Shared Counters and Parallelism

Companion slides for
The Art of Multiprocessor Programming
by Maurice Herlihy & Nir Shavit

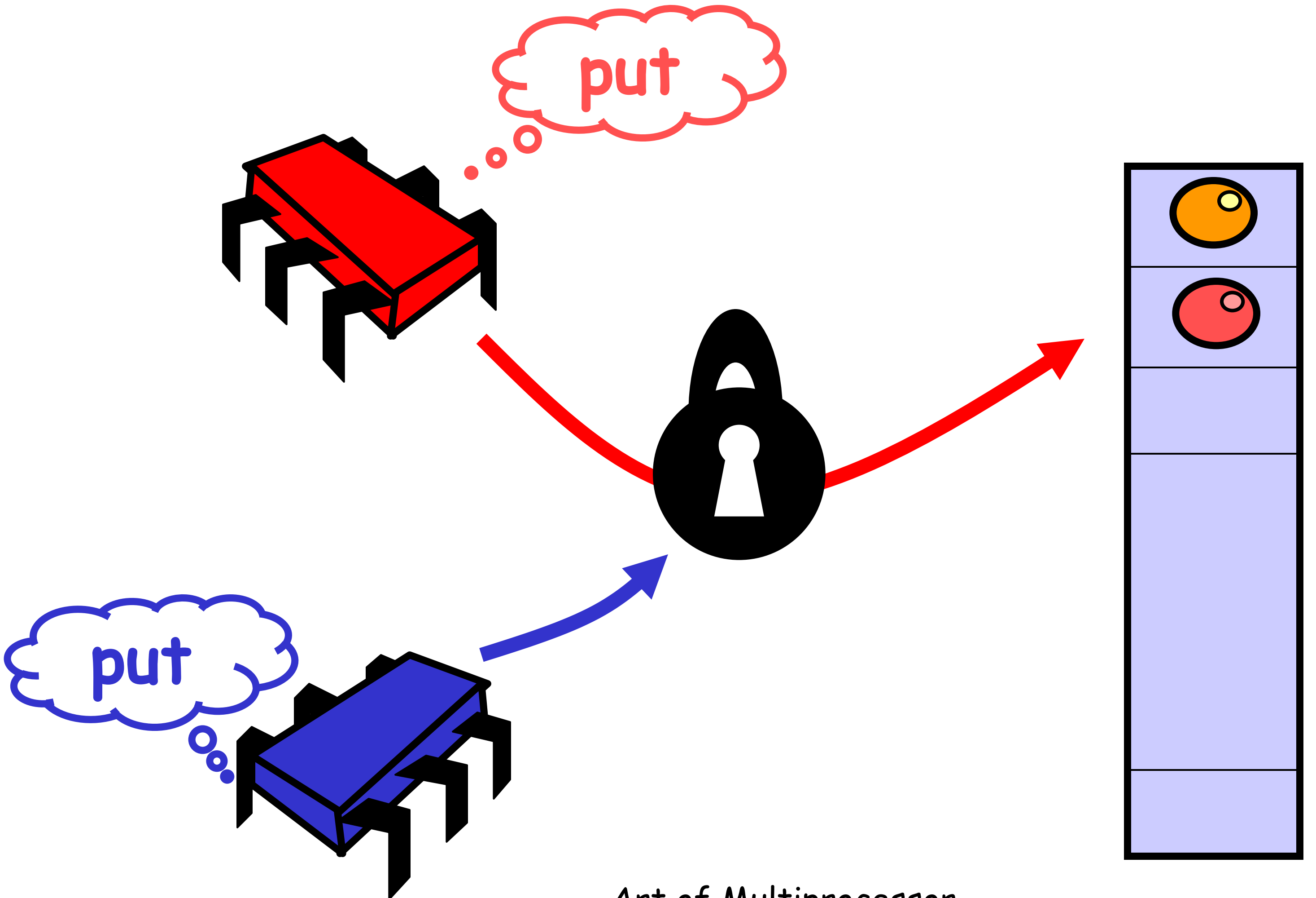
A Shared Pool

```
public interface Pool {
  public void put(Object x);
  public Object remove();
}
```

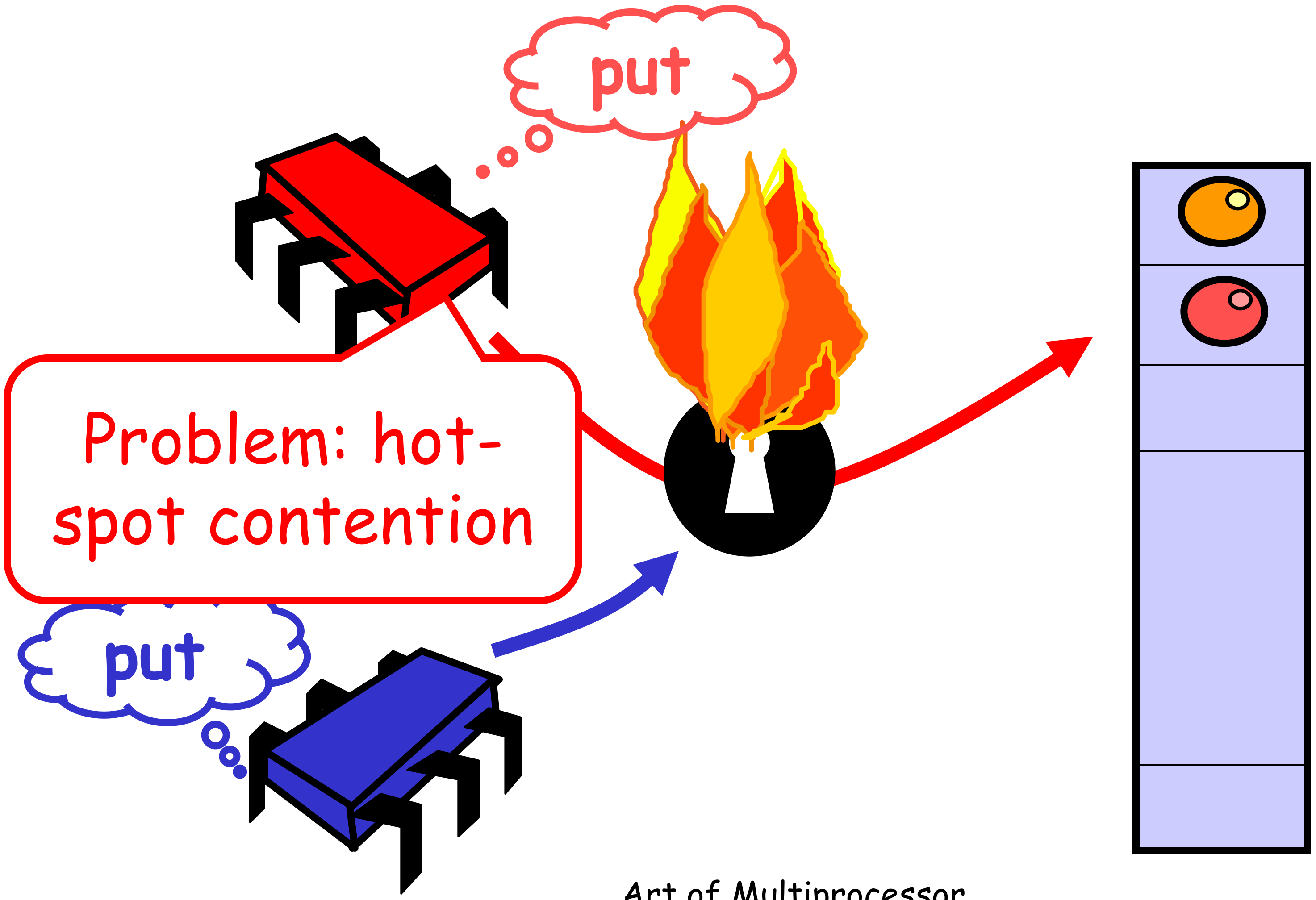
Unordered set of objects

- Put
  - Inserts object
  - blocks if full
- Remove
  - Removes & returns an object
  - blocks if empty

