Variance Reduction: Practices

SNU/NUKE/EHK

Variance Reduction Methods

- Variance reduction by
  (1) stratification of using sub-regions in the domains of certain variables, which does not change the distribution from which samples are selected; or
  (2) importance sampling or biasing, which changes the distributions from which the samples are selected.

  - The key issue for such change is to leave the mean value of the result unchanged while reducing the variance of the sample values.

Modification of Distribution

- One can select the values of a variable from a modified distribution function

$$V'(x) = V(x)f(x)/g(x), \qquad (1)$$

  - The modified distribution V'(x) guarantees the same answer as the original distribution V(x).
  - Careful selection of g(x) can leads to a reduction in variance of the result or a gain in calculational efficiency.

- Efficiency ε is defined as "inversely proportional to the product of the sampling variance and the amount of labor expended in obtaining the estimate":

$$\varepsilon = \frac{1}{T\,\sigma^2}\,, \qquad (2)$$

where T = the run time and σ^2 = variance of result.

Importance Sampling: Source Biasing

- To sample from a biased source distribution in which the probability of selecting a source particle of high importance, one that makes a relatively large contribution to the result, is greater than that of selecting a particle of low importance.

Ex. Source Biasing: leakage of particles from a slab

- Problem Definition:
  - an isotropic source of monoenergetic particles emitted uniformly throughout a homogeneous slab of scattering and absorbing material.
  - Calculate the probability of a particle escaping from the slab.
- Key points in simulation:
  - The probability varies strongly with the source location, which requires the source biasing in location.
  - Focus only on the Z-coordinates of particles' locations.

Importance Sampling: Survival Biasing

- To avoid killing particles by absorption, while still playing a fair game.
  - Particles are never killed by absorption.
  - To ensure a fair game, a weight W is set to the particle leaving the collision:

$$W_{after} = W_{before} \times (1 - \Sigma_a/\Sigma_t)$$

- May take quite a time in computation for tracking particles with small weights, which would make small contributions to the result.

Ex. Survival Biasing: particles passing through a slab

- Problem Definition:
  - a parallel beam of monoenergetic neutrons normally incident on a homogeneous slab.
  - Assume the isotropic scattering in the L system.
  - Calculate the probability of a particle being transmitted through or reflected from the slab.
- Keypoints in simulation:
  - A particle track is terminated by transmission, reflection, or absorption. → by transmission and reflection only.
  - The variance reduction must be sufficient to compensate the effects of increased run time.

Importance Sampling: Russian Roulette

- To spend the computation effort on tracking particles that have a chance of contributing significantly to the result.
  - The score contribution of a particle is always proportional to its weight.
  - RR is a means of killing light-weight particles while maintaining a fair-game.
  - The total weight of particles that are tracked is conserved by assigning the weight carried by the particles that are killed to that of the surviving particles.

Importance Sampling: Russian Roulette (cont.)

Method 1

- Set the lower limit of particle weight for tracking, W_L.
- The particle is killed with some fixed probability p_k.
- Compare the weight W of each particle entering a zone or experiencing a collision with W_L .
  - If $W \geq W_L$, the tracking continues with no change.
  - If $W < W_L$, take a random number ξ.

    if $\xi < p_k$, the particle is killed.

    if not, the particle survives and is assigned with a new weight

$$W' = W/(1-p_k) \qquad p_k \cdot 0 + q_k \cdot W' = W;\ W' = W/q_k = W/(1-p_k)$$

- If the new weight is still below W_L, the process is repeated.

Importance Sampling: Russian Roulette (cont.)

Method 2

- Set the lower limit of particle weight for tracking, W_L.
- The weight of the surviving particles are fixed as W_A.
- Compare the weight W of each particle entering a zone or experiencing a collision with W_L .
  - If $W \geq W_L$, the tracking continues with no change.
  - If $W < W_L$, the particle is subject to Russian roulette and is killed with the probability.

$$p_k = 1 - W/W_A \quad (p_k \cdot 0 + q_k \cdot W_A = W;\ q_k = W/W_A;\ p_k = 1 - q_k)$$

Take a random number ξ. If $\xi < p_k$, the particle is killed. if not, the particle is assigned the weight W_A.

  - $W_A > W_L$ compensated with large p_k.

Importance Sampling: Russian Roulette (cont.)

Method 2 (cont.)

- Since the probability of the particle surviving is equal to W/W_A and the game is played only if $W < W_L$, the ratio W_L/W_A controls the probability with which a particle survives the game.
  - If $(W \leq)\ W_L \ll W_A$, few particles subjected to RR would survive.

    Hence it is customary to set W_A within an order of magnitude of W_L.
  - Set $p_k = 1 - W/W_A \sim 0.9\ (\geq 1 - W_L/W_A)$ for W_L of choice.

- The value of W_L and thus the value of W_A can vary throughout the geometry.

Ex. Russian Roulette: particles passing through a slab

- Problem Definition:
  - a parallel beam of monoenergetic neutrons normally incident on a homogeneous slab.
  - Assume the isotropic scattering in the L system.
  - Calculate the probability of a particle being transmitted through or reflected from the slab.

- Keypoints in simulation:
  - For thin regions, playing the RR game only after collisions would save the run time by eliminating the expenditure of computation for the particles that have a high probability of passing through the region without suffering a collision.

Importance Sampling: Splitting

- To induce particles to have roughly equal weights
  - A wide disparity in particle weight leads to a wide disparity in scores contributed by these particles.
  - Approximately equal scores produce low variance.

