, O_i	Expected Frequency, E_i	$\frac{(O_i - E_i)^2}{E_i}$
Three different digits	680	720	2.24
Three like digits	31	10	44.10
Exactly one pair	289	270	1.33
	1000	1000	47.65

- The appropriate degrees of freedom are one less than the number of class intervals.
- Since $\chi^2_{0.05,\,2}$ (= 5.99) < X^2 (=47.65), the independence of the numbers is rejected on the basis of this test.

- *Chi-Squared Distribution*

chi-square distribution table

χ^2_p = the χ^2 value such that the area to its right is *p*.

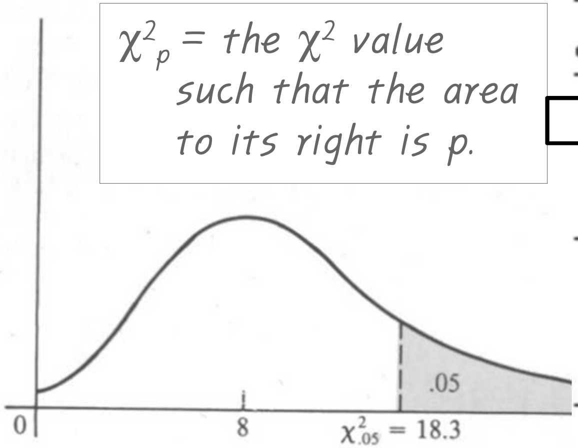


FIGURE 4.5
$P[X^2_{10} \geq \chi^2_{.05}] = .05$ and $P[X^2_{10} < \chi^2_{.05}] = .95$.

df \ p	0.995	0.975	0.9	0.5	0.1	0.05	0.025	0.01	0.005	df
1	.000	.000	0.016	0.455	2.706	3.841	5.024	6.635	7.879	1
2	0.010	0.051	0.211	1.386	4.605	5.991	7.378	9.210	10.597	2
3	0.072	0.216	0.584	2.366	6.251	7.815	9.348	11.345	12.838	3
4	0.207	0.484	1.064	3.357	7.779	9.488	11.143	13.277	14.860	4
5	0.412	0.831	1.610	4.351	9.236	11.070	12.832	15.086	16.750	5
6	0.676	1.237	2.204	5.348	10.645	12.592	14.449	16.812	18.548	6
7	0.989	1.690	2.833	6.346	12.017	14.067	16.013	18.475	20.278	7
8	1.344	2.180	3.490	7.344	13.362	15.507	17.535	20.090	21.955	8
9	1.735	2.700	4.168	8.343	14.684	16.919	19.023	21.666	23.589	9
10	2.156	3.247	4.865	9.342	15.987	18.307	20.483	23.209	25.188	10
11	2.603	3.816	5.578	10.341	17.275	19.675	21.920	24.725	26.757	11
12	3.074	4.404	6.304	11.340	18.549	21.026	23.337	26.217	28.300	12
13	3.565	5.009	7.042	12.340	19.812	22.362	24.736	27.688	29.819	13
14	4.075	5.629	7.790	13.339	21.064	23.685	26.119	29.141	31.319	14
15	4.601	6.262	8.547	14.339	22.307	24.996	27.488	30.578	32.801	15

Correlation Test

- ✓ Generate $U_0, U_1, \ldots, U_n$
- ✓ Compute an estimate for the (serial correlation)

$$\hat{\rho}_1 = \frac{\sum_{i=1}^{n}(U_i - \bar{U}(n))(U_{i+1} - \bar{U}(n))}{\sum_{i=1}^{n}(U_i - \bar{U}(n))^2}$$

where $\bar{U}(n)$ is the sample mean.

- ✓ If the U_is are really i.i.d. U(0.1), then $\hat{\rho}_1$ should be close to zero.

* "Consequential up, down and then up"s makes $\hat{\rho}_1$ be negatively large. Also, "consequential ups" or "consequential downs" make $\hat{\rho}_1$ be positively large!

- ✓ For large n, $p(-2/\sqrt{n} \le \hat{\rho}_1 \le 2/\sqrt{n}) \approx 0.95$.

Thus reject H_0 at the 5% significance level

if $\hat{\rho}_1 < -2/\sqrt{n}$ or $\hat{\rho}_1 > 2/\sqrt{n}$.