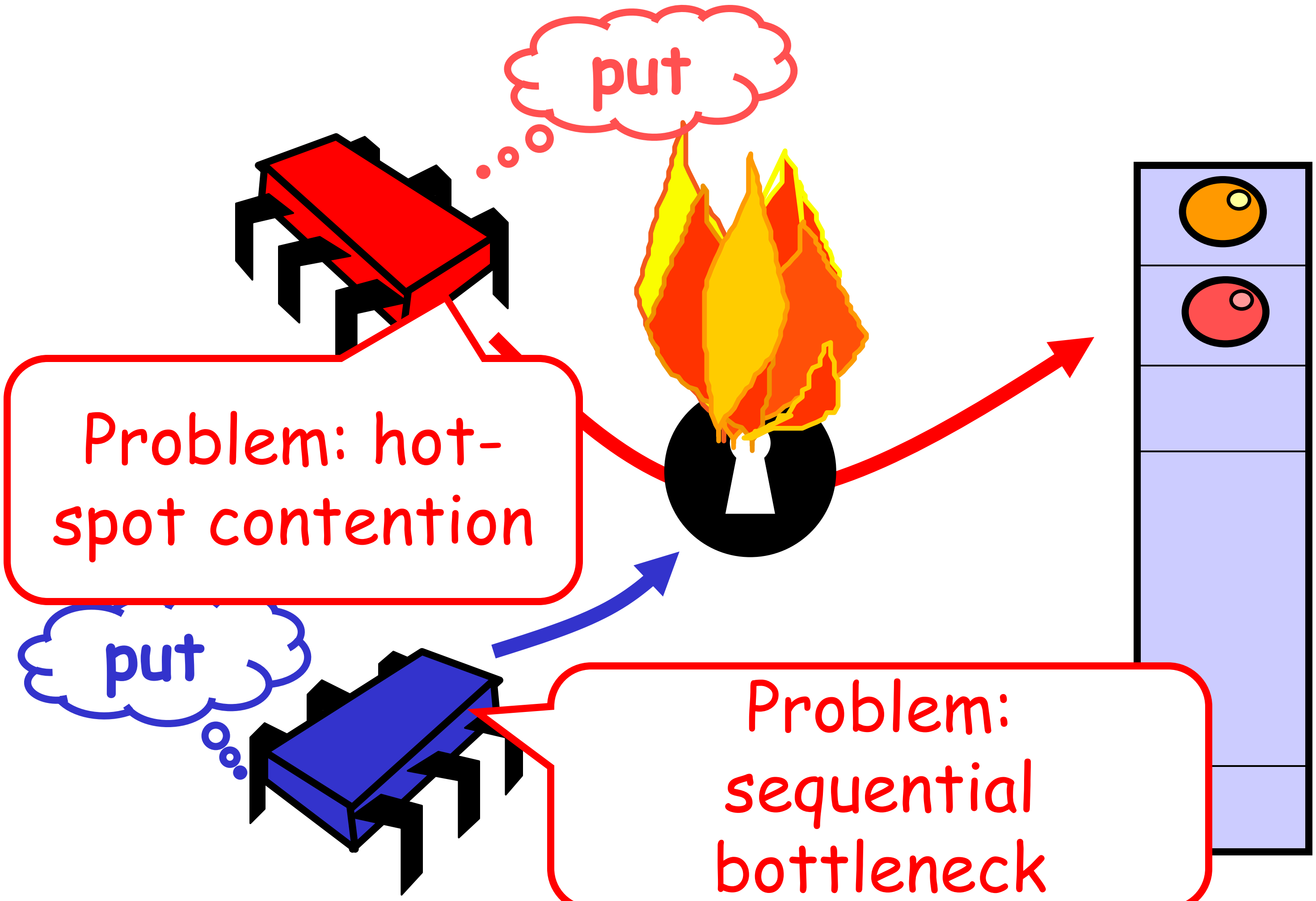
Simple Locking Implementation



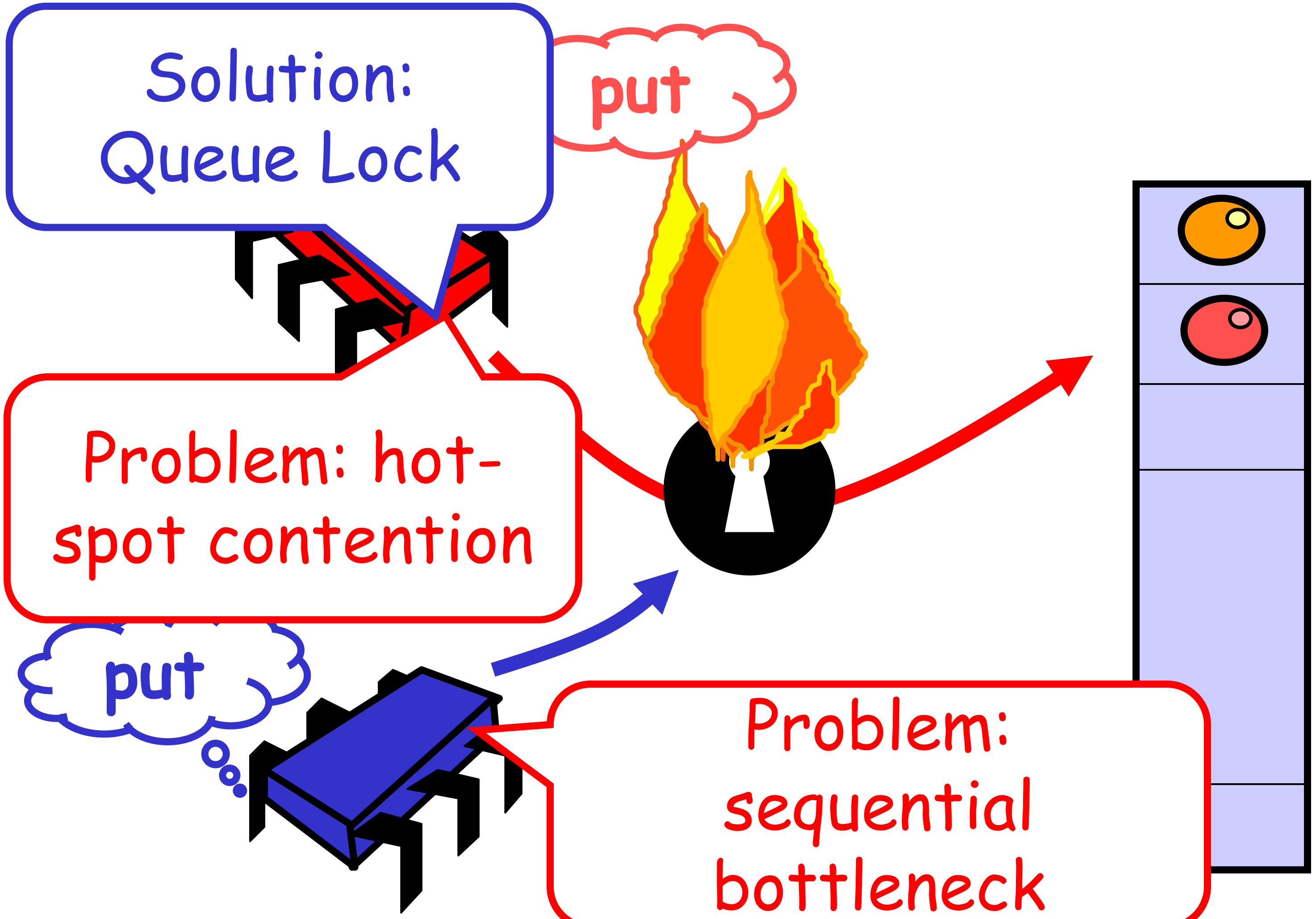
Simple Locking Implementation



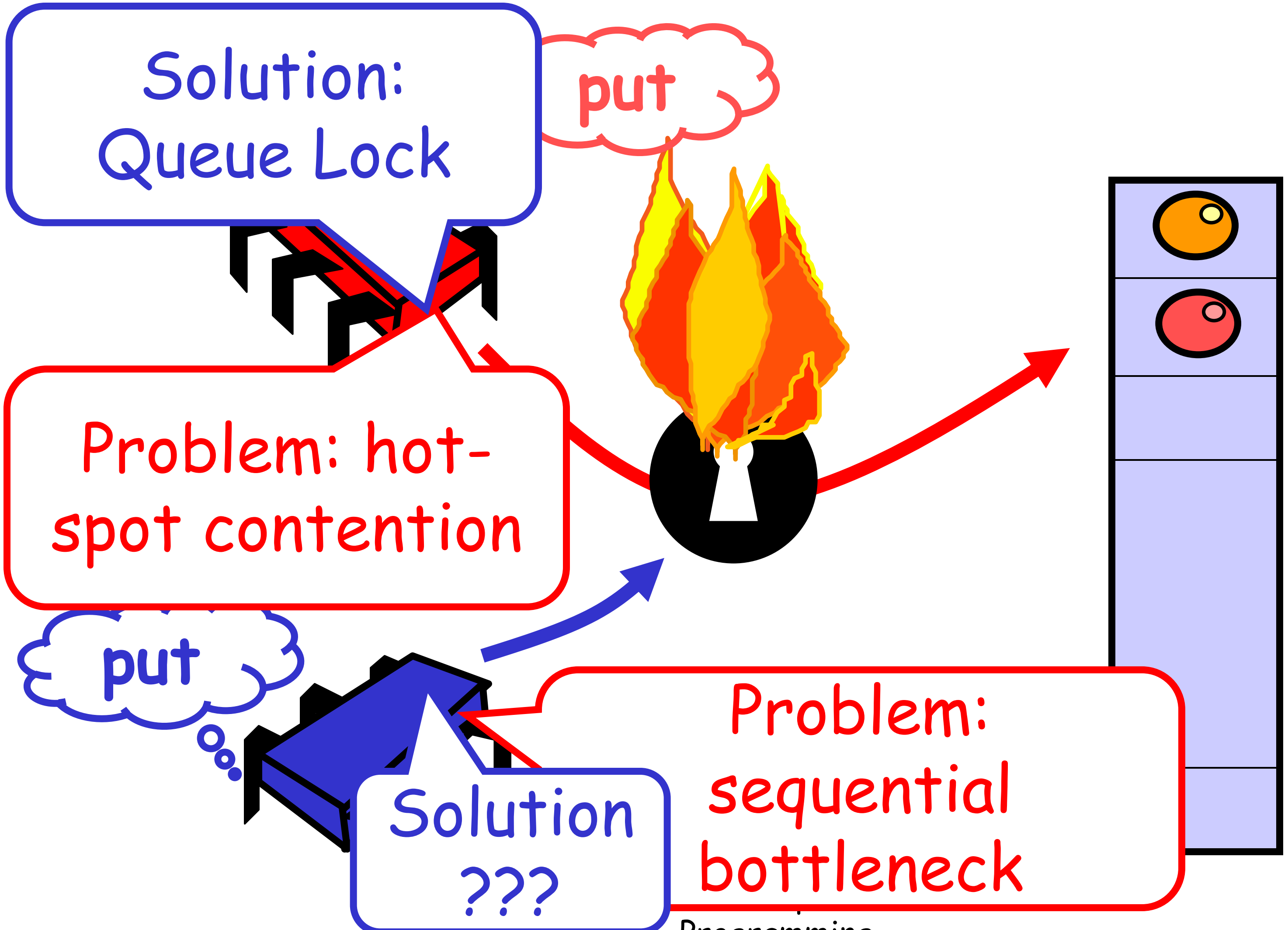
Simple Locking Implementation



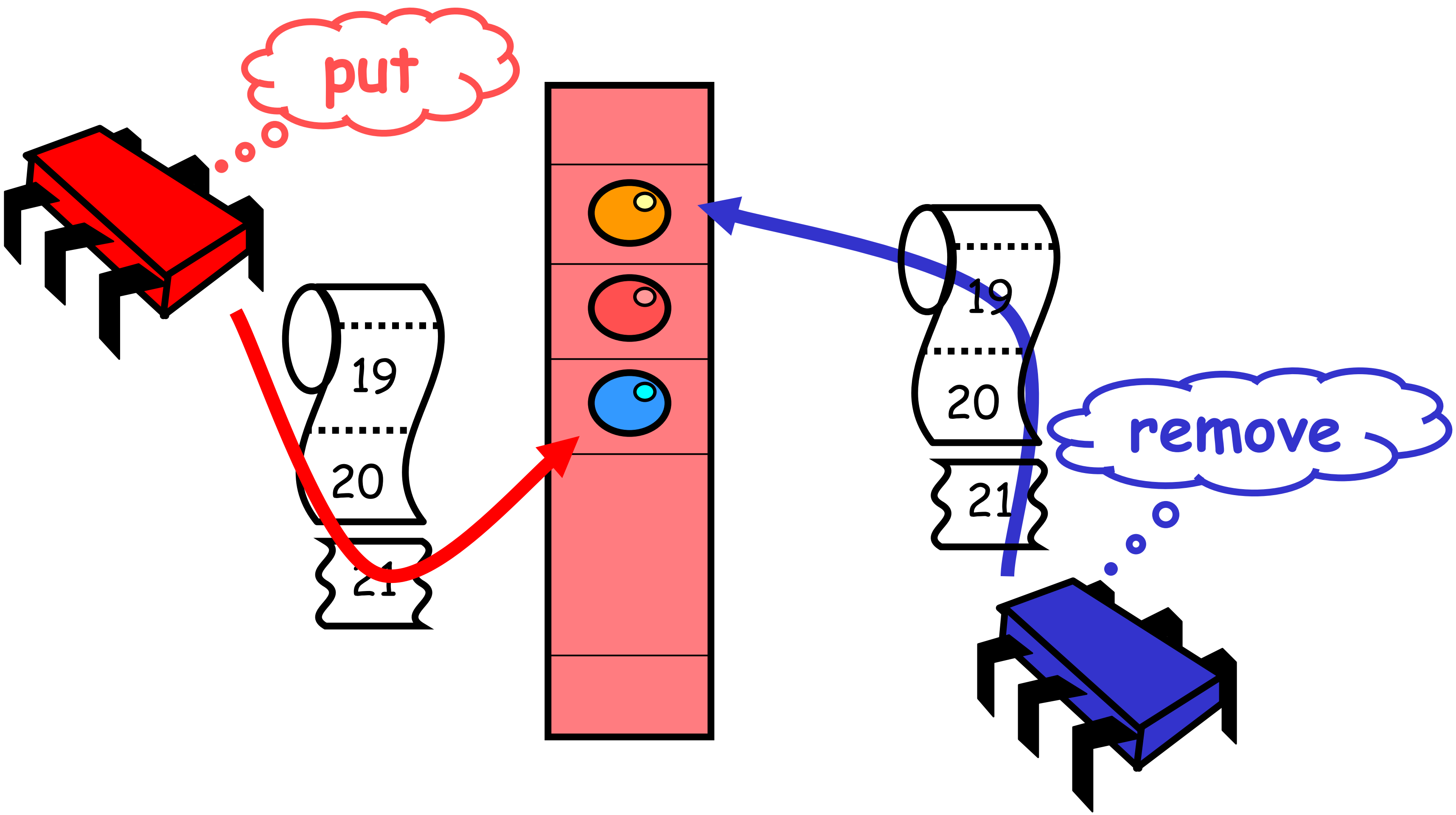
Simple Locking Implementation



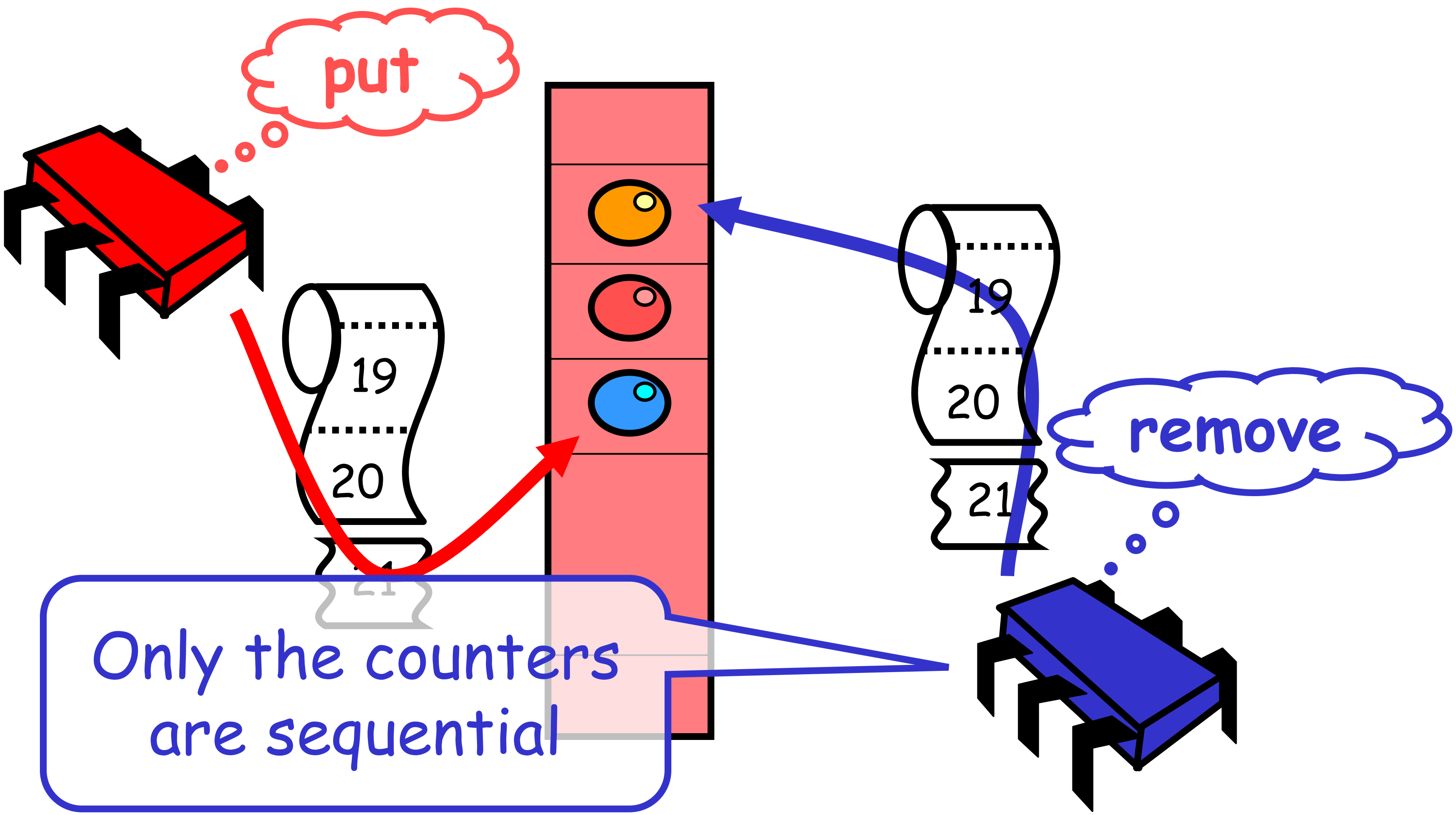
Simple Locking Implementation



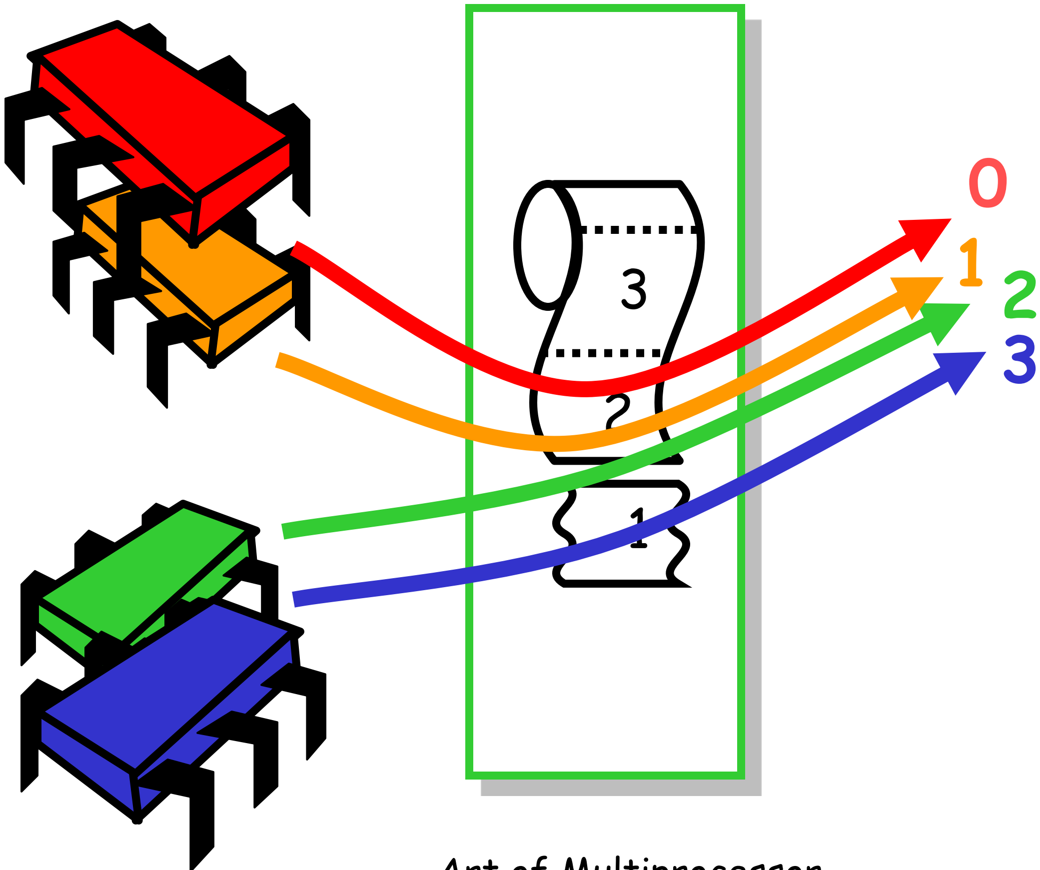
Counting Implementation



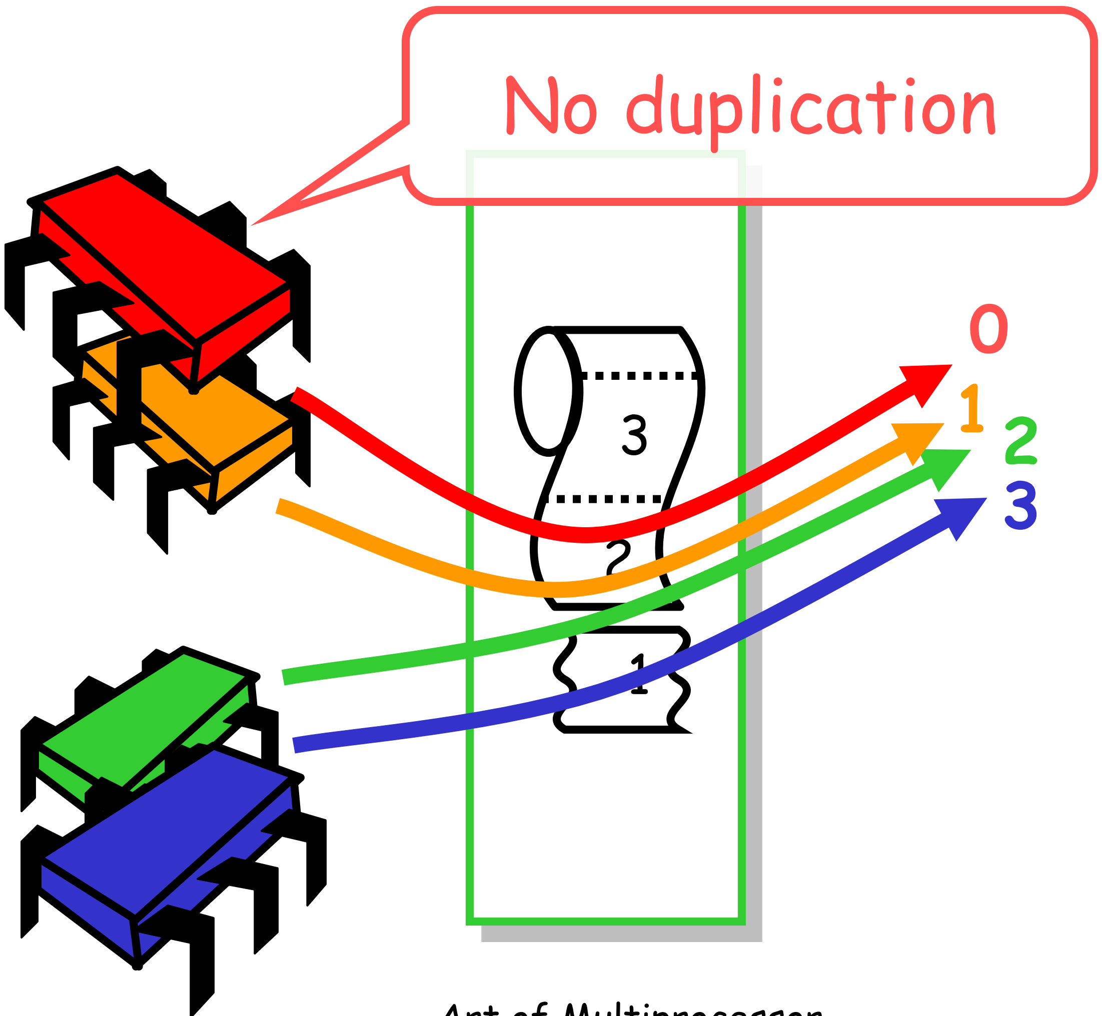
Counting Implementation



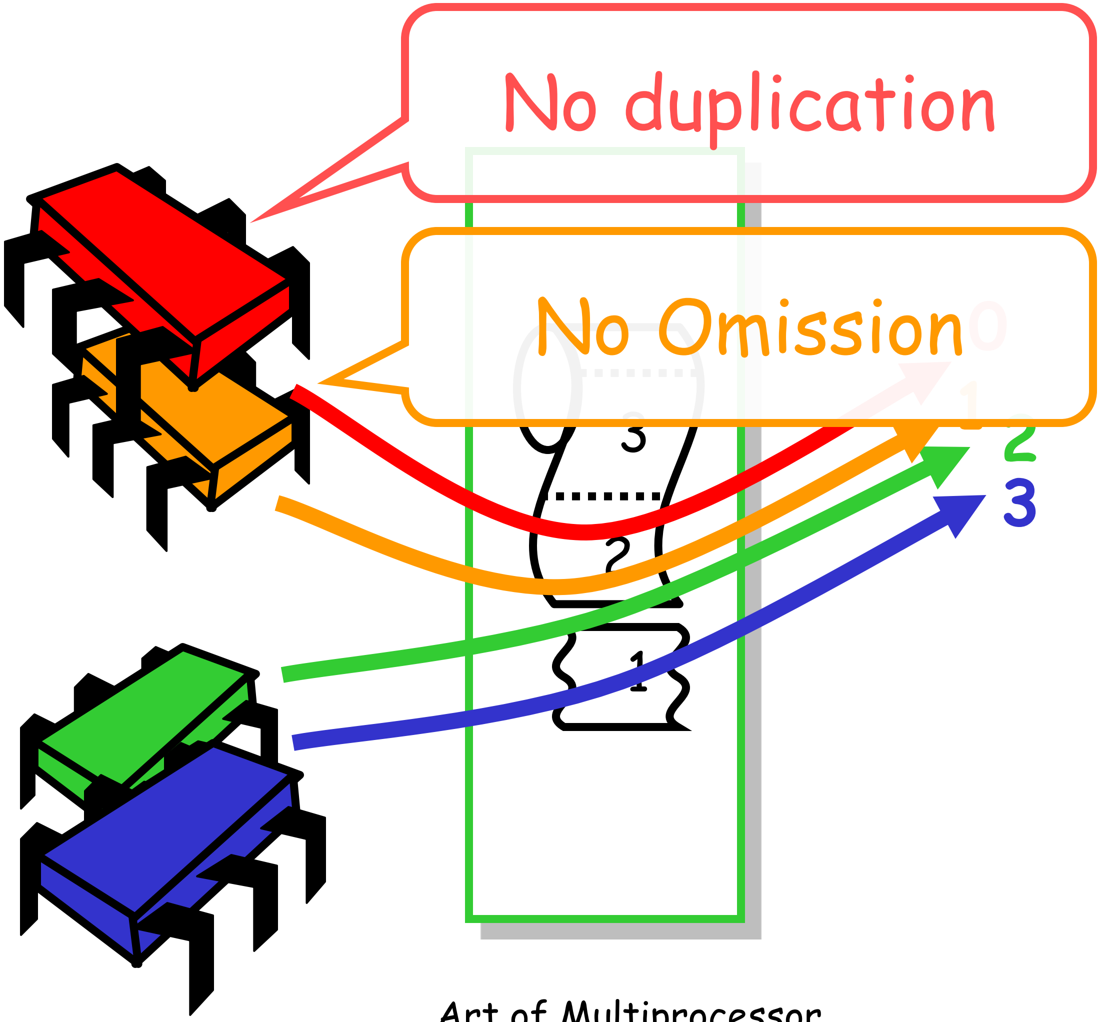
Shared Counter



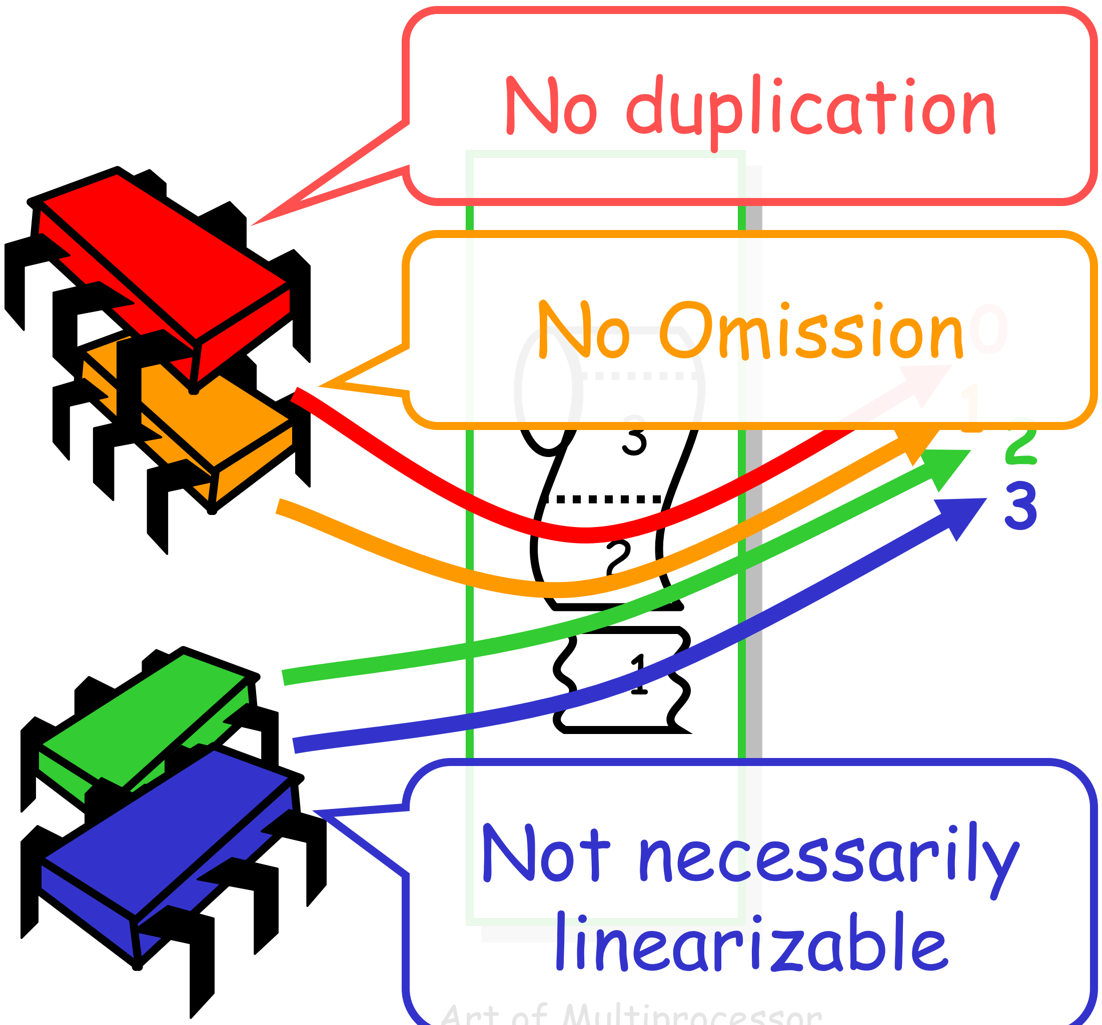
Shared Counter



Shared Counter



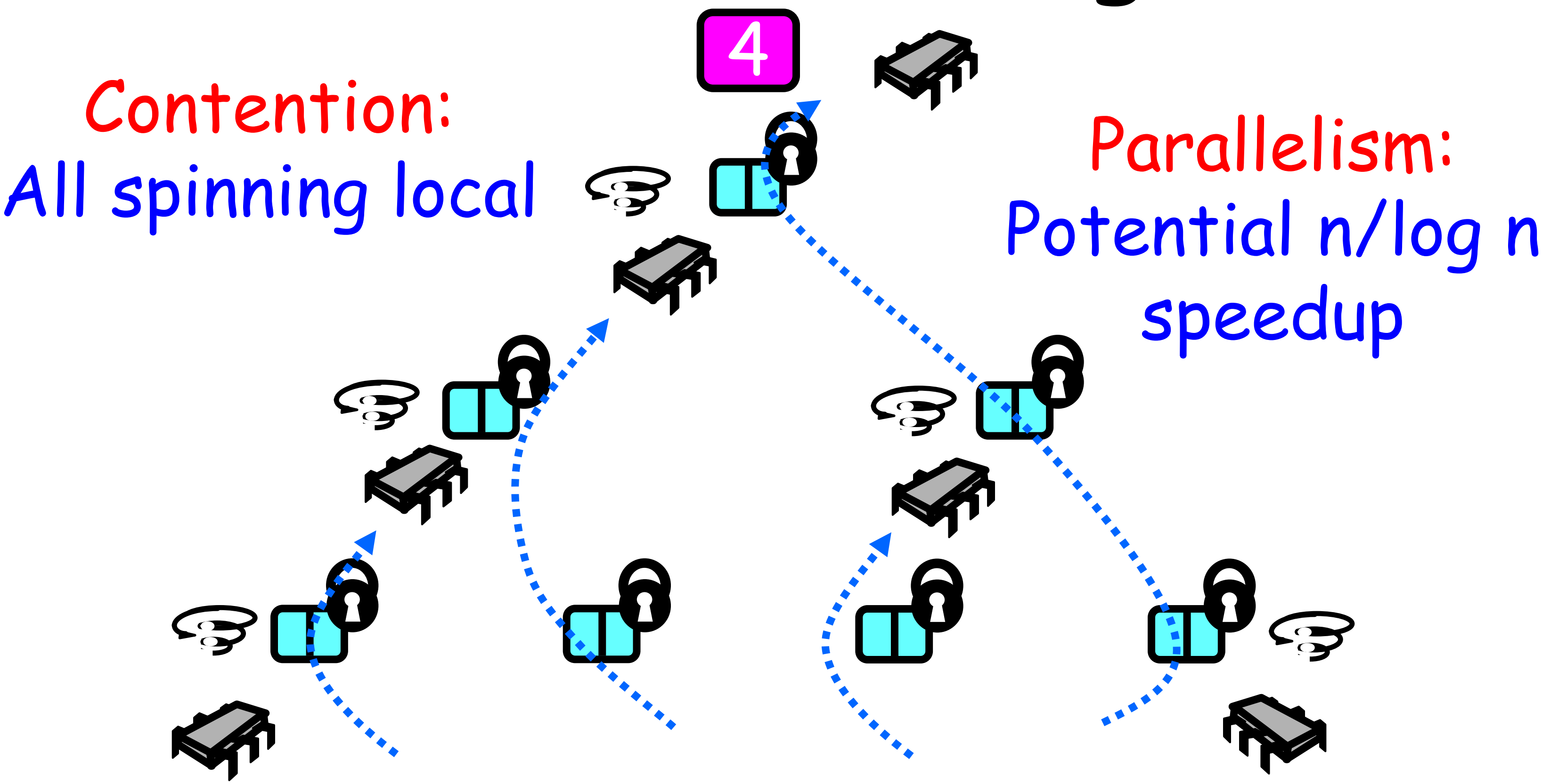
Shared Counter



Shared Counters

- Can we build a shared counter with
  - Low memory contention, and
  - Real parallelism?
- Locking
  - Can use queue locks to reduce contention
  - No help with parallelism issue …