- Splitting is to keep the weights of the particles below some maximum value W_H while RR is to keep the weights of particles above some minimum value W_L.
  - When RR and splitting are used in combination, one can define a "weight window" such that the weights of particles are restricted to values in this range:

$$W_L \le W \le W_H$$

Importance Sampling: Splitting (cont.)

- If $W > W_H$ (some higher weight limit),
  - The particle is split into a fixed number n_k with a new weight W':
    $$W' = W/n_k .$$
    If W' is still greater than W_H, splitting is played again until W' become less than W_H.
  - The particle may be split into $\{[W/W_H] + 1\}$ particles, by which the new weight W' is always less than W_H.
  - If one has a target value of $W' = W_T$, there are produced n particles of weight $W' = W_T$ and one particle of weight $W' = W_R = W - nW_T$ where $nW_T \leq W \leq (n+1)W_T$.

Ex. Splitting & RR: particles passing through a slab

- Keypoints in simulation:
  - The best choice of W_L and W_H depend on the size of an importance region.
  - A narrow weight window in a large region might result in more time spent in splitting and killing particles than in tracking them.
  - With a large number of regions, the efficiency may be decreased by imposing excessive boundary crossings.

Importance Sampling: Exponential Transformation

In a deep-penetration transport

- An accurate answer requires a thorough sampling of the phase space near the detector.
- A fair-game plays by biasing particle flow toward the area of interest and thus increasing the fraction of computation devoted to the sampling of the important region of phase space.

- Exponential transformation or path stretching
  - to sample flight paths greater than one mfp when a particle is moving toward the detector while sampling flight paths shorter than one mfp when a particle is moving away from the detector.

Importance Sampling: Exp. Transformation (cont.)

<u>In an unbiased calculation</u>

- The flight path of a particle is selected from the distribution $p(\eta)$ given by

$$p(\eta)d\eta = e^{-\Sigma_t \eta}\Sigma_t d\eta \ . \qquad (3)$$

- Such a sampling results in a path length x as given by

$$x = \frac{-\ln \xi}{\Sigma_t}. \qquad (4)$$

Importance Sampling: Exp. Transformation (cont.)

<u>In a biased calculation</u>

- Define an artificial interaction probability $p^*(\eta)$ such that

$$p^*(\eta)d\eta = e^{-\Sigma_t^*\eta}\Sigma_t^* d\eta \ , \qquad (5) \quad \text{where } \Sigma_t^* = \Sigma_t - g(r,E,\vec{\Omega}) \qquad (6)$$

where $g(r,E,\vec{\Omega})$ can be positive or negative depending on location, energy, and direction of the particle as long as one maintains $\Sigma_t^* > 0$ at locations where $\Sigma_t > 0$.

- Select a new flight path by using the modified cross section

$$x^* = \frac{-\ln \xi}{\Sigma_t^*}. \qquad (7)$$

Importance Sampling: Exp. Transformation (cont.)

In a biased calculation (cont.)

- If $g(r,E,\vec{\Omega}) > 0$, x^* is longer than the unmodified flight path x (path stretching): $x^* > x$.
- To make a fair game at the selected x^*,

$$p(x^*)W = p^*(x^*)\,W^* \tag{8}$$

or

$$W^* = W \cdot \frac{\Sigma_t}{\Sigma_t - g(r,E,\vec{\Omega})} e^{-g(r,E,\vec{\Omega})x^*} \tag{9}$$

  – For $g(r,E,\vec{\Omega}) > 0$, $W^* < W$ for larger x^* and $W^* > W$ for smaller x^*.

- necessary to restrict $g(r,E,\vec{\Omega}) < \Sigma_t$.

Importance Sampling: Exp. Transformation (cont.)

In a biased calculation (cont.)

- Define a normalized exponential transform parameter ρ

$$\rho = g(r,E,\vec{\Omega})/\Sigma_t \ , \qquad (10)$$

such that $-1 < \rho < 1$ and thus $-\Sigma_t < g(r,E,\vec{\Omega}) < \Sigma_t$.

- One can define B as

$$\boxed{B} = \frac{1}{1-\rho} = \boxed{\frac{\Sigma_t}{\Sigma_t^*}}. \qquad \left(\frac{1}{2} < B < \infty\right) \qquad (11)$$

Then $g(r,E,\vec{\Omega}) = \Sigma_t\left(1-\frac{1}{B}\right)$ (12) and $\boxed{x^* = Bx.}$ (13)

from (4) and (7)

Importance Sampling: Exp. Transformation (cont.)

<u>In a biased calculation (cont.)</u>

- Substituting (12) into (9) gives

$$W^* = WBe^{-\Sigma_t x^*\left(1-\frac{1}{B}\right)} . \qquad (14)$$

not Σ_t but Σ_t^*

- It is common to define ρ as follows:
  - If the direction of travel of a particle following a collision is $\vec{\Omega}$ and the unit vector from the collision point to a detector is $\vec{\Omega}'$, one may define

$$\rho = \rho_0 \vec{\Omega} \cdot \vec{\Omega}' \qquad \text{where } \rho_0 > 1. \qquad (15)$$

  - The maximum value of the stretching parameter $\rho_{max} = \rho_0$
  - $\rho = \rho_0$ when the particle is directed toward the detector; $\rho = -\rho_0$ when it is directed away from the detector.

Homework #4