Software Combining Tree

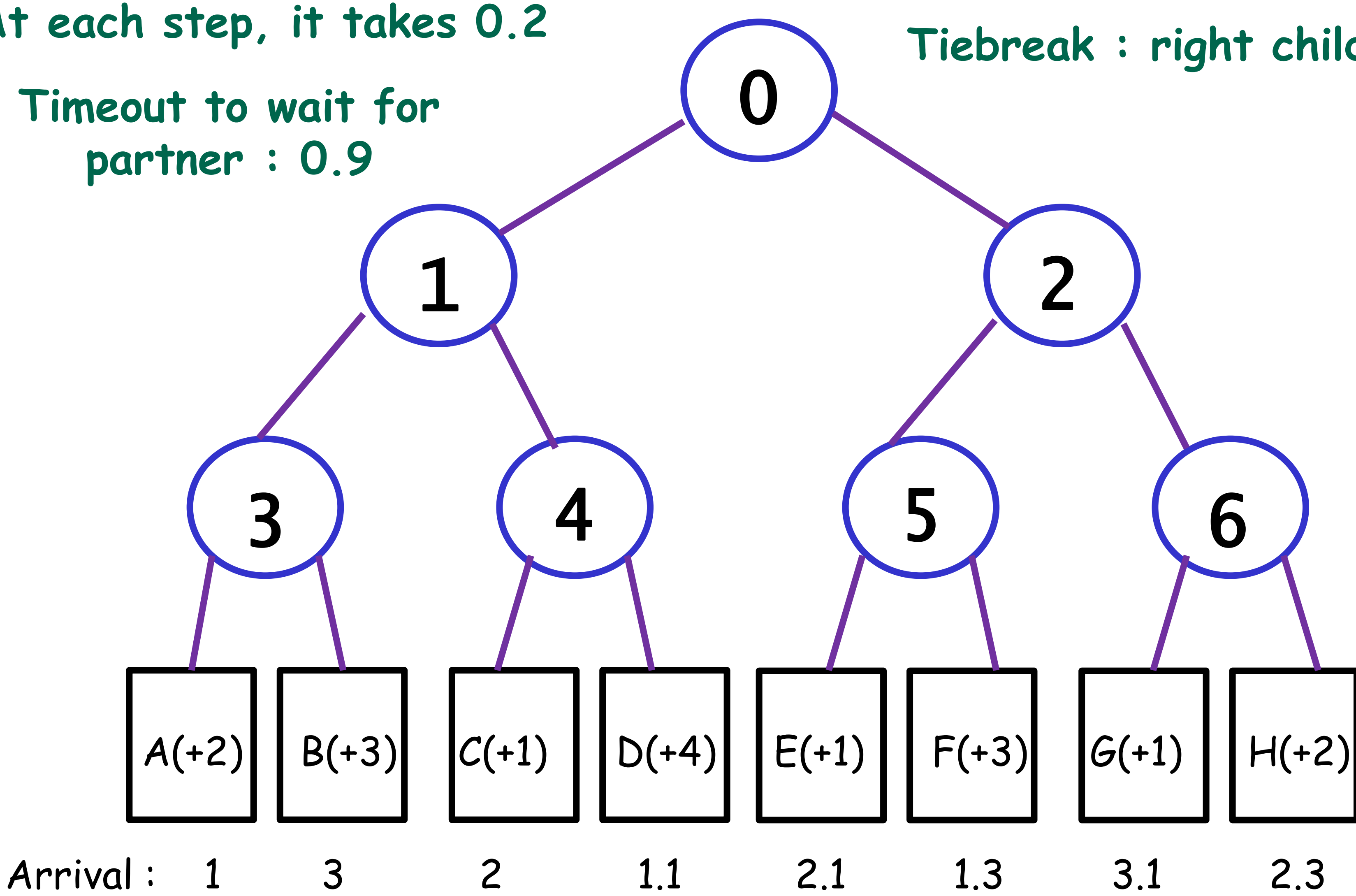


Combining Tree

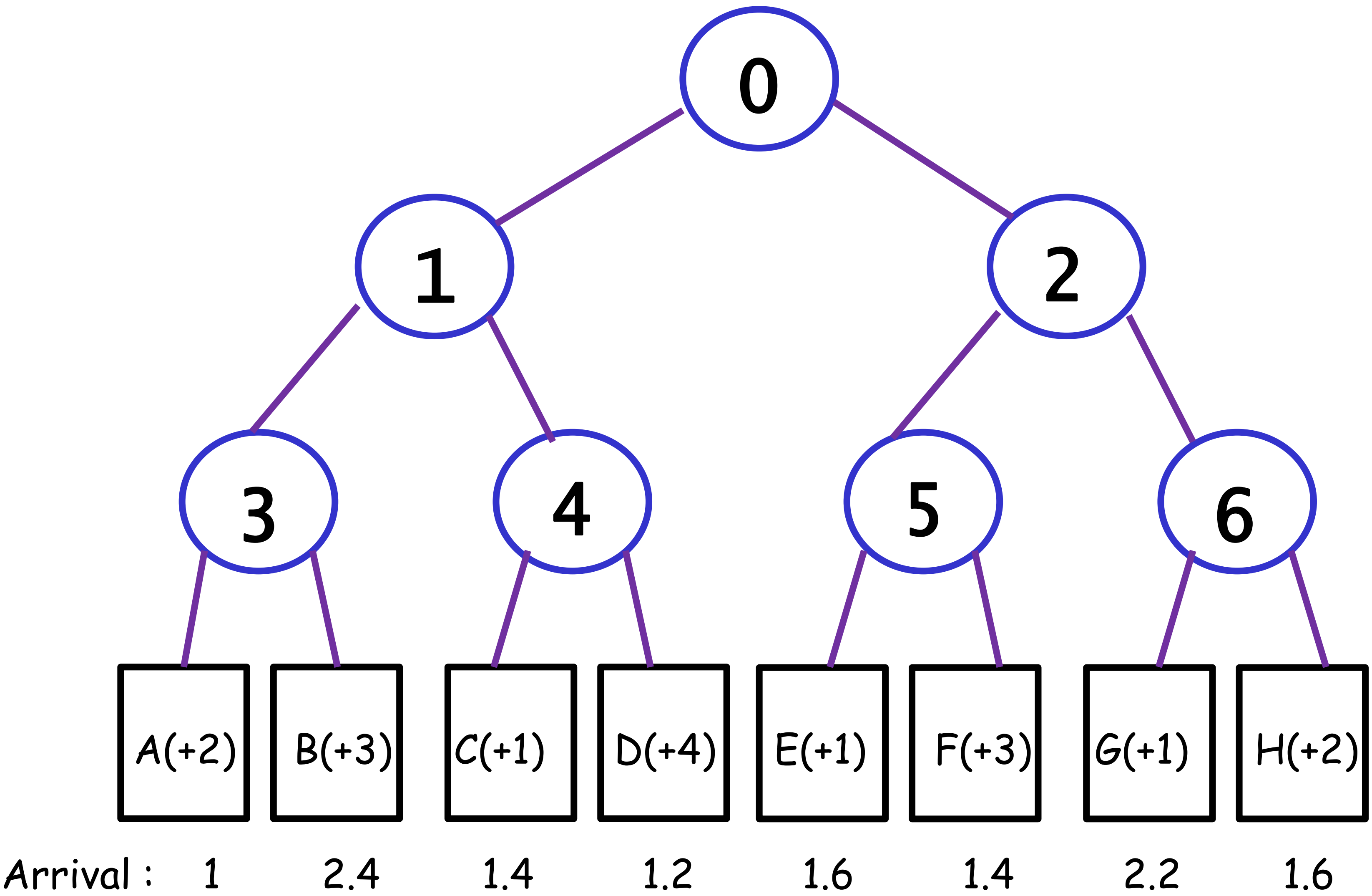
At each step, it takes 0.2

Timeout to wait for partner : 0.9

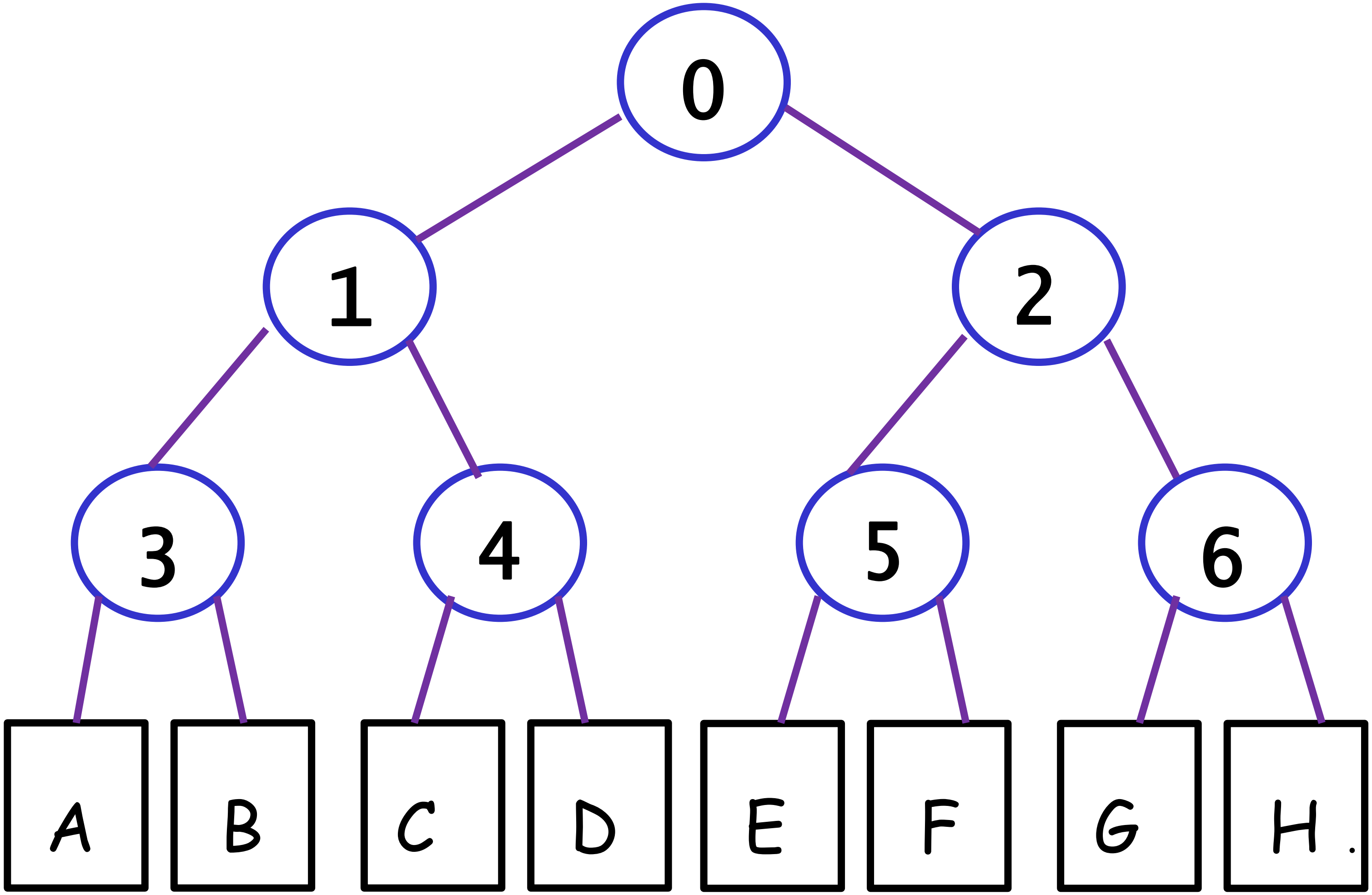
Tiebreak : right child wins



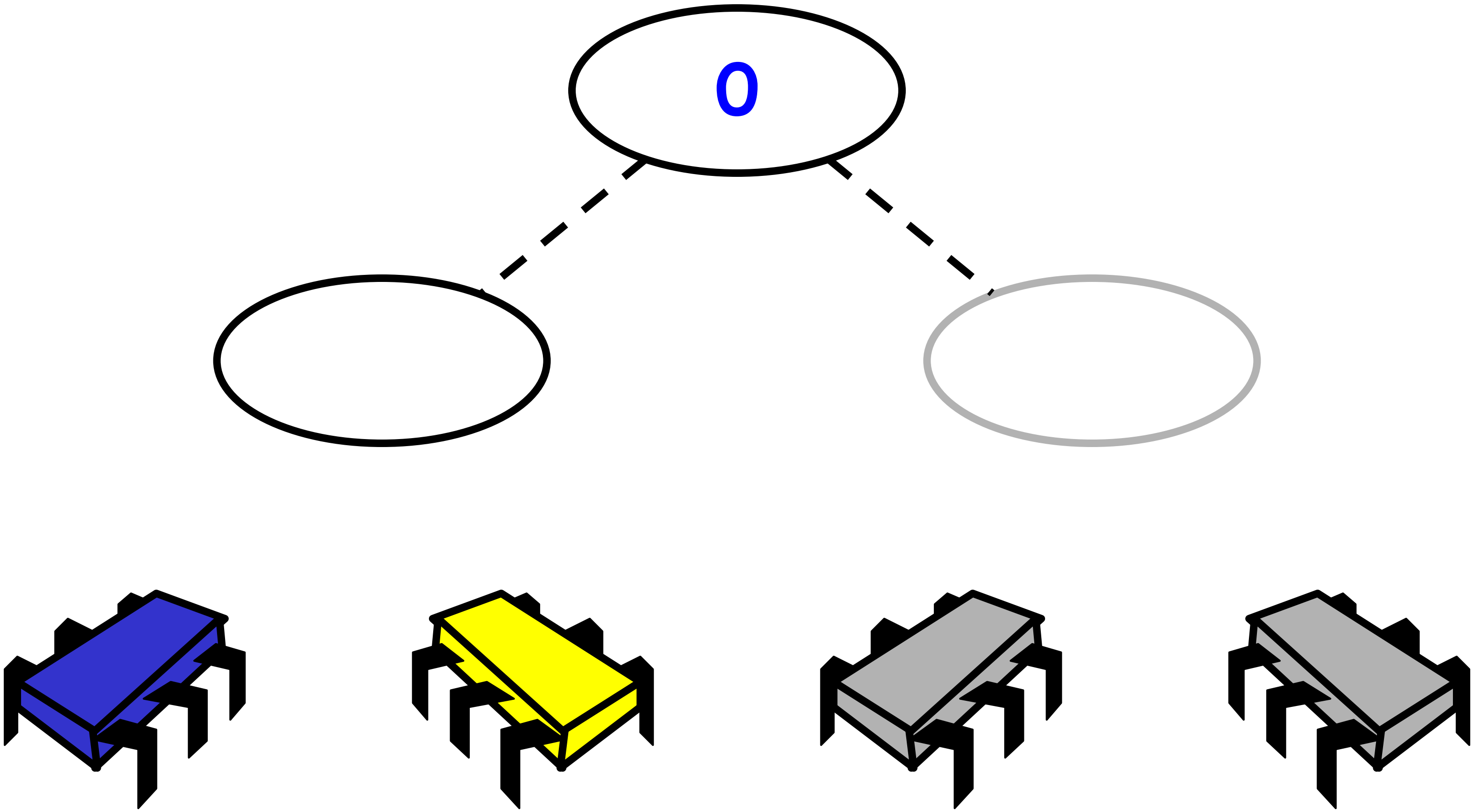
Combining Tree



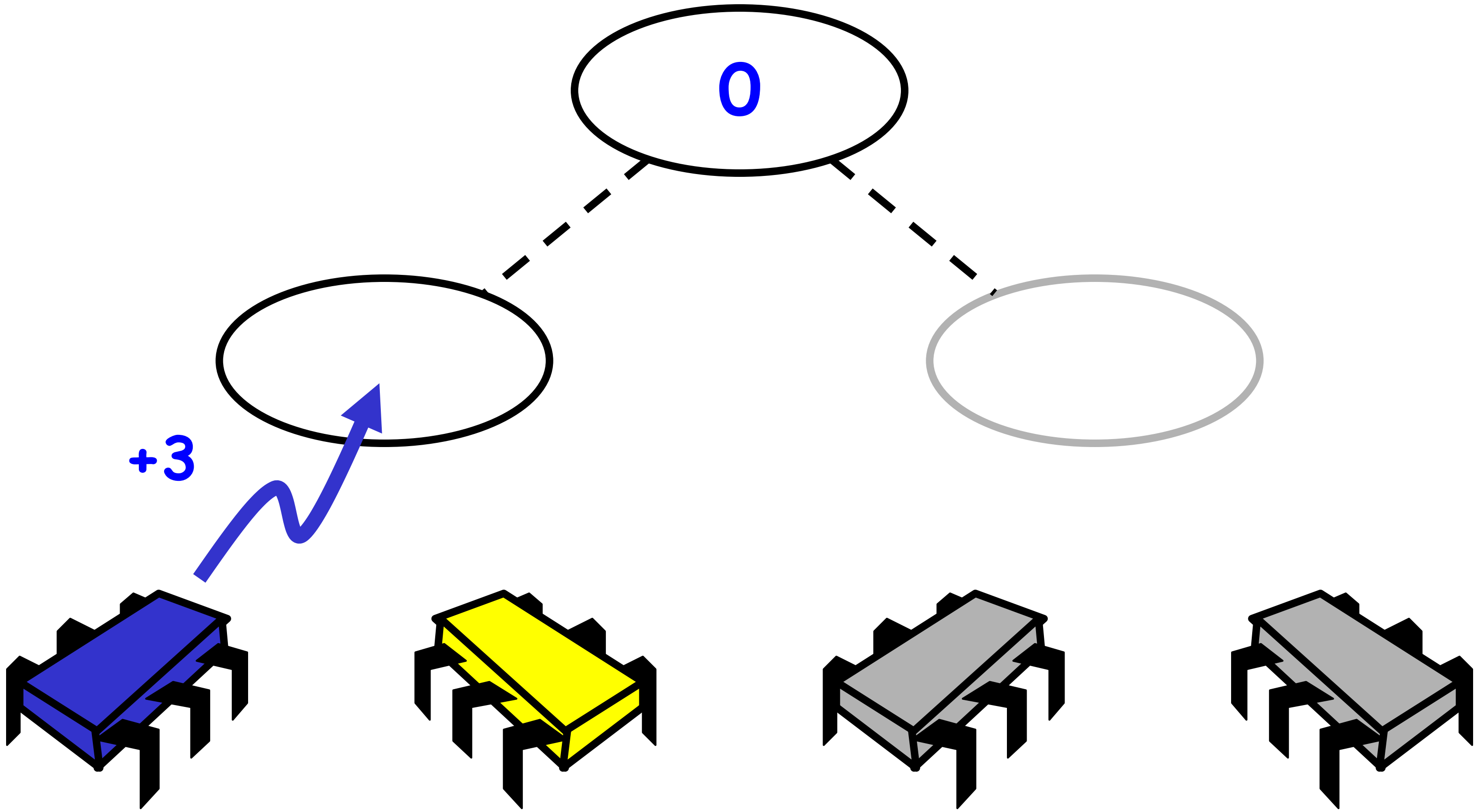
Combining Tree



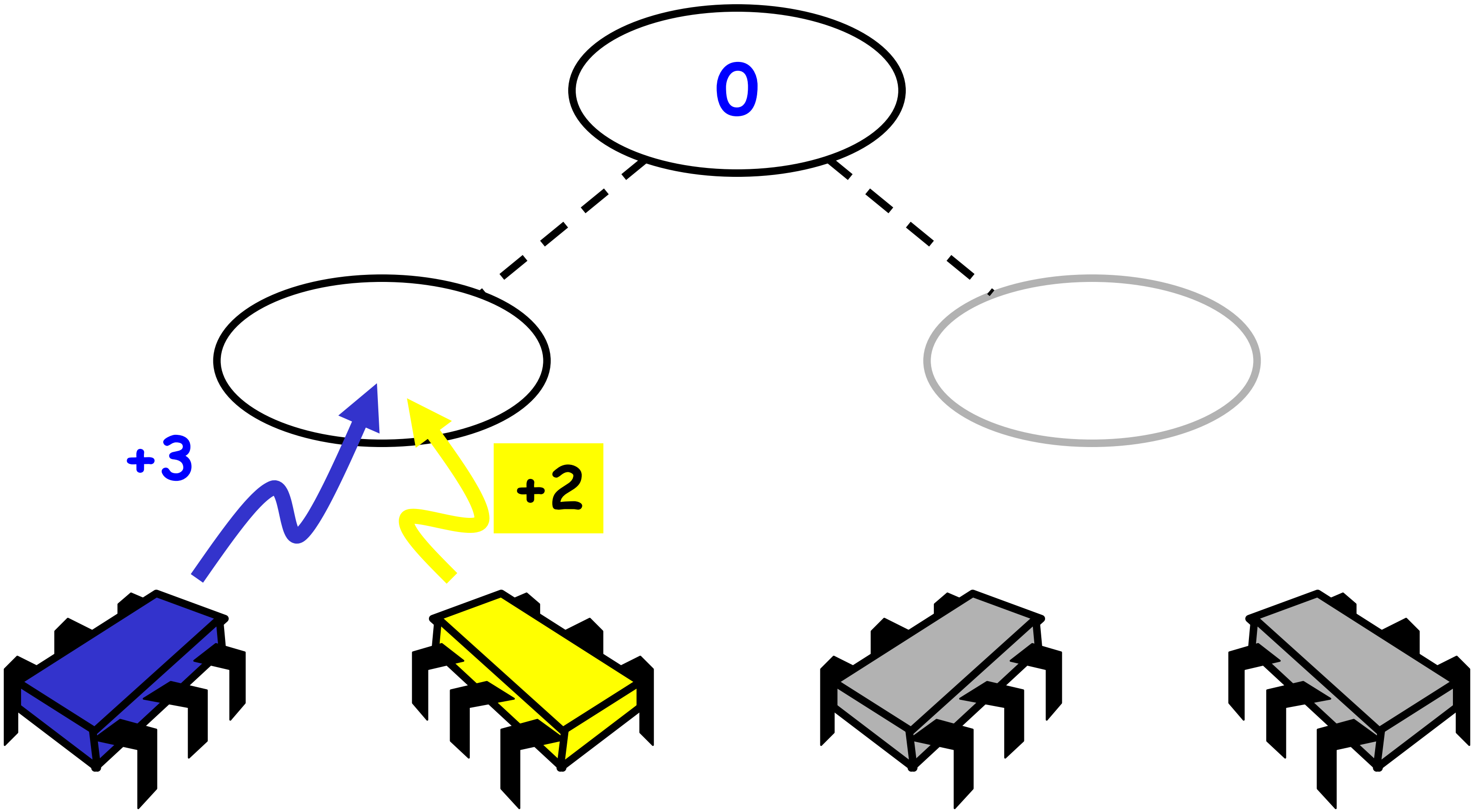
Combining Trees



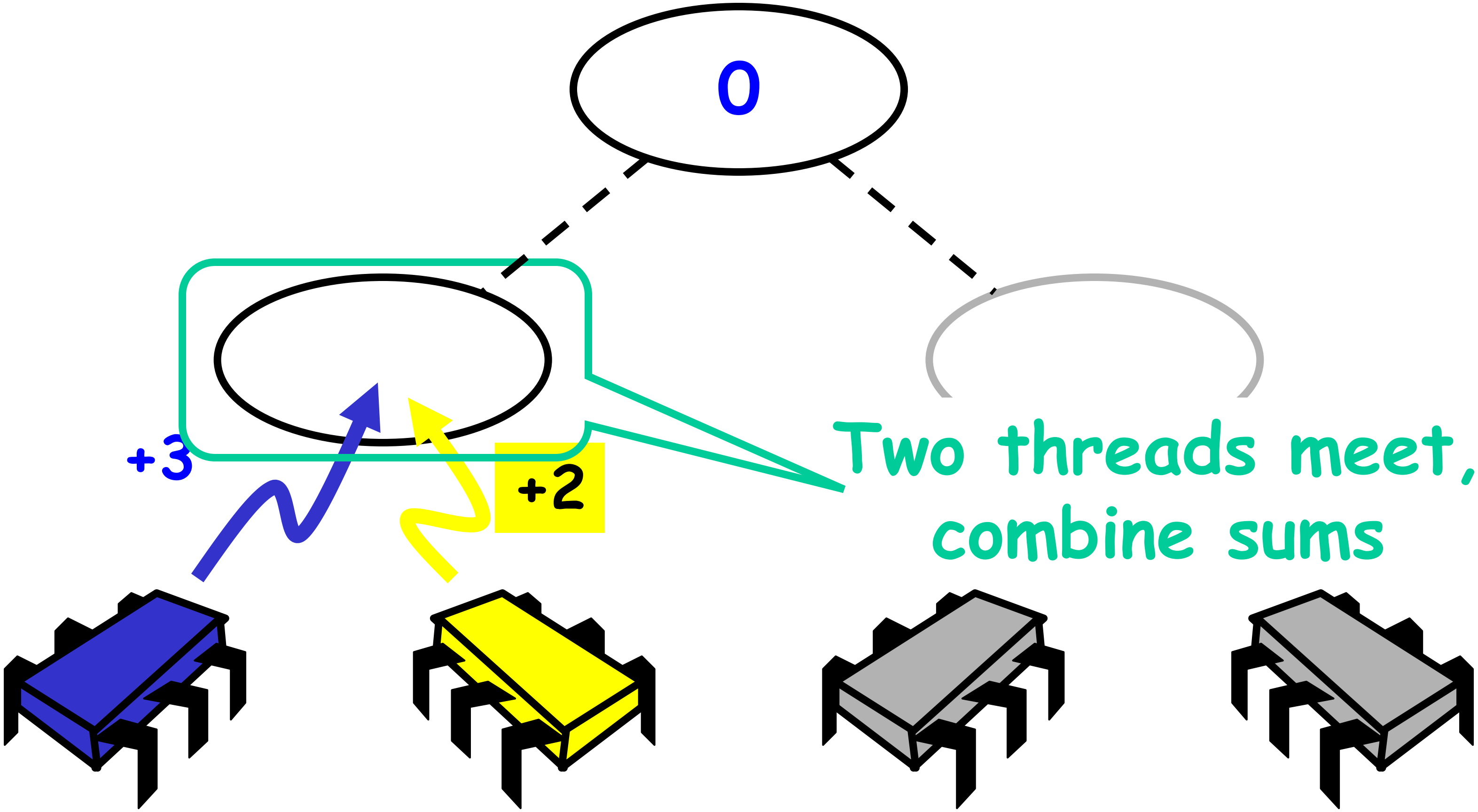
Combining Trees



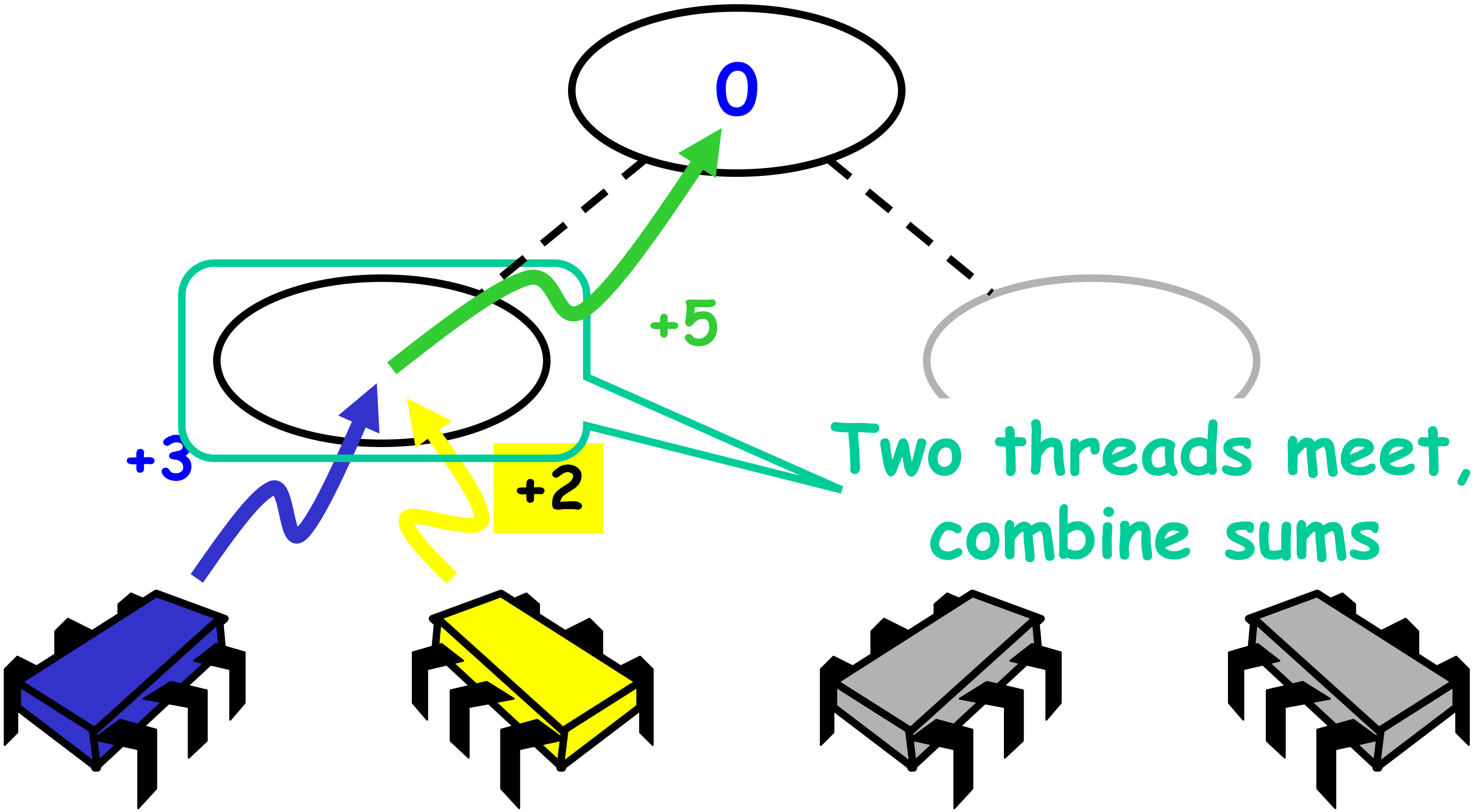
Combining Trees



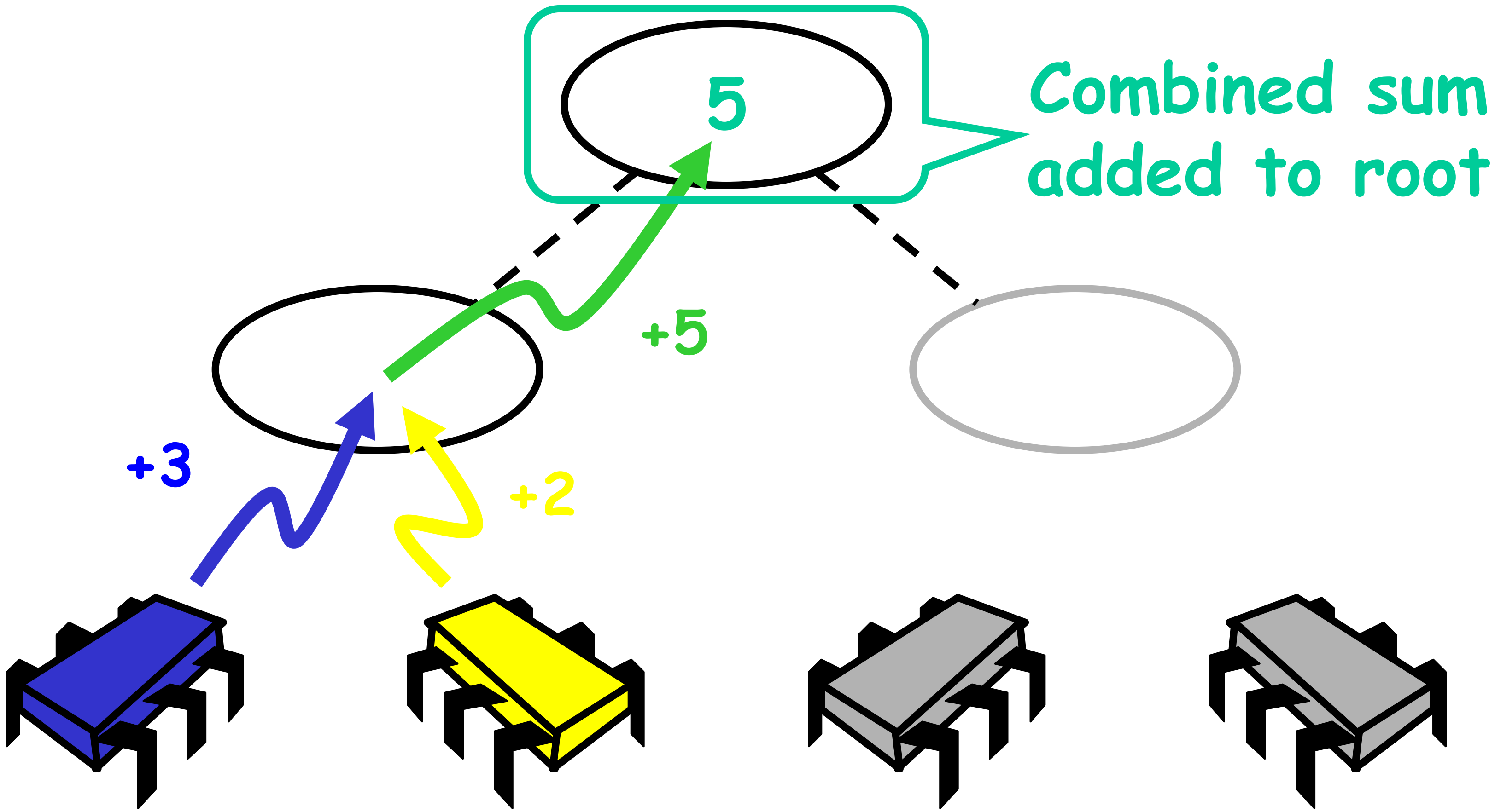
Combining Trees



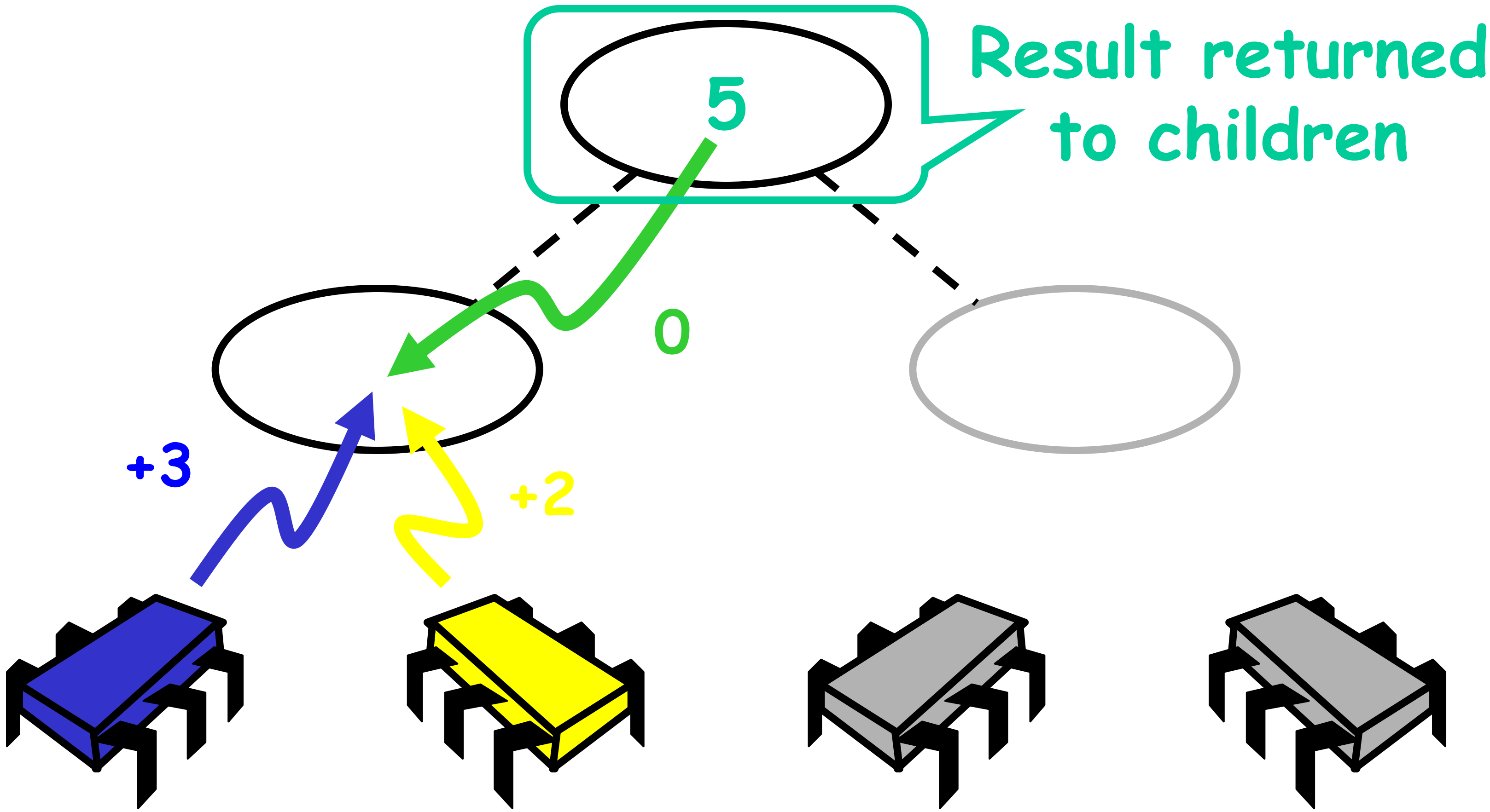
Combining Trees



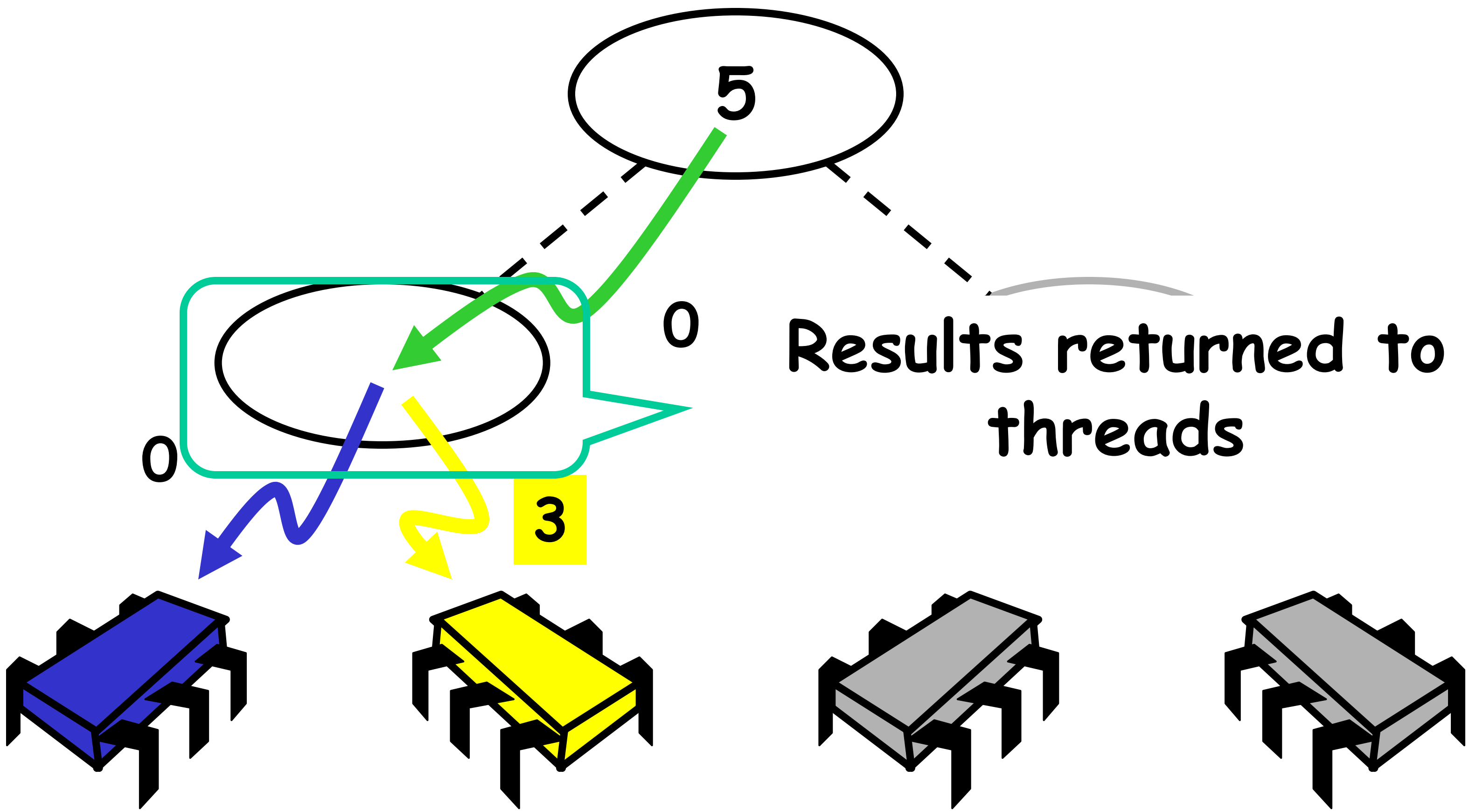
Combining Trees



Combining Trees



Combining Trees



Devil in the Details

- What if
  - threads don't arrive at the same time?
- Wait for a partner to show up?
  - How long to wait?
  - Waiting times add up …
- Instead
  - Use multi-phase algorithm
  - Try to wait in parallel …

Combining Status

```
enum CStatus{
 IDLE, FIRST, SECOND, DONE, ROOT};
```

Combining Status

```
enum CStatus{
 IDLE, FIRST, SECOND, DONE, ROOT};
```

Nothing going on

Combining Status

```
enum CStatus{
 IDLE, FIRST, SECOND, DONE, ROOT};
```

1st thread ISO partner for combining, will return soon to check for 2nd thread

Combining Status

```
enum CStatus{
  IDLE, FIRST, SECOND, DONE, ROOT};
```

2nd thread arrived with value for combining

Combining Status

```
enum CStatus{
 IDLE, FIRST, SECOND, DONE, ROOT};
```

1st thread has completed operation & deposited result for 2nd thread

Combining Status

```
enum CStatus{
 IDLE, FIRST, SECOND, DONE, ROOT};
```

Special case: root node

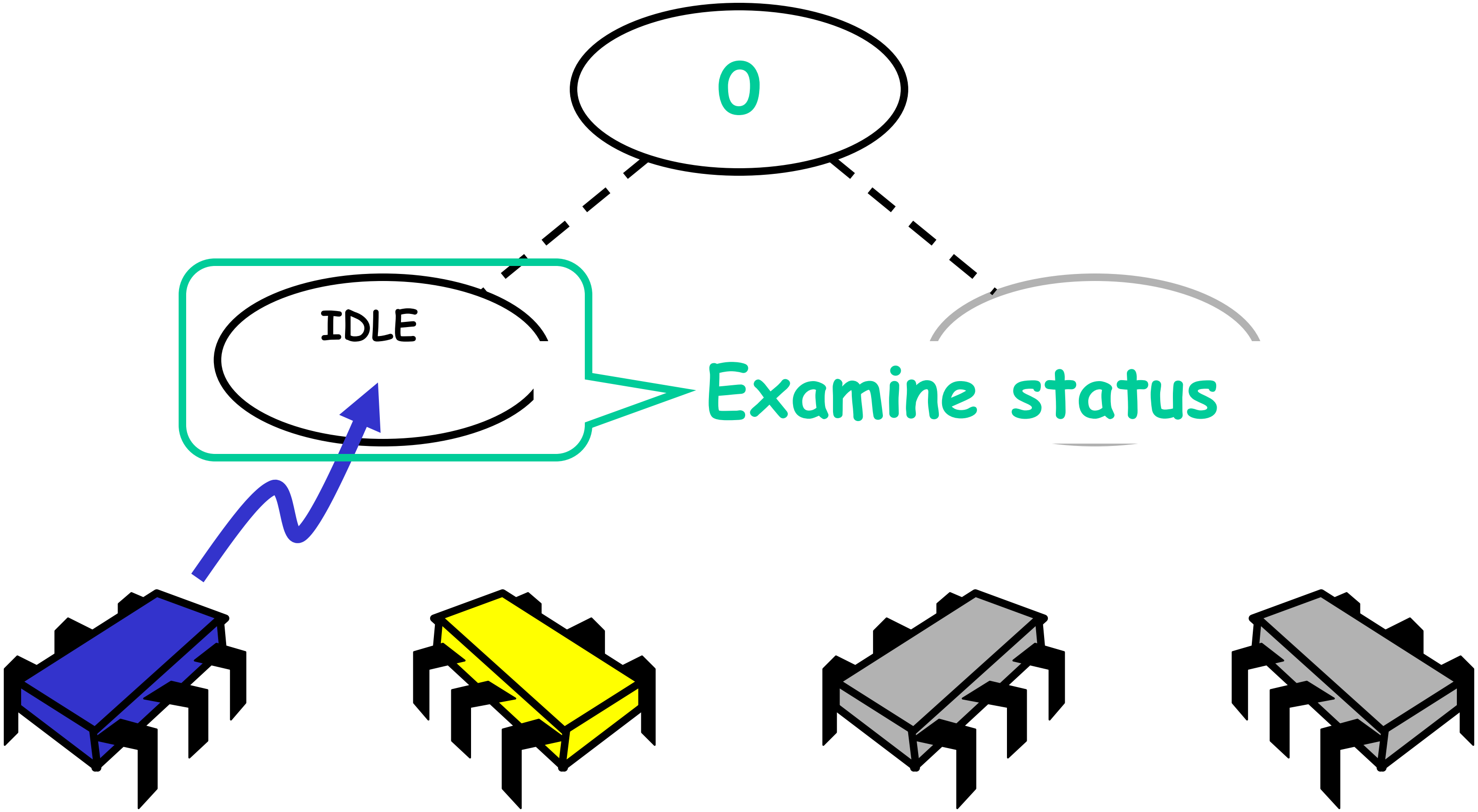
Node Synchronization

- Short-term
  - Synchronized methods
  - Consistency during method call
- Long-term
  - Boolean locked field
  - Consistency across calls

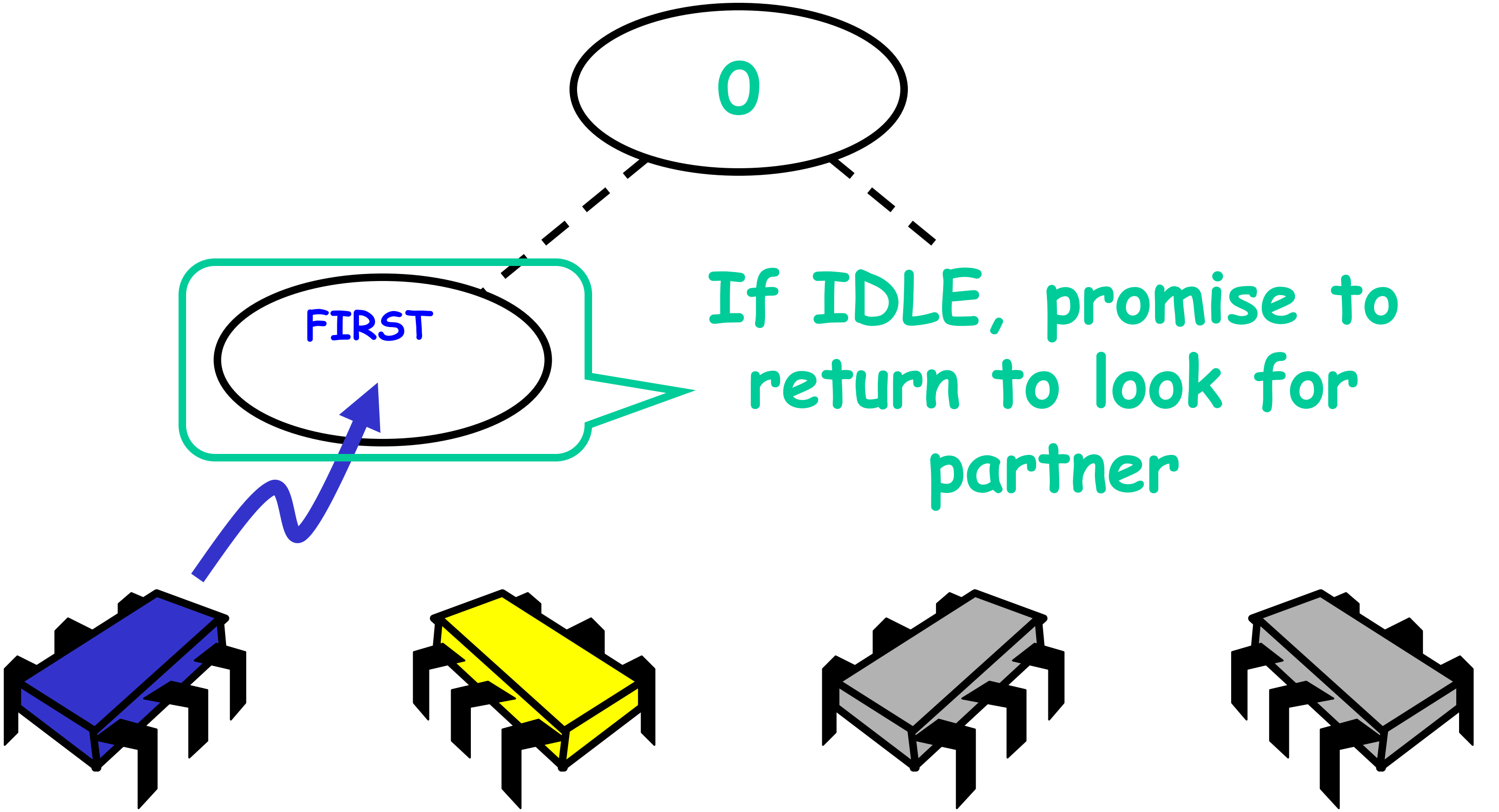
Phases

- Precombining
  - Set up combining rendez-vous
- Combining
  - Collect and combine operations
- Operation
  - Hand off to higher thread
- Distribution
  - Distribute results to waiting threads

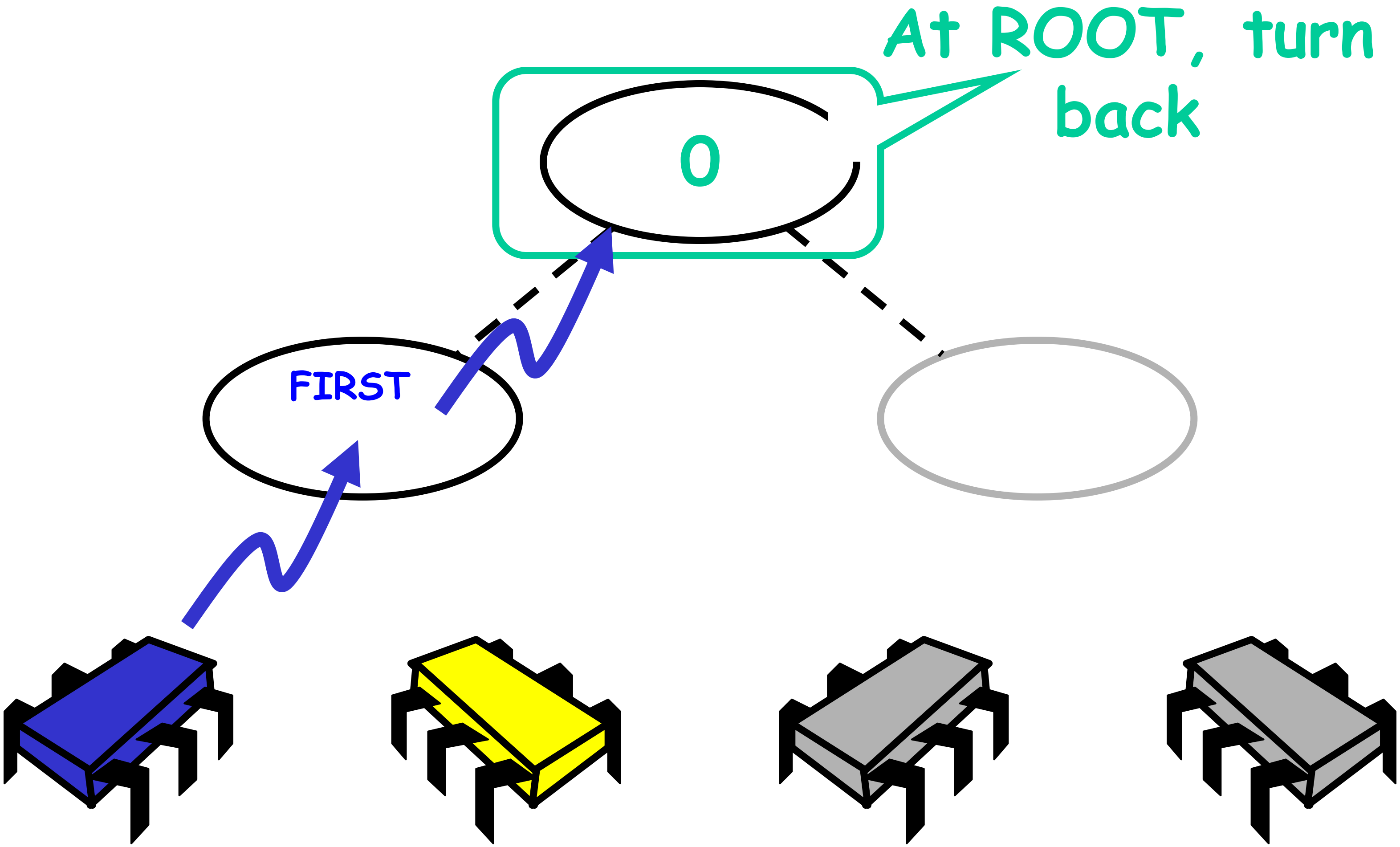
Precombining Phase



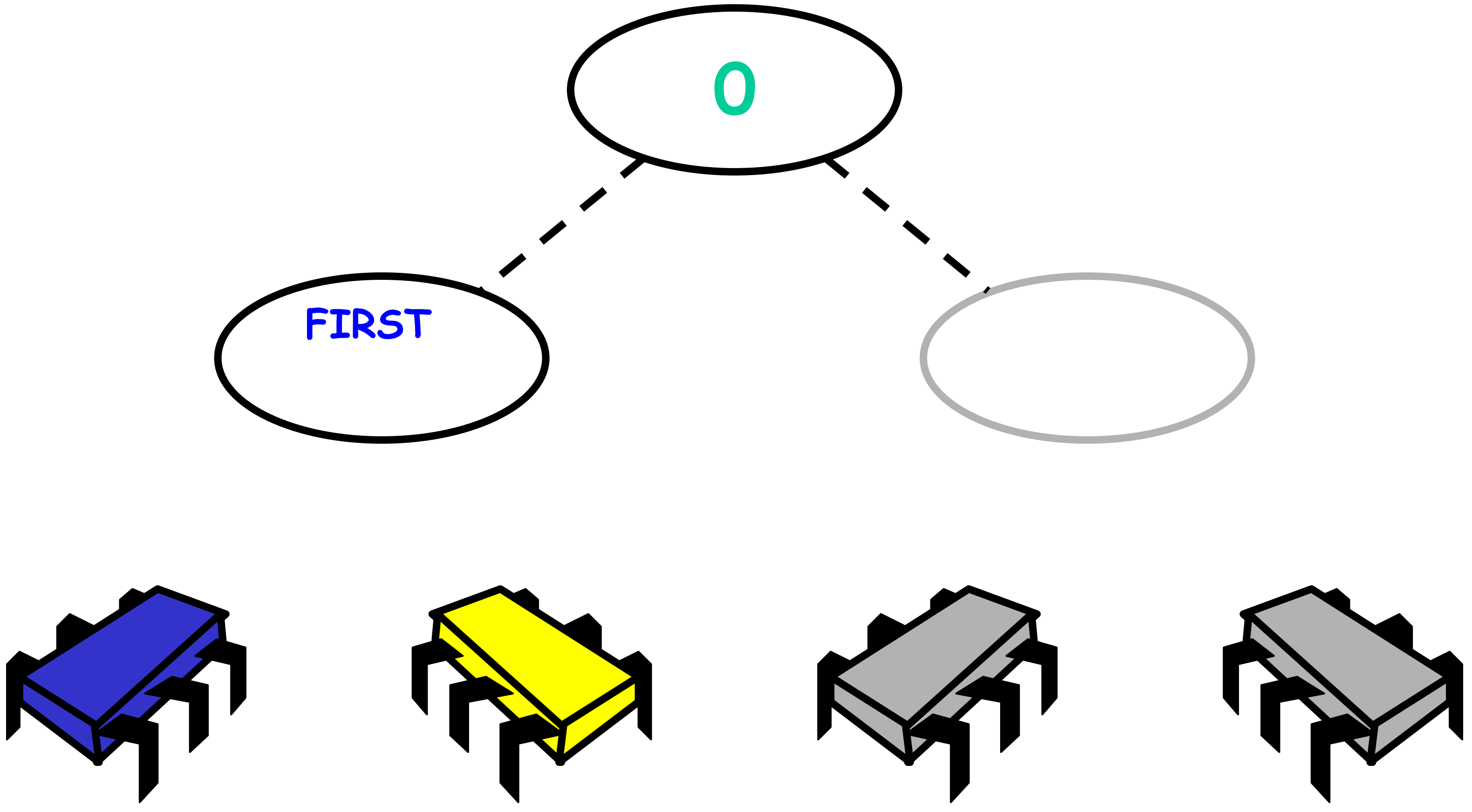
Precombining Phase



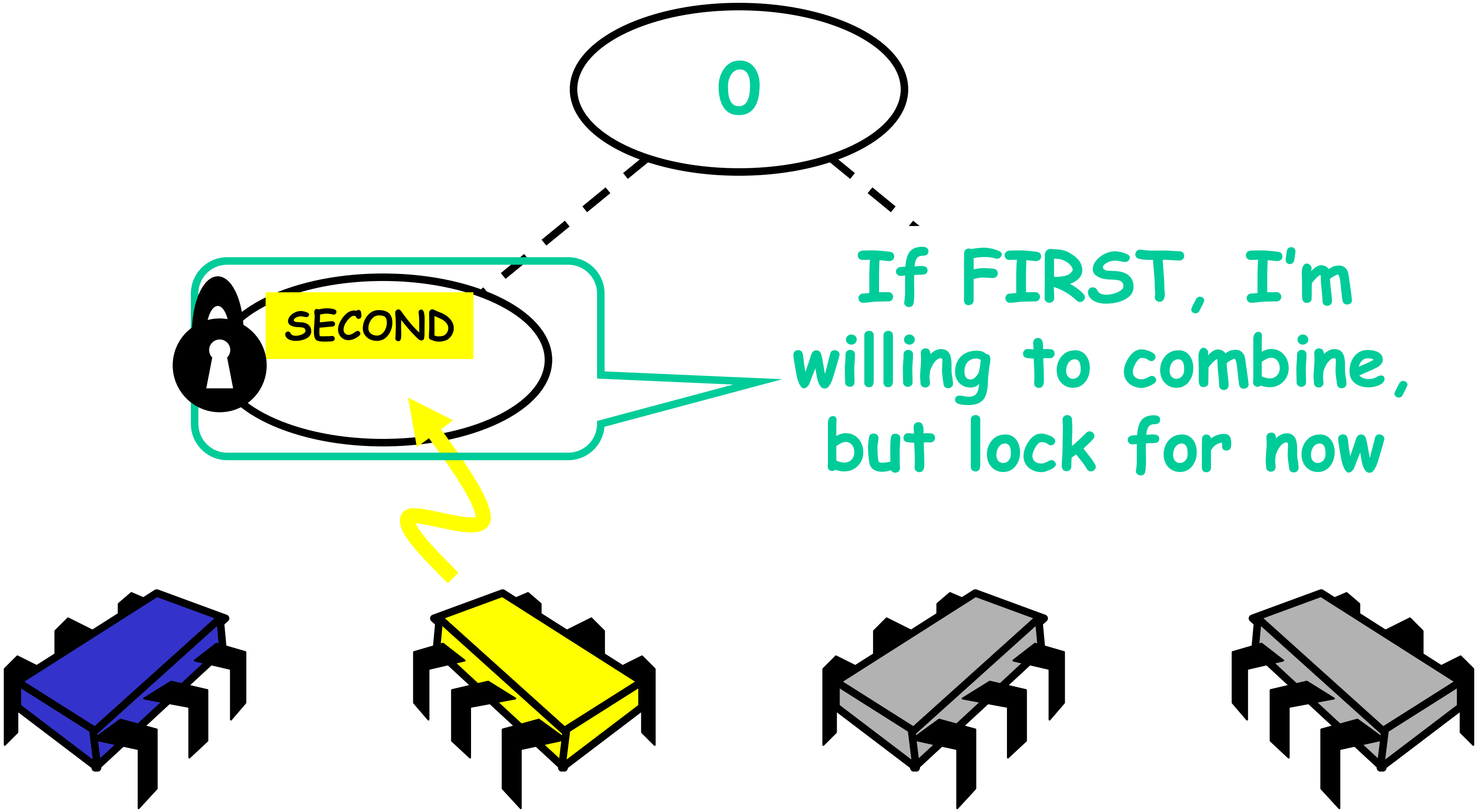
Precombining Phase



Precombining Phase



Precombining Phase



Code

- Tree class
  - In charge of navigation
- Node class
  - Combining state
  - Synchronization state
  - Bookkeeping

Precombining Navigation

```
Node node = myLeaf;
while (node.precombine()) {
  node = node.parent;
  }
Node stop = node;
```

Precombining Navigation

```
Node node = myLeaf;
while (node.precombine()) {
  node = node.parent;
  }
Node stop = node;
```

Start at leaf

Precombining Navigation

```
Node node = myLeaf;
while (node.precombine()) {
   node = node.parent;
   }
Node stop = node;
```

Move up while instructed to do so

Precombining Navigation

```
Node node = myLeaf;
while (node.precombine()) {
  node = node.parent;
  }
Node stop = node;
```

Remember where we stopped

Precombining Node

```
synchronized boolean precombine() {
 while (locked) wait();
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Precombining Node

```
synchronized boolean precombine() {
 while (locked) wait();
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Short-term synchronization

Synchronization

```
synchronized boolean precombine() {
 while (locked) wait();
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Wait while node is locked

Precombining Node

```
synchronized boolean precombine() {
 while (locked) wait();
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Check combining status

Node was IDLE

```
synchronized boolean precombine() {
 while (locked) {wait();}
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
```

I will return to look for combining value

Precombining Node

```
synchronized boolean precombine() {
 while (locked) {wait();}
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              retur
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Continue up the tree

I'm the 2nd Thread

```
synchronized boolean precombine() {
 while (locked) {wait();}
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
```

If 1st thread has promised to return, lock node so it won't leave without me

Precombining Node

```
synchronized boolean precombine() {
 while (locked) {wait();}
 switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Prepare to deposit 2nd value

Precombining Node

End of phase 1, don't continue up tree

```
) {
Y) {wait();}
switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Node is the Root

If root, phase 1 ends, don't continue up tree

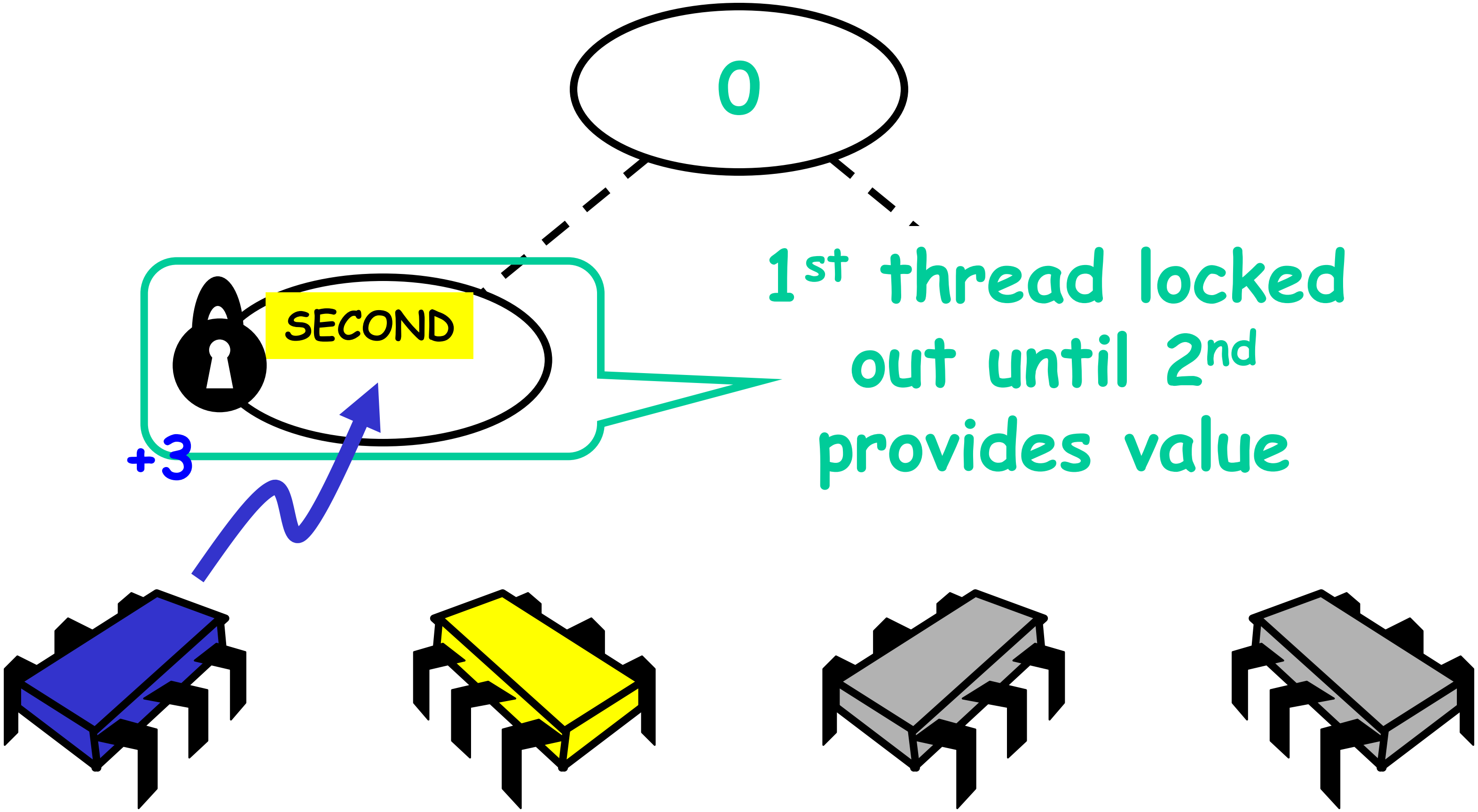
```
) {
Y) {wait();}
switch (cStatus) {
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

Precombining Node

```
synchronized boolean phase1() {
 while (locked) {w
 switch (cStatus)
  case IDLE: cStatus = CStatus.FIRST;
             return true;
  case FIRST: locked = true;
              cStatus = CStatus.SECOND;
              return false;
  case ROOT: return false;
  default: throw new PanicException()
  }
}
```

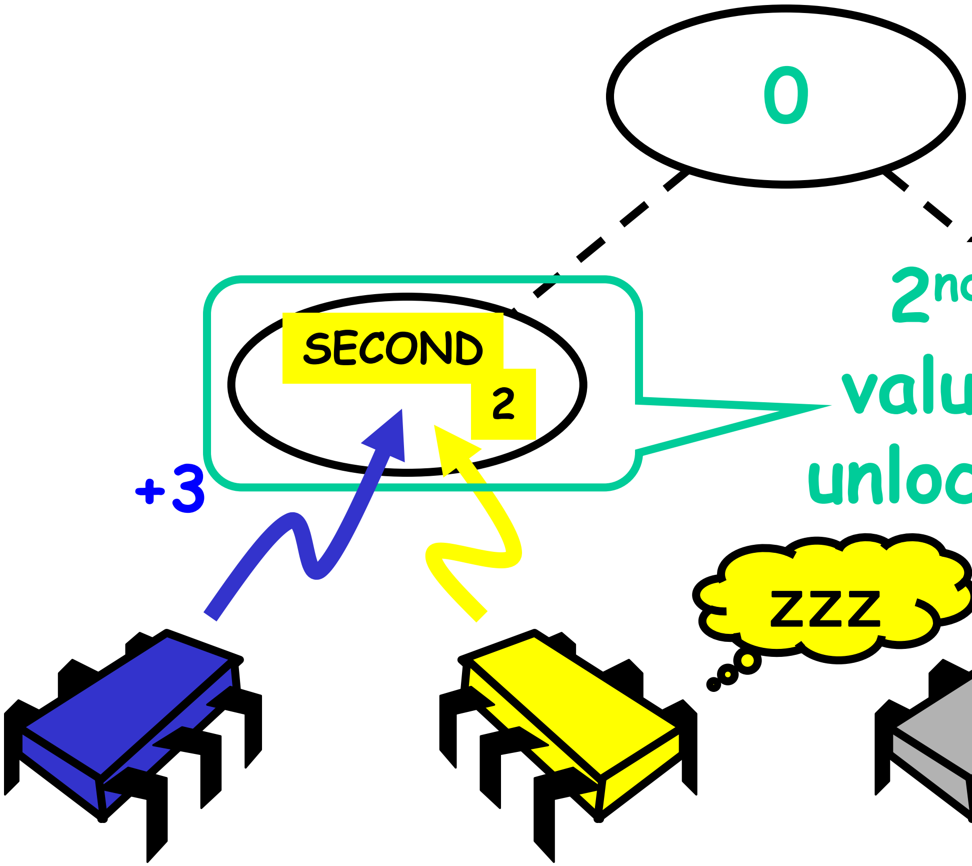
Always check for unexpected values!

Combining Phase



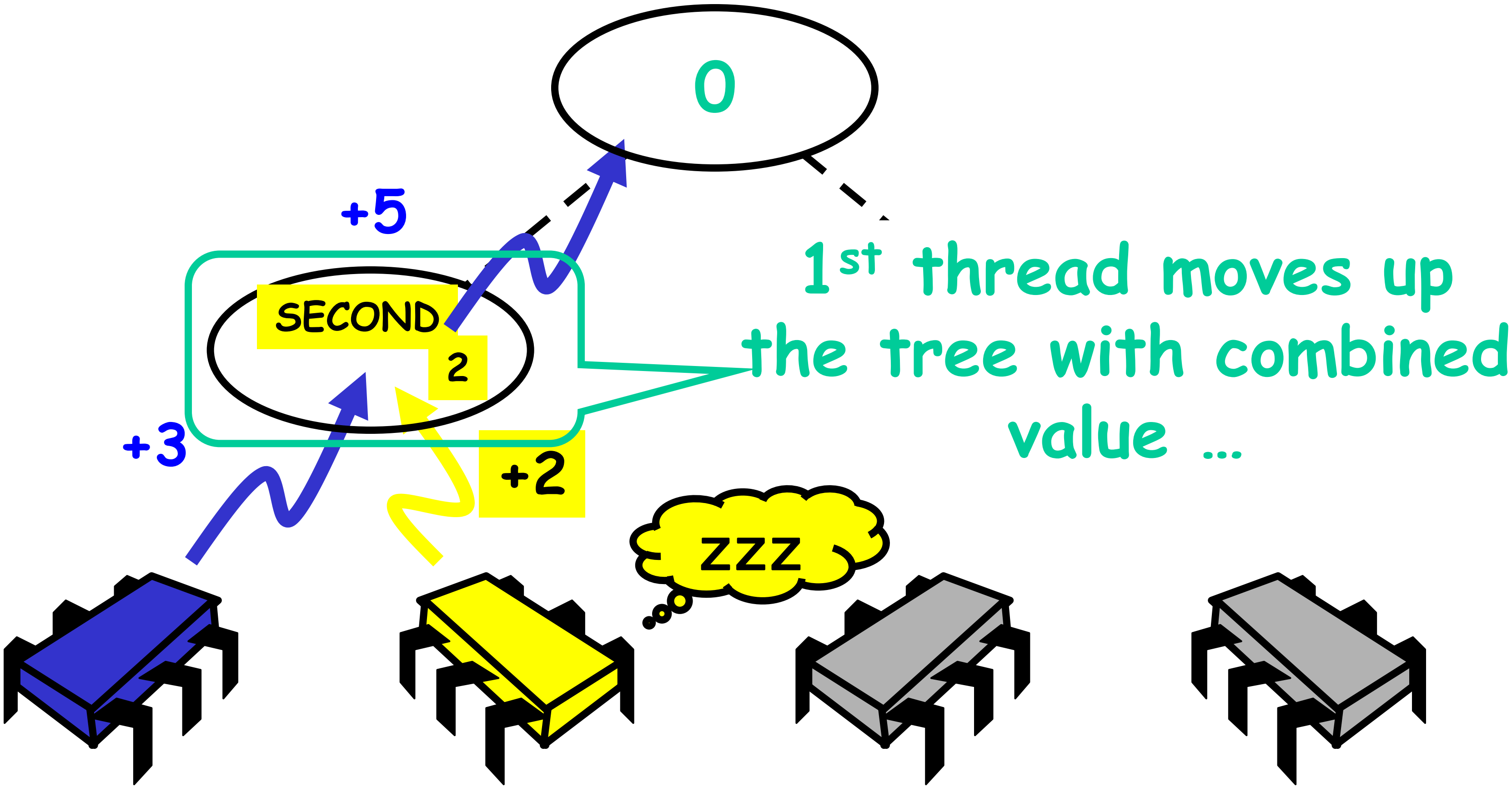
1st thread locked out until 2nd provides value

Combining Phase

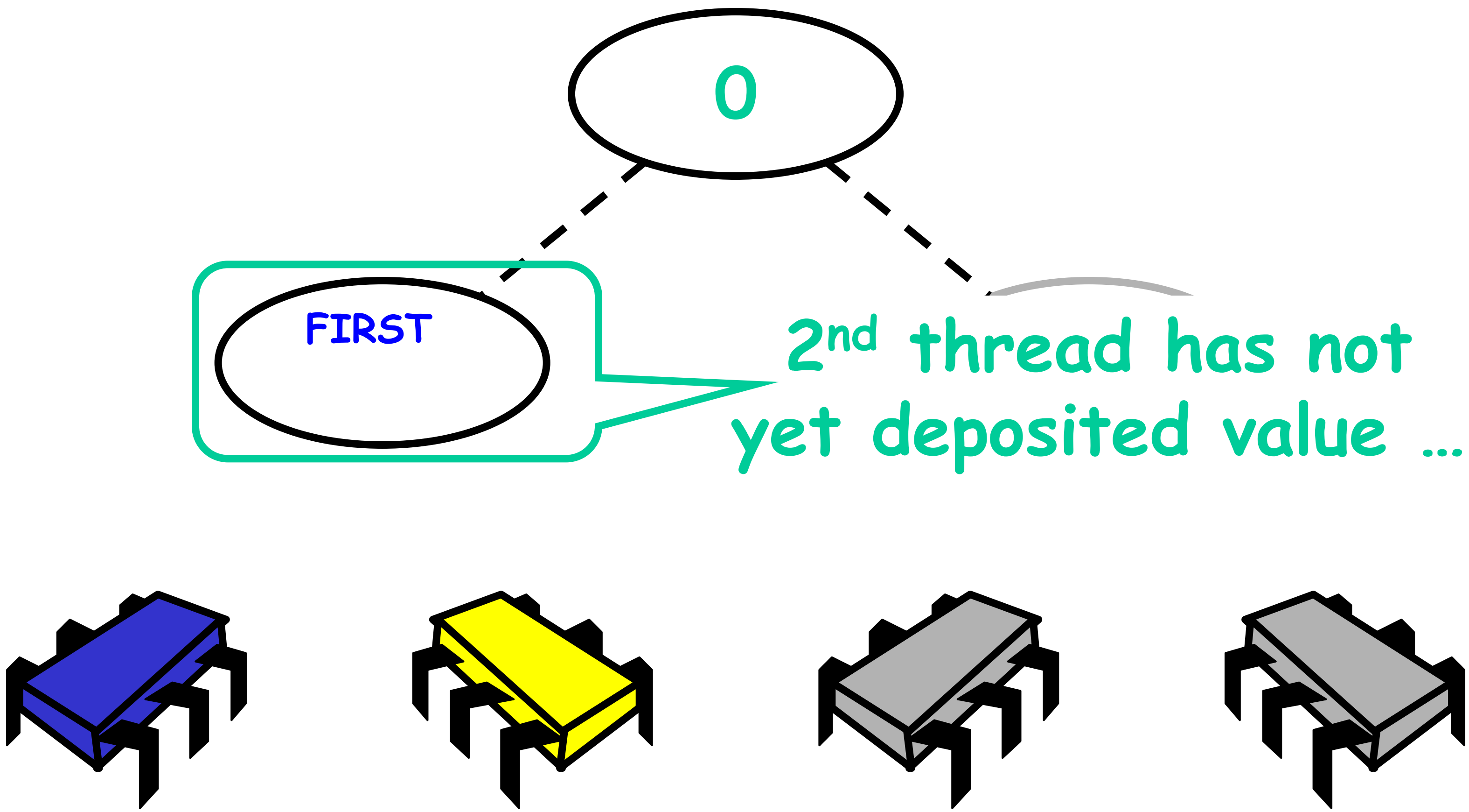


2nd thread deposits value to be combined, unlocks node, & waits …

Combining Phase



Combining (reloaded)



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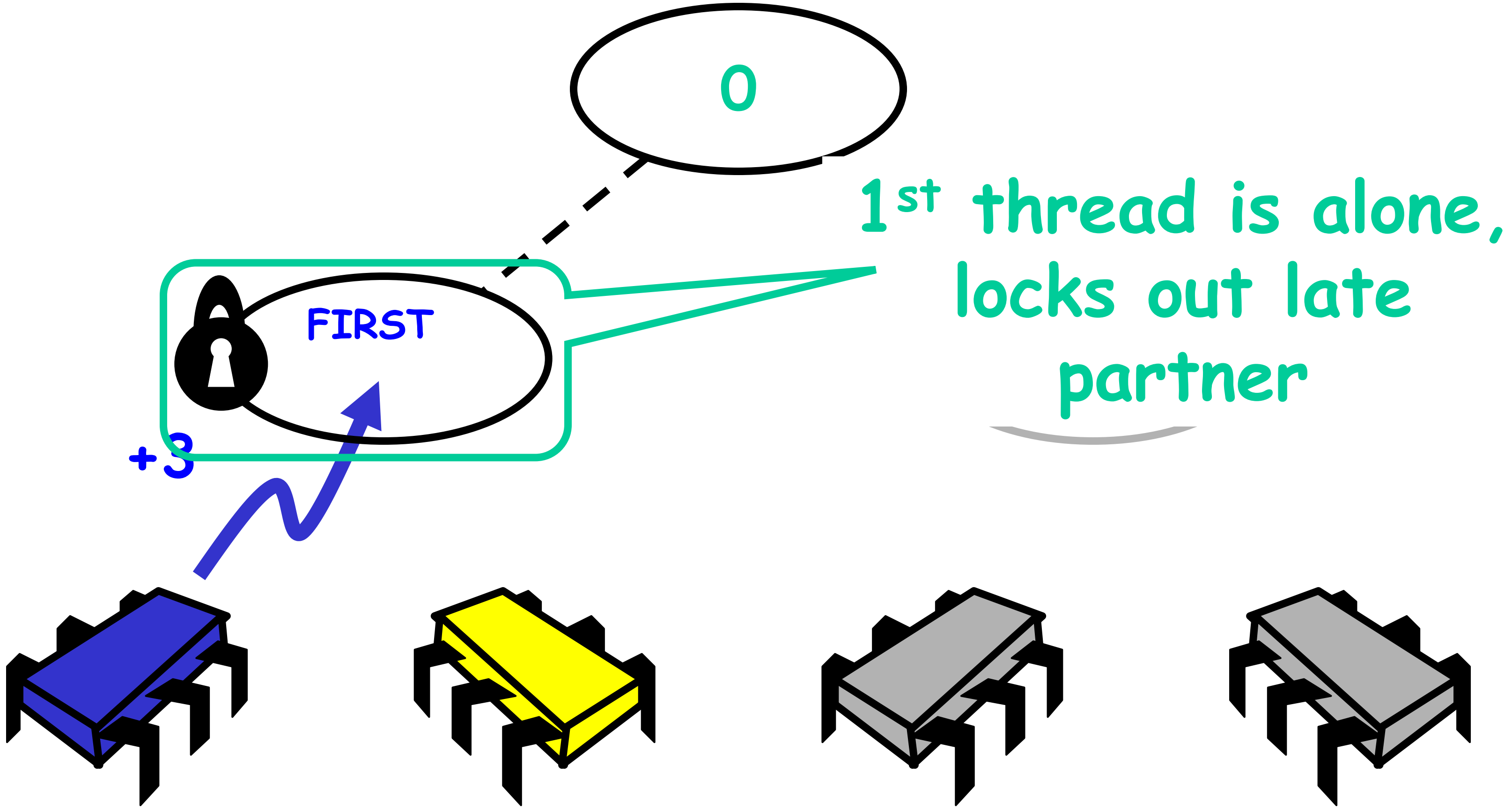
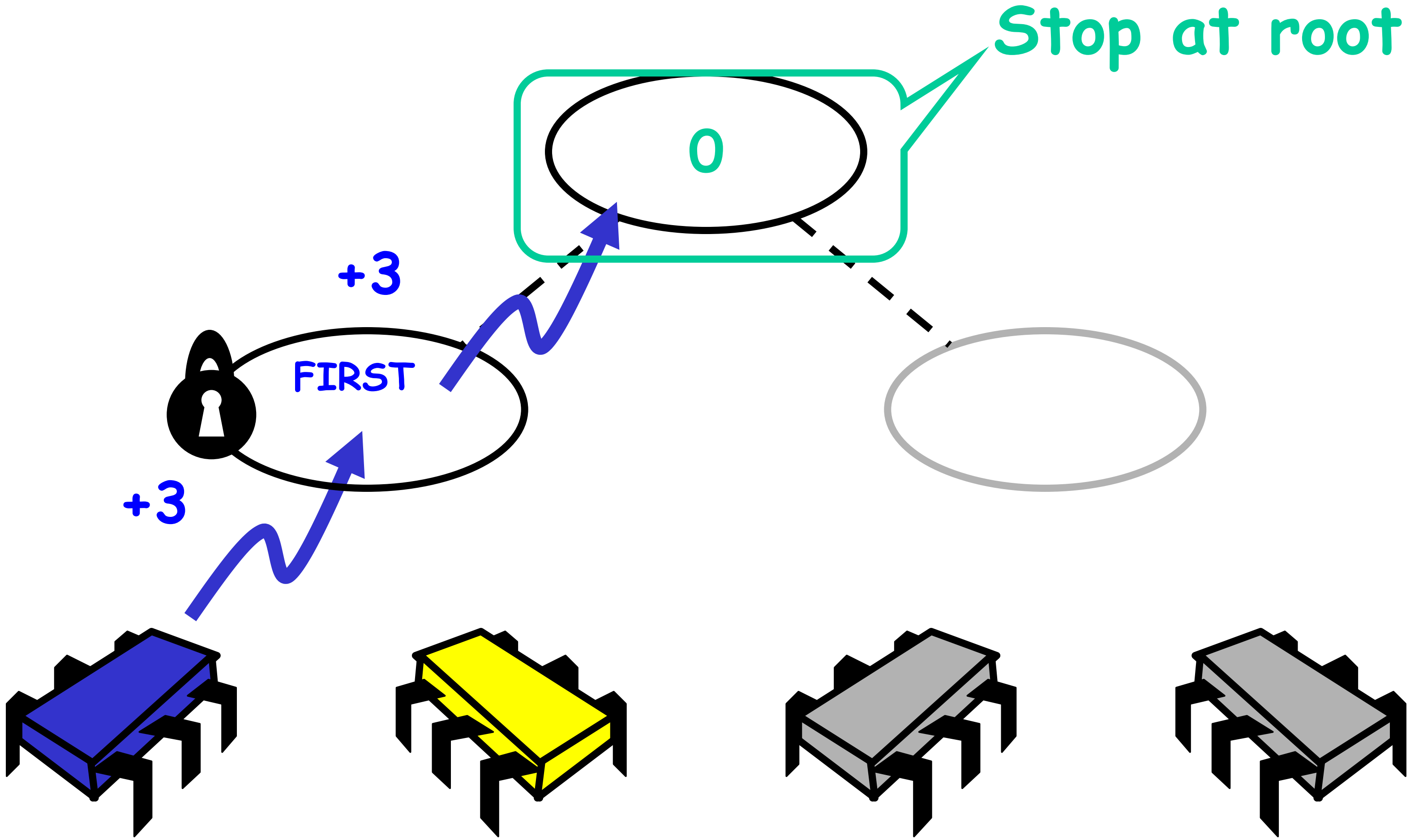
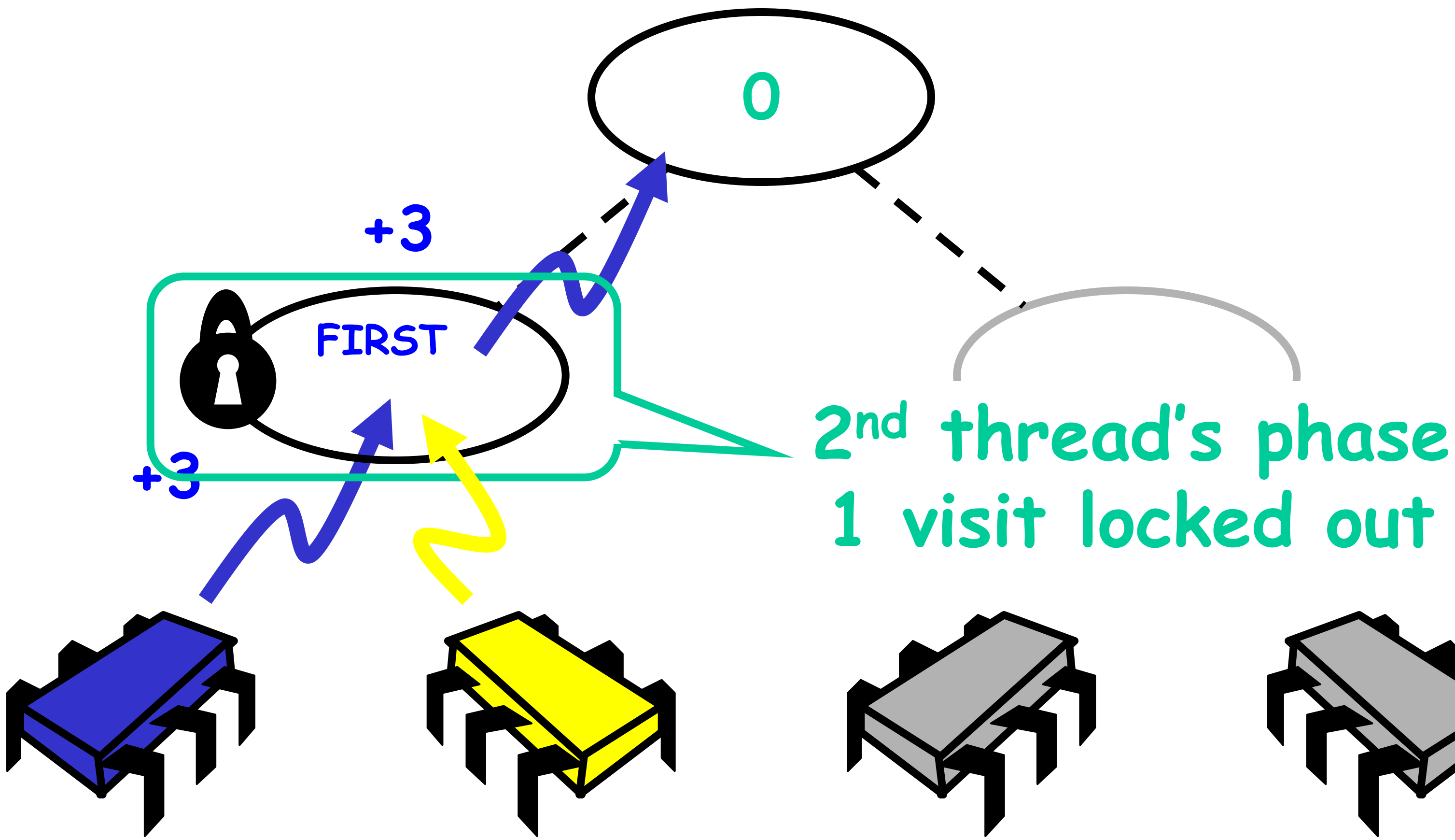


Combining (reloaded)



Combining (reloaded)



Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
  combined = node.combine(combined);
  stack.push(node);
  node = node.parent;
  }
```

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
   combined = node.combine(combined);
   stack.push(node);
   node = node.parent;
   }
```

Start at leaf

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
   combined = node.combine(combined);
   stack.push(node);
   node = node.parent;
   }
```

Add 1

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
  combined = node.combine(combined);
  stack.push(node);
  node = node.parent;
  }
```

Revisit nodes visited in phase 1

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
  combined = node.combine(combined);
  stack.push(node);
  node = node.parent;
  }
```

Accumulate combined values, if any

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
  combined = node.combine(combined);
  stack.push(node);
  node = node.parent;
  }
```

We will retraverse path in reverse order …

Combining Navigation

```
node = myLeaf;
int combined = 1;
while (node != stop) {
  combined = node.combine(combined);
  stack.push(node);
  node = node.parent;
  }
```

Move up the tree

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Wait until node is unlocked

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Lock out late attempts to combine

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Remember our contribution

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Check status

Combining Phase Node

```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

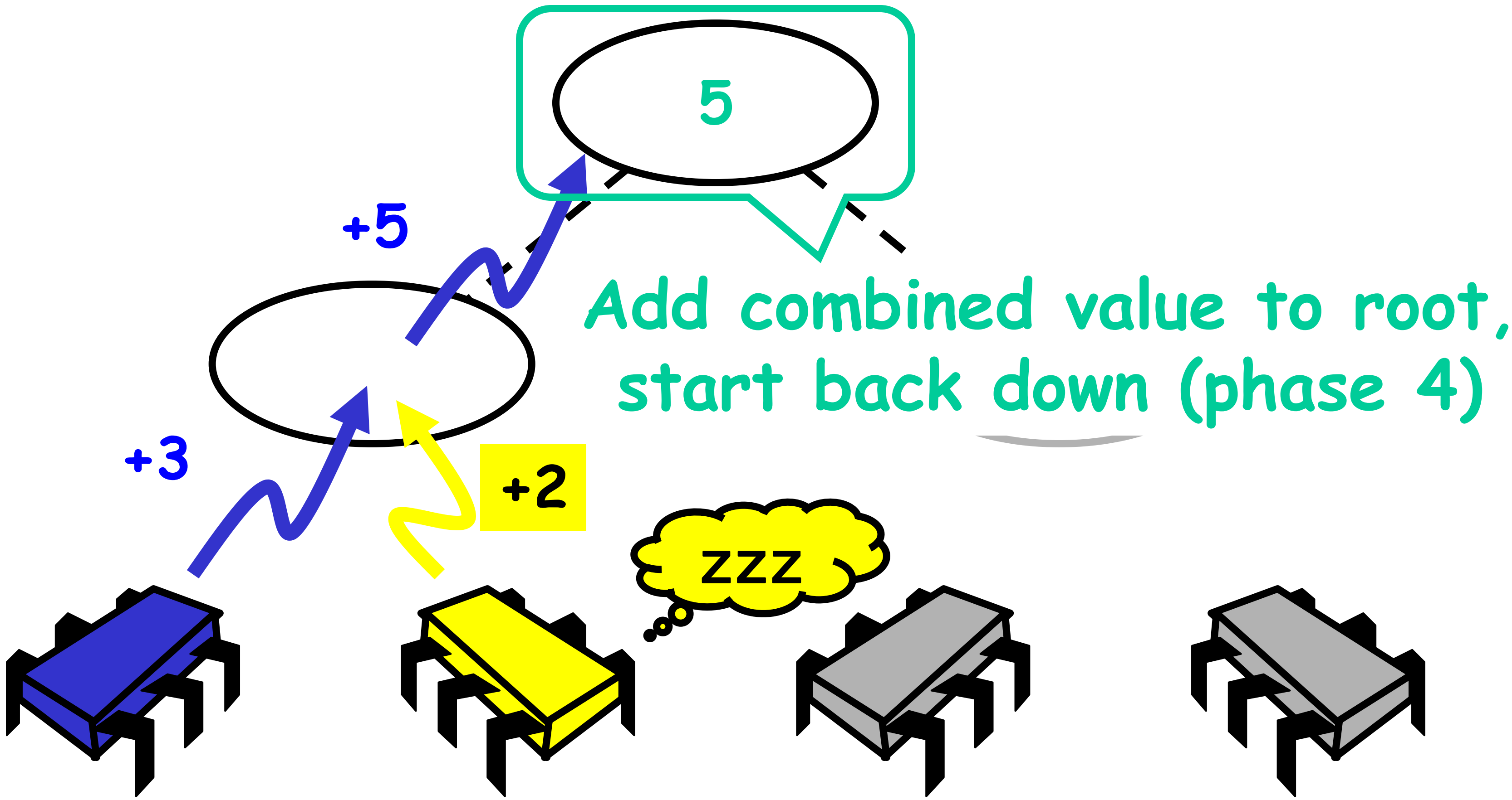
1st thread is alone

Combining Node

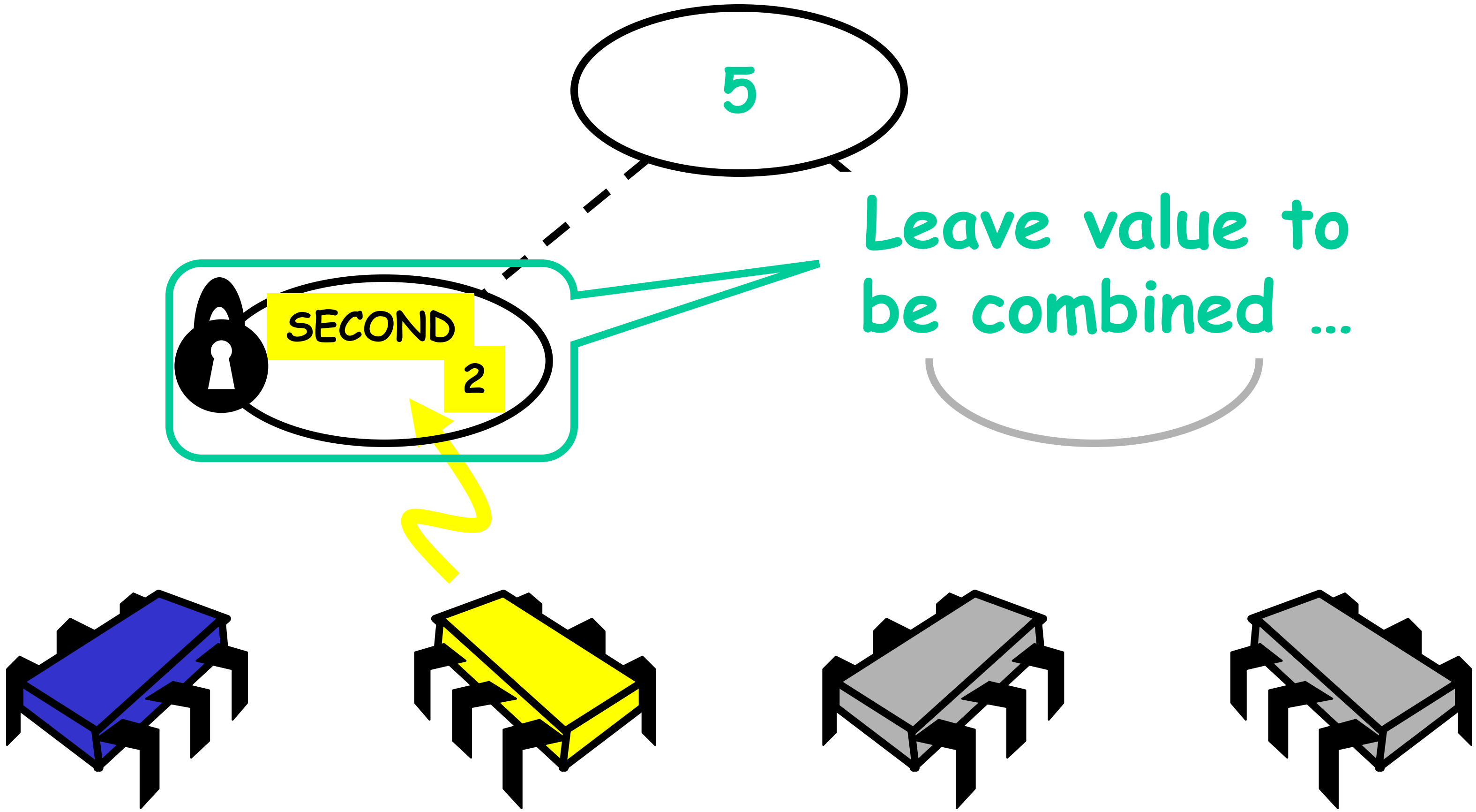
```
synchronized int combine(int combined) {
  while (locked) wait();
  locked = true;
  firstValue = combined;
  switch (cStatus) {
    case FIRST:
      return firstValue;
    case SECOND:
      return firstValue + secondValue;
    default: …
    }
  }
```

Combine with 2nd thread

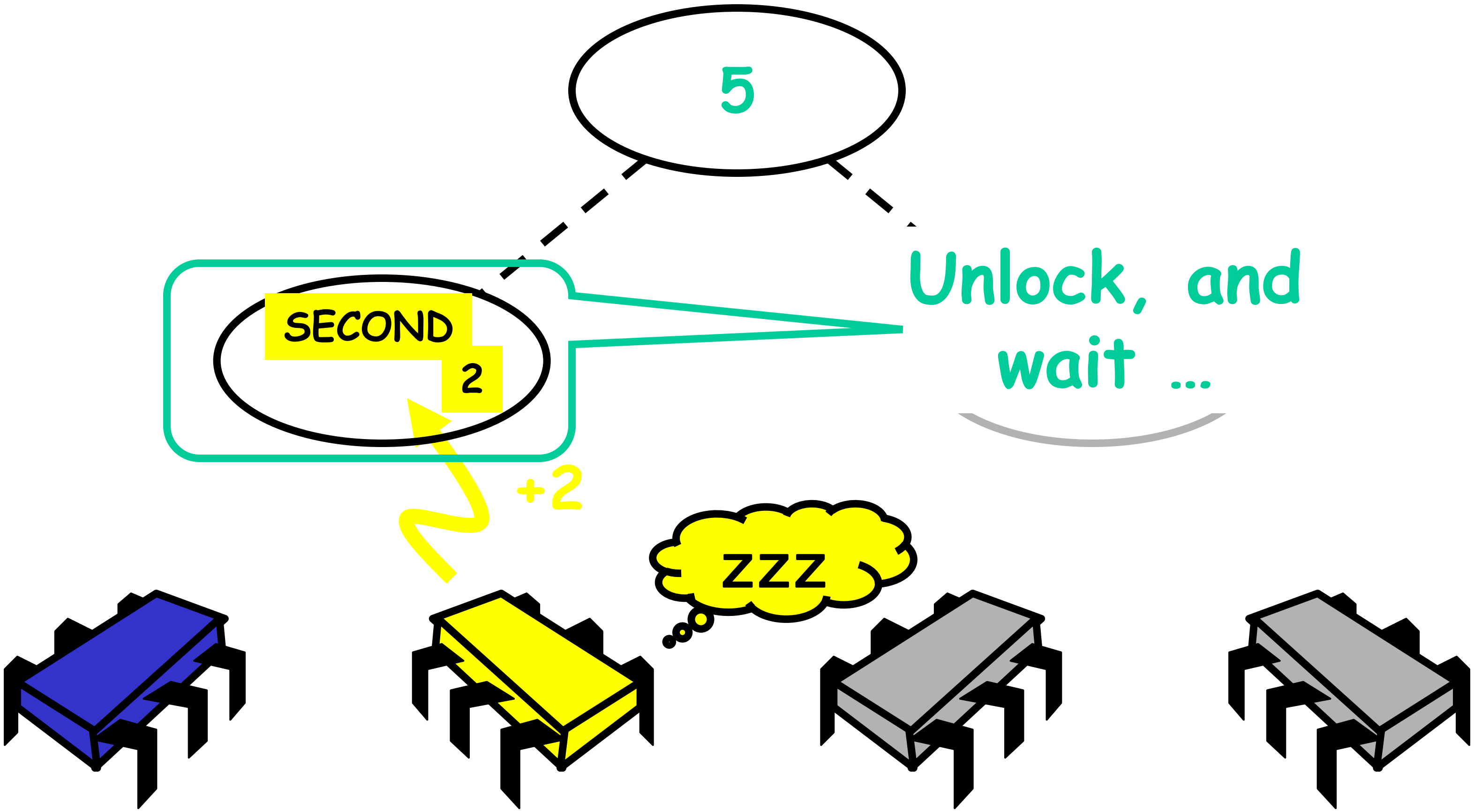
Operation Phase



Operation Phase (reloaded)



Operation Phase (reloaded)



Operation Phase Navigation

```
prior = stop.op(combined);
```

Operation Phase Navigation

```
prior = stop.op(combined);
```

Get result of combining

Operation Phase Node

```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue = result;
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

At Root

```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue = result;
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

Add sum to root, return prior value

Intermediate Node

```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue = result;
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

Deposit value for later combining …

Intermediate Node

```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue = result;
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

Unlock node, notify 1st thread

Intermediate Node

```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

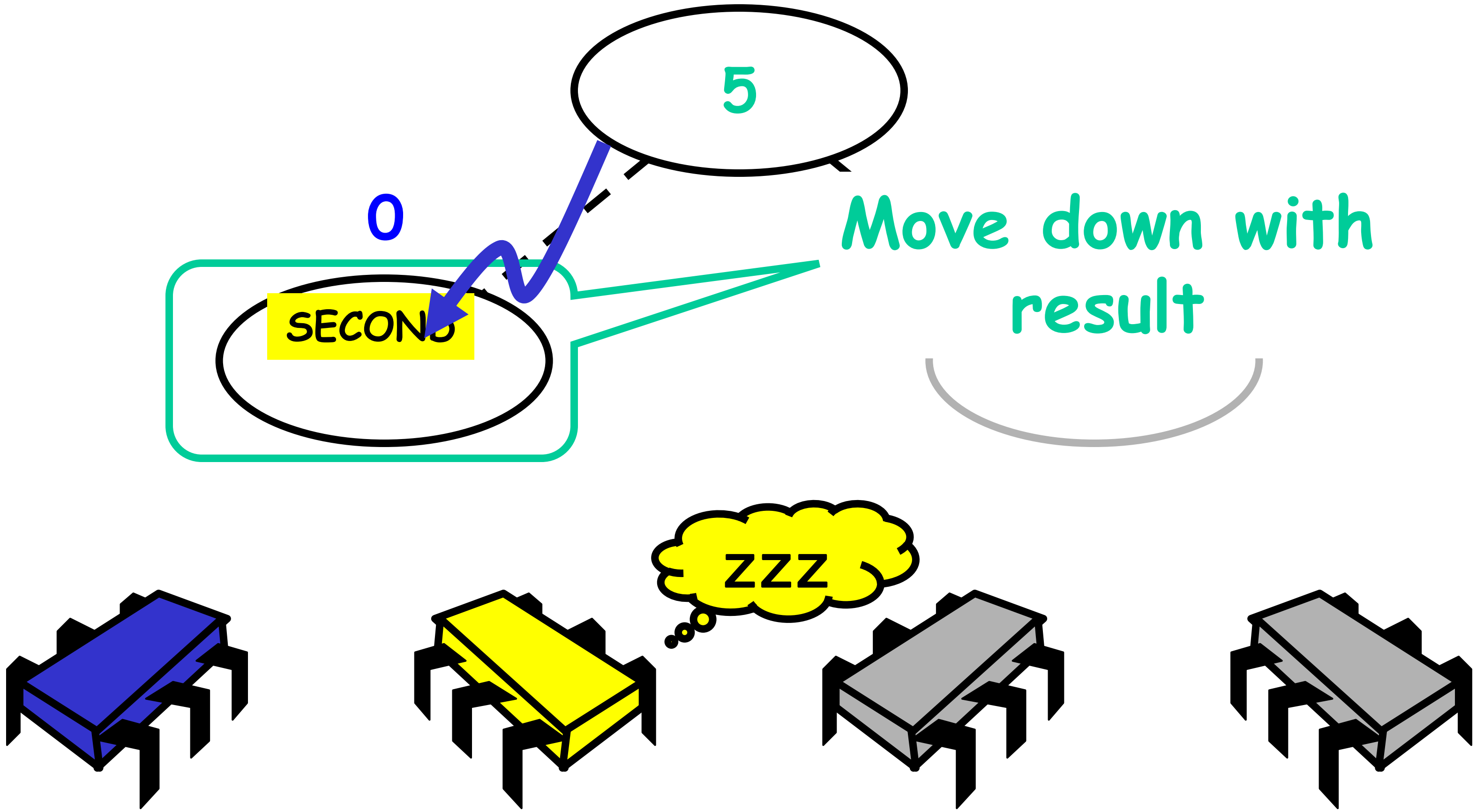
Wait for 1st thread to deliver results

Intermediate Node

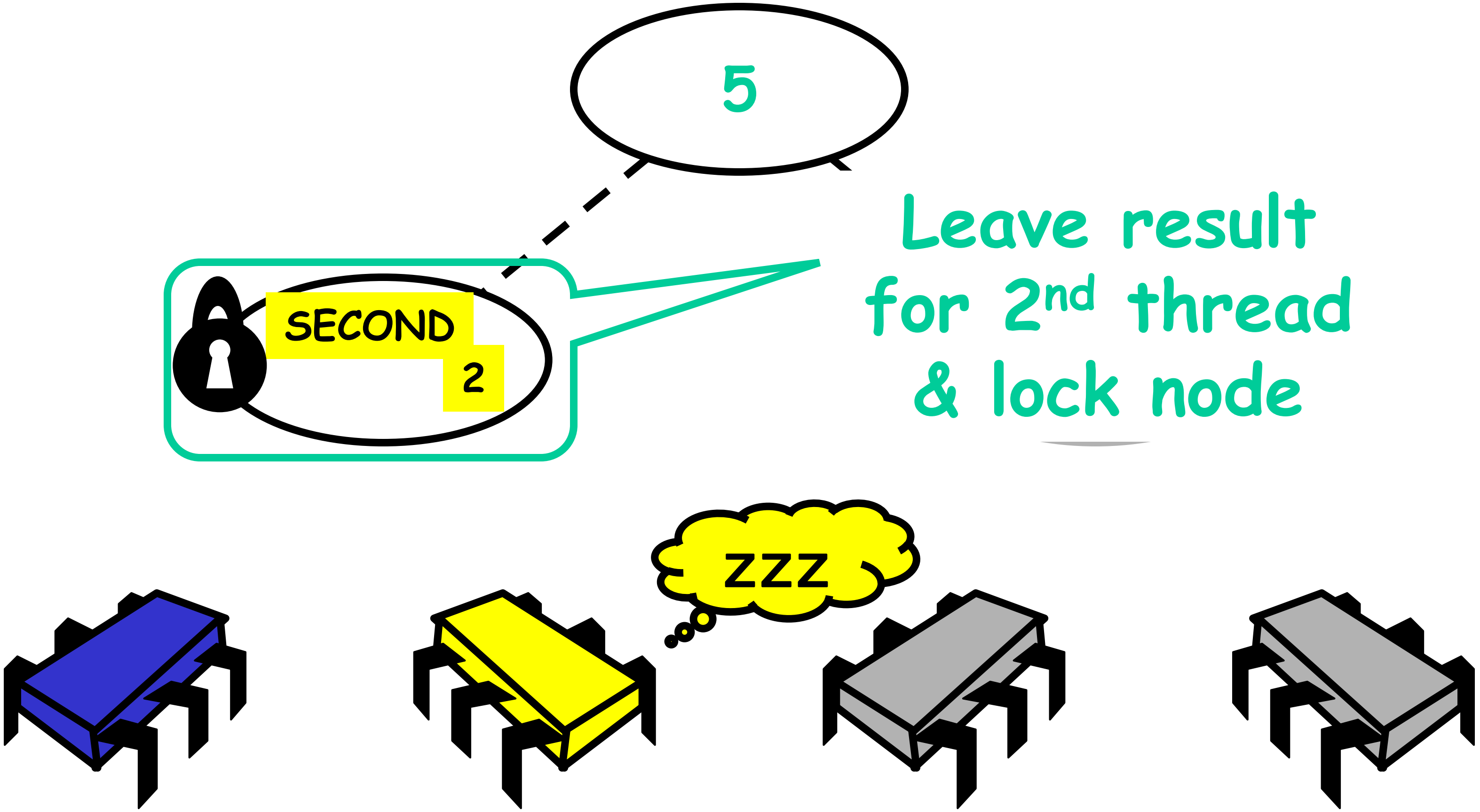
```
synchronized int op(int combined) {
  switch (cStatus) {
    case ROOT: int oldValue
      result += combined;
      return oldValue;
    case SECOND: secondValue = combined;
      locked = false; notifyAll();
      while (cStatus != CStatus.DONE) wait();
      locked = false; notifyAll();
      cStatus = CStatus.IDLE;
      return result;
    default: …
```

Unlock node & return

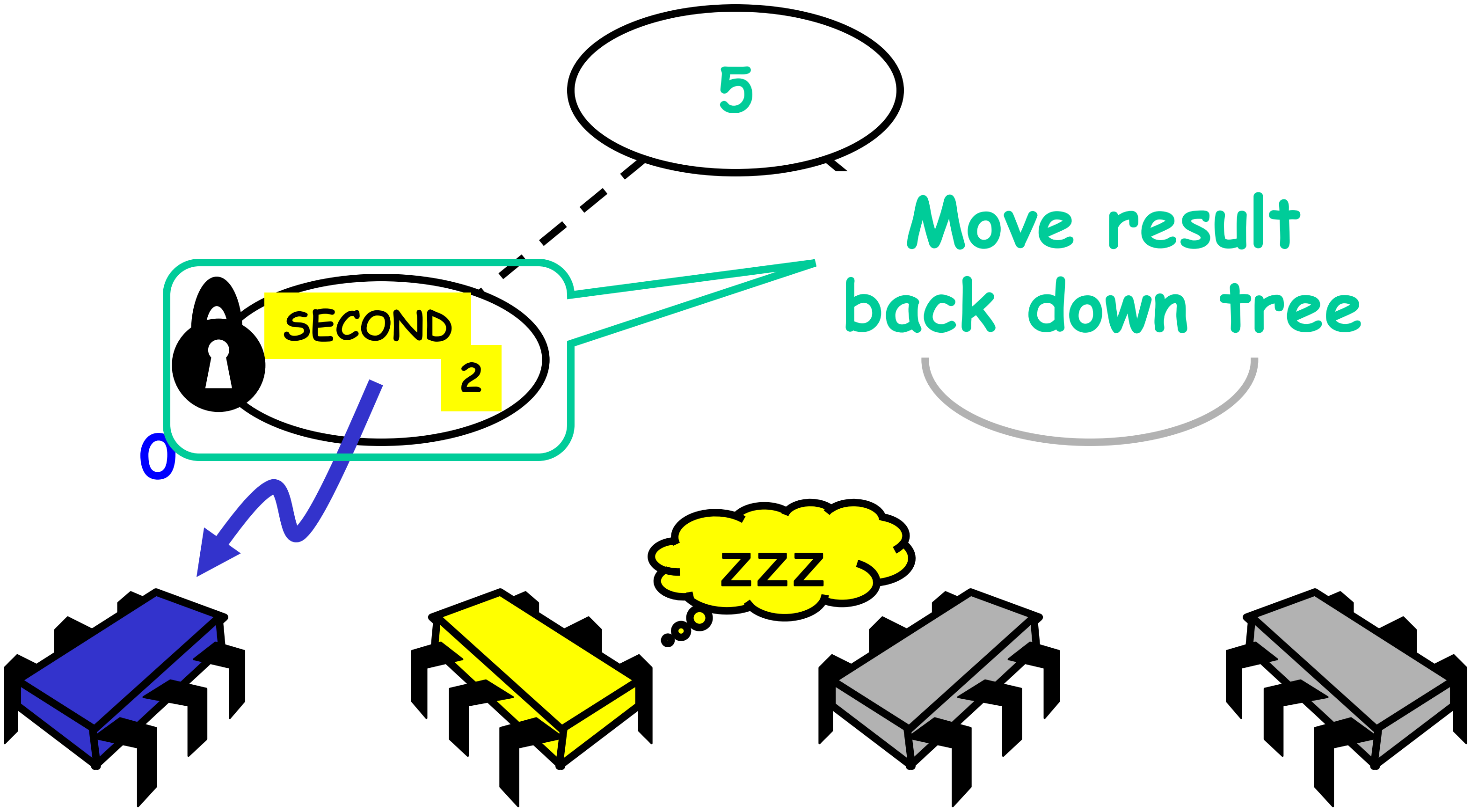
Distribution Phase



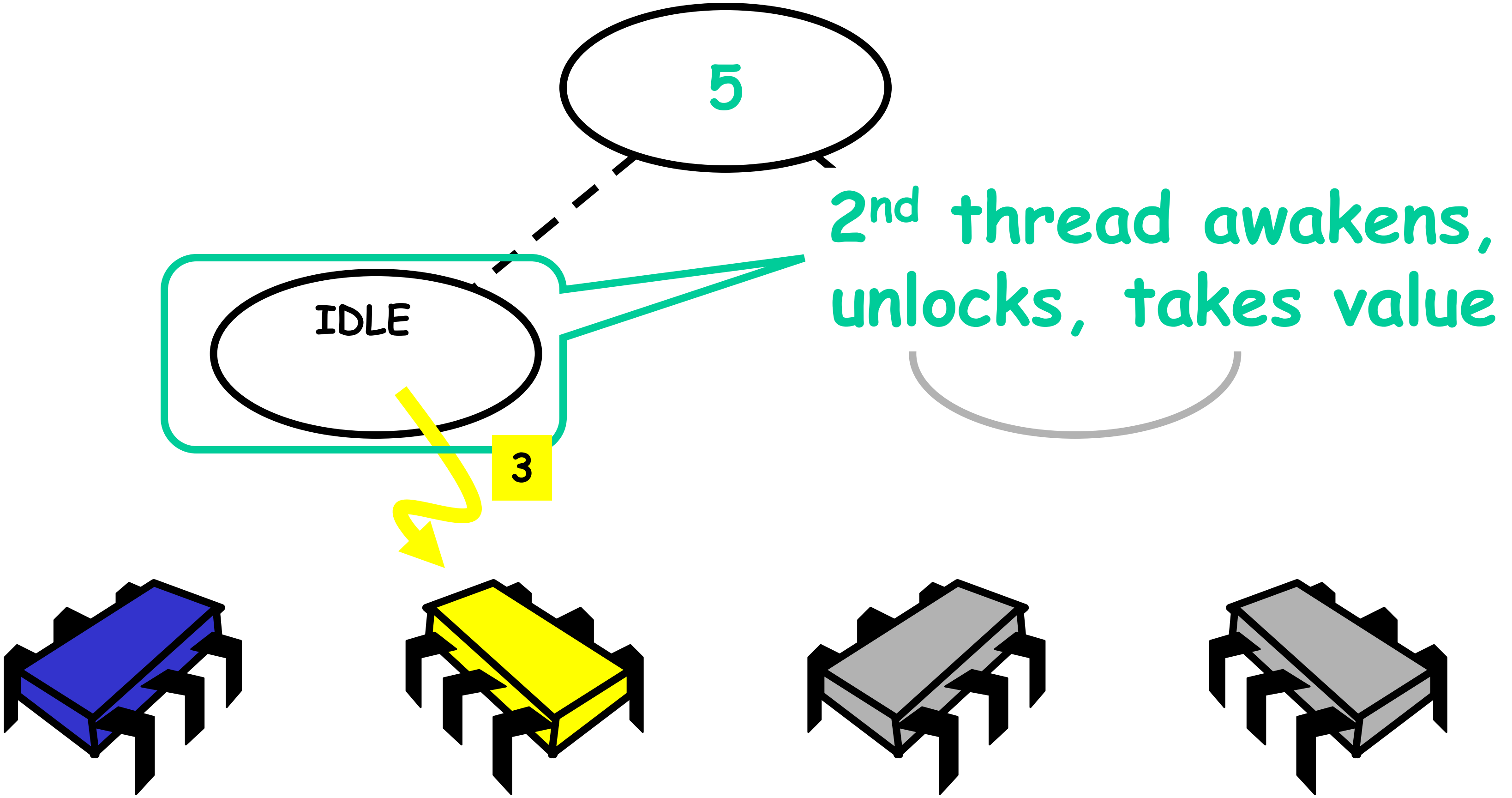
Distribution Phase



Distribution Phase



Distribution Phase



Distribution Phase Navigation

```
while (!stack.empty()) {
  node = stack.pop();
  node.distribute(prior);
  }
return prior;
```

Distribution Phase Navigation

```
while (!stack.empty()) {
  node = stack.pop();
  node.distribute(prior);
  }
return prior;
```

Traverse path in reverse order

Distribution Phase Navigation

```
while (!stack.empty()) {
  node = stack.pop();
  node.distribute(prior);
  }
return prior;
```

Distribute results to waiting 2nd threads

Distribution Phase Navigation

```
while (!stack.empty()) {
  node = stack.pop();
  node.distribute(prior);
  }
return prior;
```

Return result to caller

Distribution Phase

```
synchronized void distribute(int prior) {
   switch (cStatus) {
     case FIRST:
       cStatus = CStatus.IDLE;
       locked = false; notifyAll();
       return;
     case SECOND:
       result = prior + firstValue;
       cStatus = CStatus.DONE; notifyAll();
       return;
     default: …
```

Distribution Phase

```
synchronized void distribute(int prior) {
  switch (cStatus) {
    case FIRST:
      cStatus = CStatus.IDLE;
      locked = false; notifyAll();
      return;
    case SECOND:
      result = prior + firstValue;
      cStatus = CStatus.DONE; notifyAll();
      return;
    default: …
```

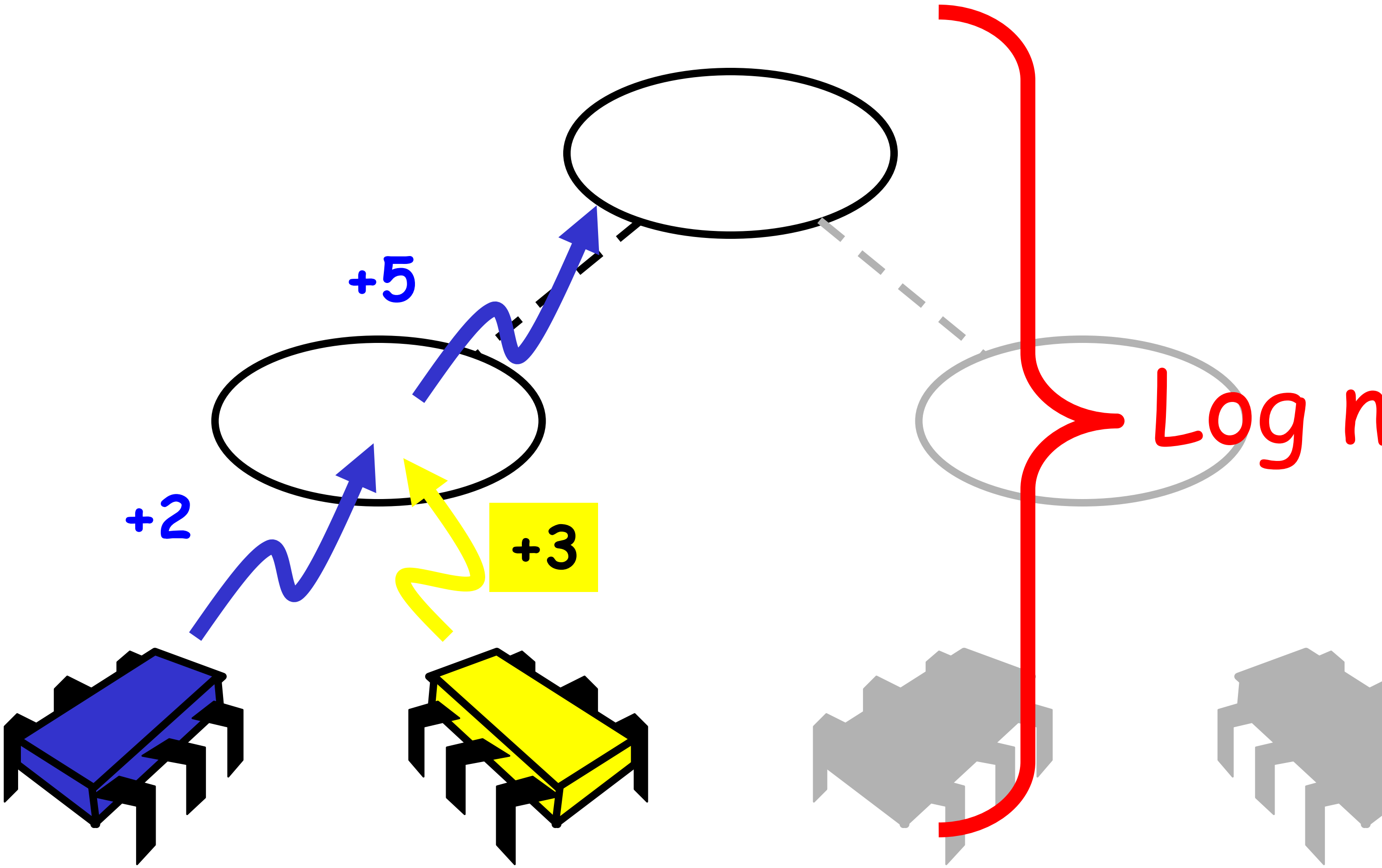
No combining, unlock node & reset

Distribution Phase

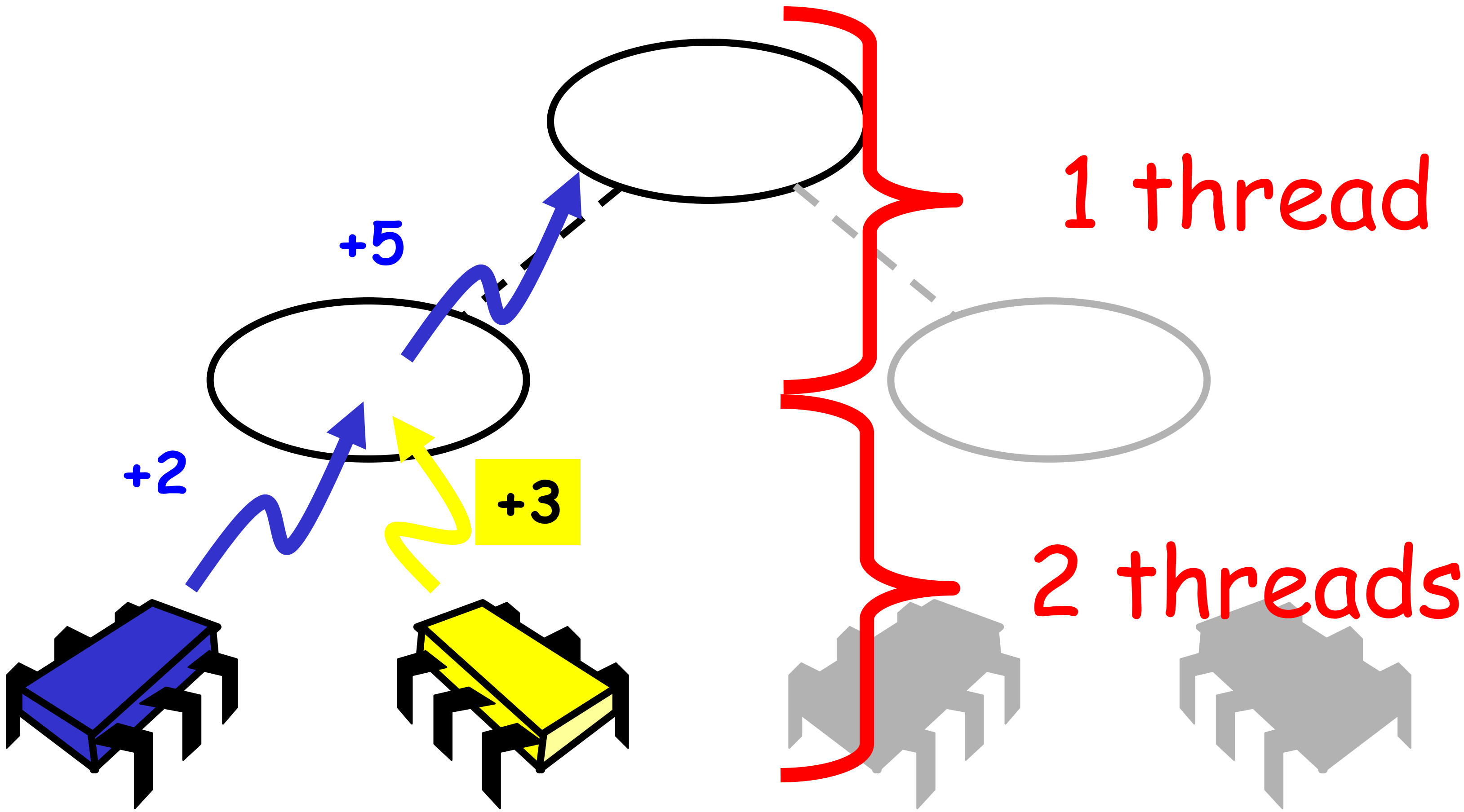
```
synchronized void distribute(int prior) {
   switch (cStatus) {
     case FIRST:
       cStatus = CStatu
       locked = false; notifyAll();
       return;
     case SECOND:
       result = prior + firstValue;
       cStatus = CStatus.DONE; notifyAll();
       return;
     default: …
```

Notify 2nd thread that result is available

Bad News: High Latency



Good News: Real Parallelism



Throughput Puzzles

- Ideal circumstances
  - All n threads move together, combine
  - n increments in $O(\log n)$ time
- Worst circumstances
  - All n threads slightly skewed, locked out
  - n increments in $O(n \cdot \log n)$ time

Index Distribution Benchmark

```
void indexBench(int iters, int work) {
 while (int i < iters) {
  i = r.getAndIncrement();
  Thread.sleep(random() % work);
 }}
```

Index Distribution Benchmark

```
void indexBench(int iters, int work) {
 while (int i < iters) {
  i = r.getAndIncrement();
  Thread.sleep(random() % work);
 }}
```

How many iterations

Index Distribution Benchmark

```
void indexBench(int iters, int work) {
 while (int i < iters) {
  i = r.getAndIncrement();
  Thread.sleep(random() % work);
 }}
```

Expected time between incrementing counter

Index Distribution Benchmark

```
void indexBench(int iters, int work) {
 while (int i < iters) {
  i = r.getAndIncrement();
  Thread.sleep(random() % work);
 }}
```

Take a number

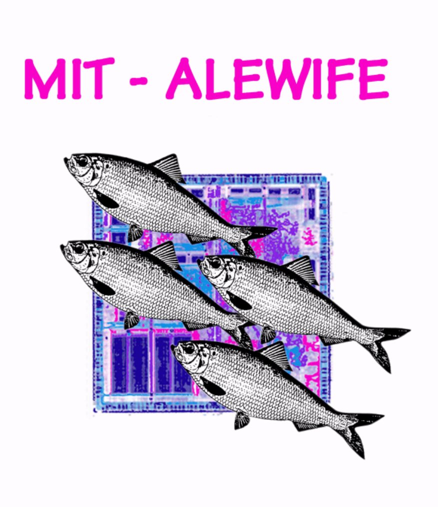
Index Distribution Benchmark

```
void indexBench(int iters, int work) {
 while (int i < iters) {
  i = r.getAndIncrement();
  Thread.sleep(random() % work);
 }}
```

Pretend to work
(more work, less concurrency)

Performance Benchmarks

- Alewife
  - NUMA architecture
  - Simulated

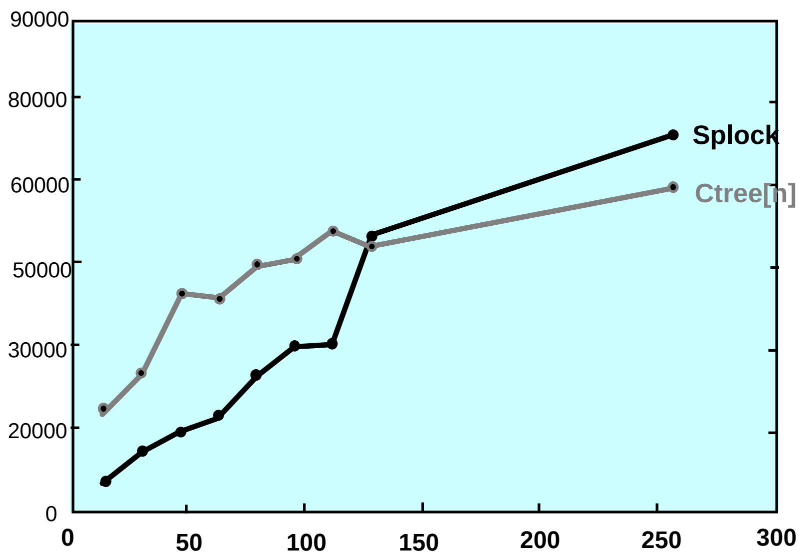


- **Throughput:**
  - average number of **inc** operations in 1 million cycle period.
- **Latency:**
  - average number of simulator cycles per **inc** operation.

Performance

cycles
per
operation

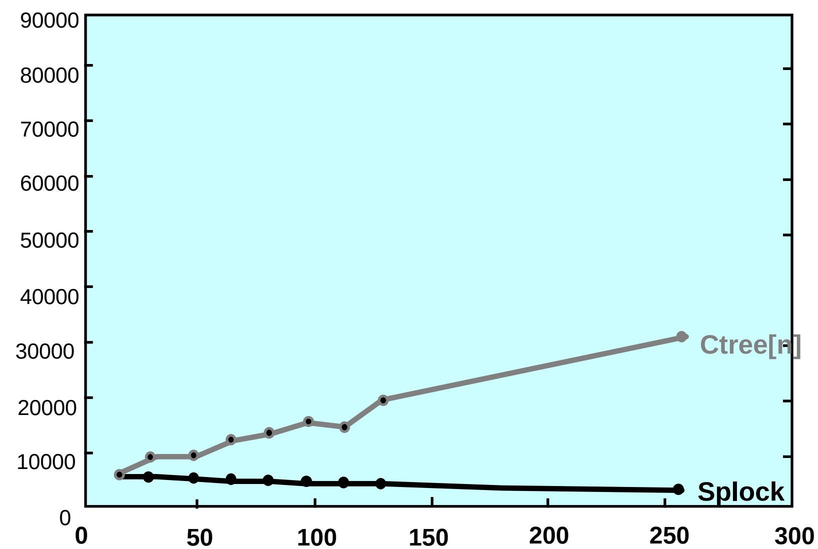
Latency:



num of processors

operations
per million
cycles

Throughput:



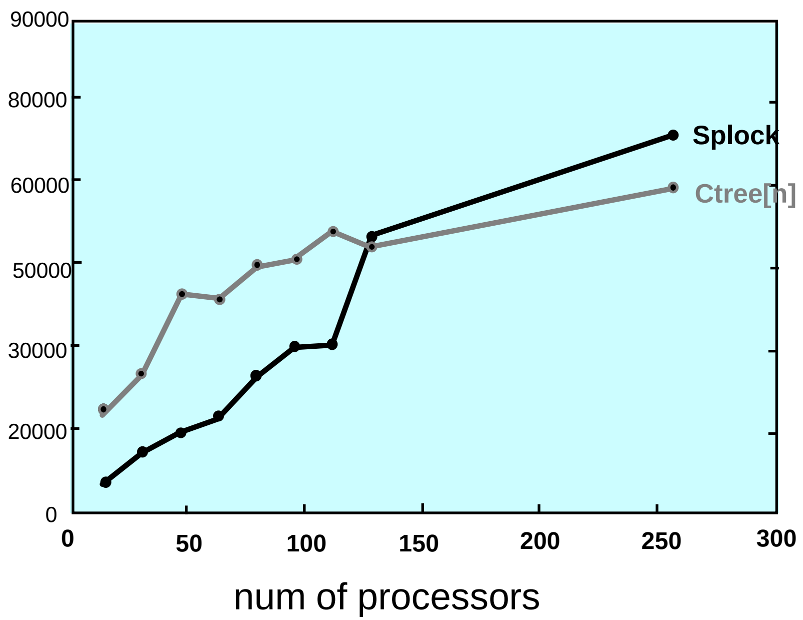
num of processors

work = 0

Performance

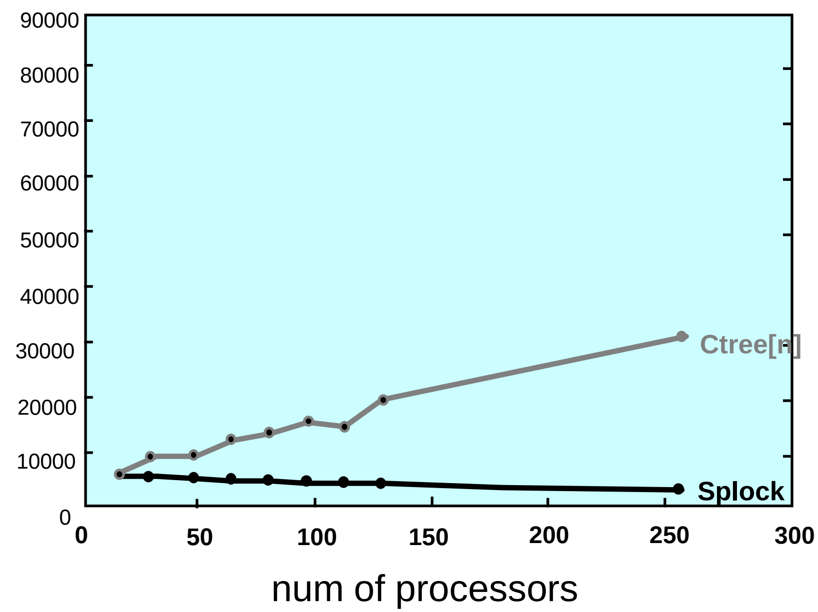
cycles per operation

Latency:



operations per million cycles

Throughput:

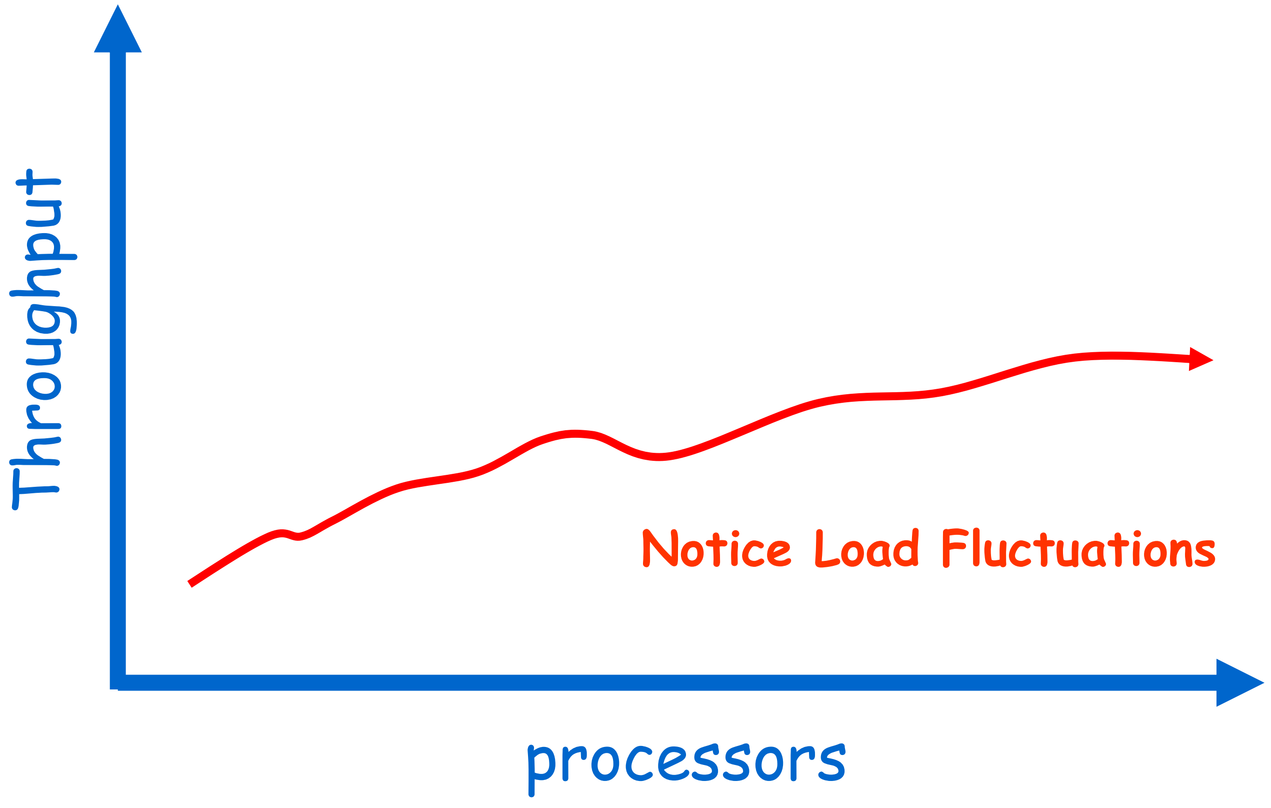


work = 0

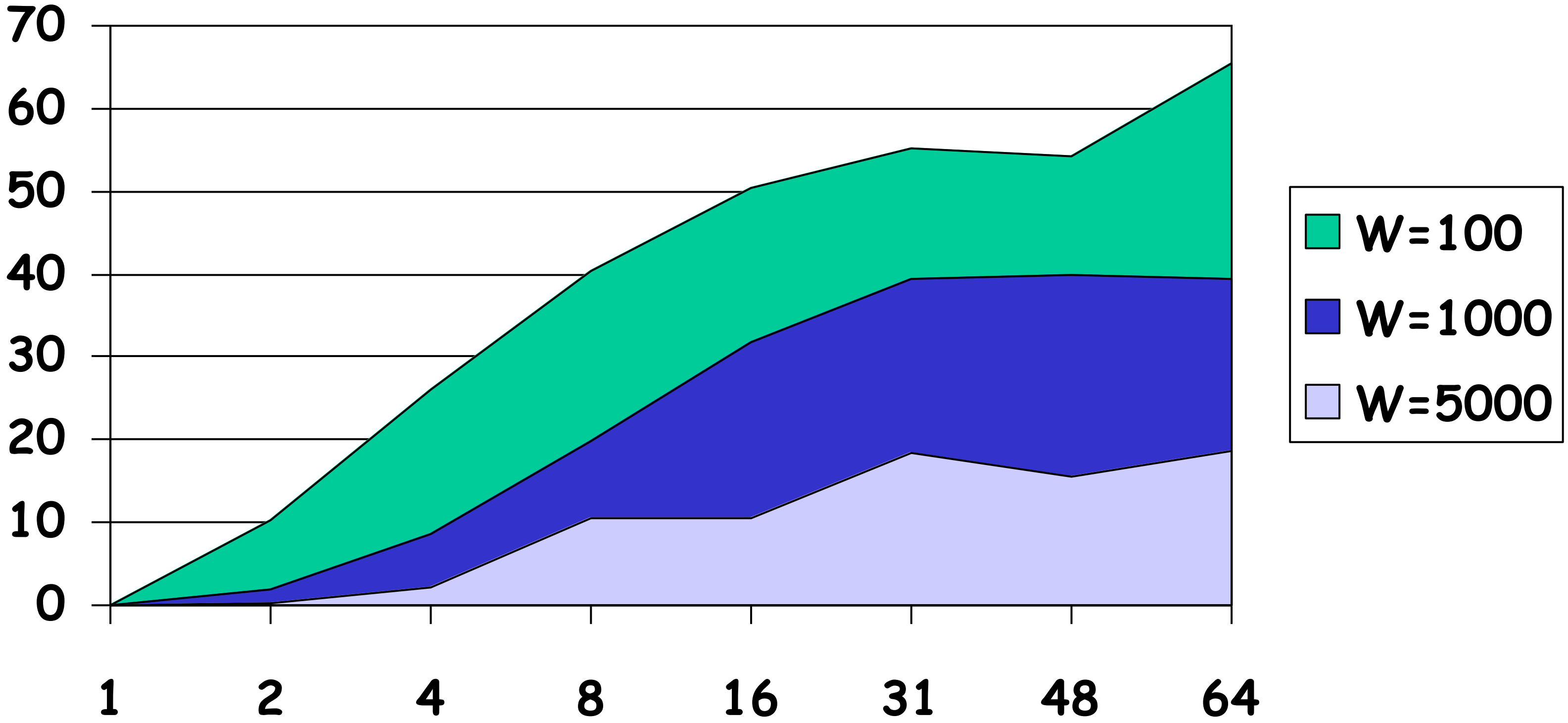
The Combining Paradigm

- Implements any RMW operation
- When tree is loaded
  - Takes 2 log n steps
  - for n requests
- Very sensitive to load fluctuations:
  - if the arrival rates drop
  - the combining rates drop
  - overall performance deteriorates!

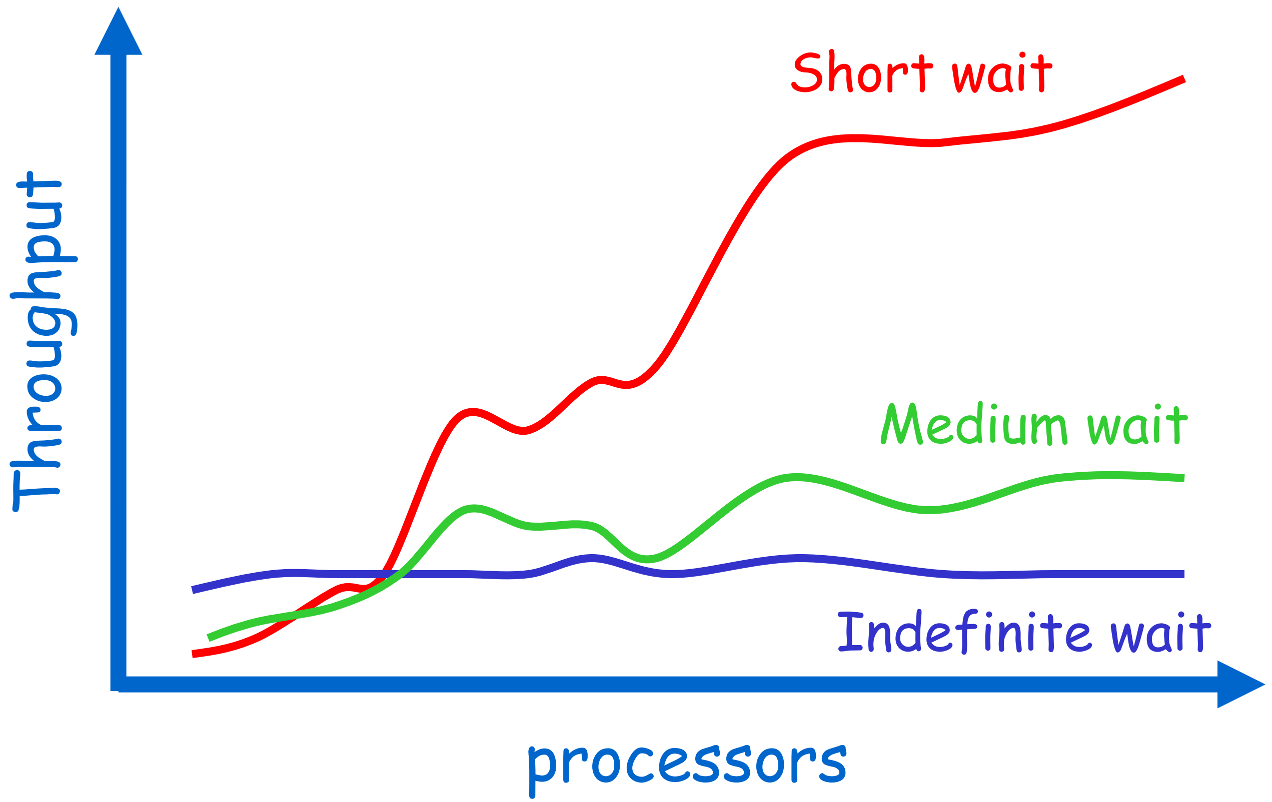
Combining Load Sensitivity



Combining Rate vs Work



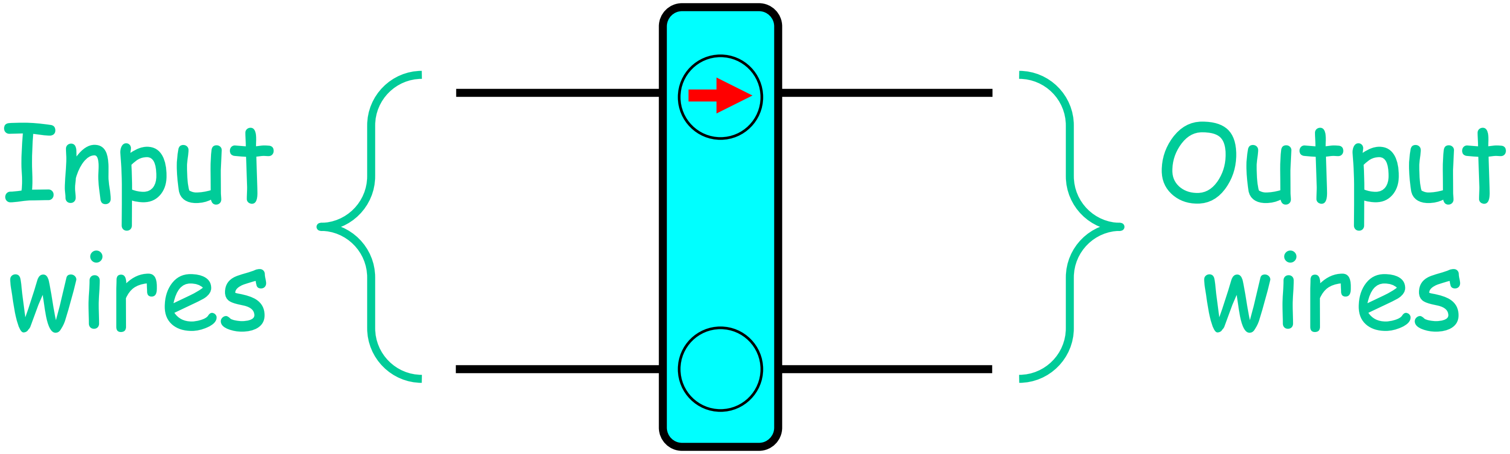
Better to Wait Longer



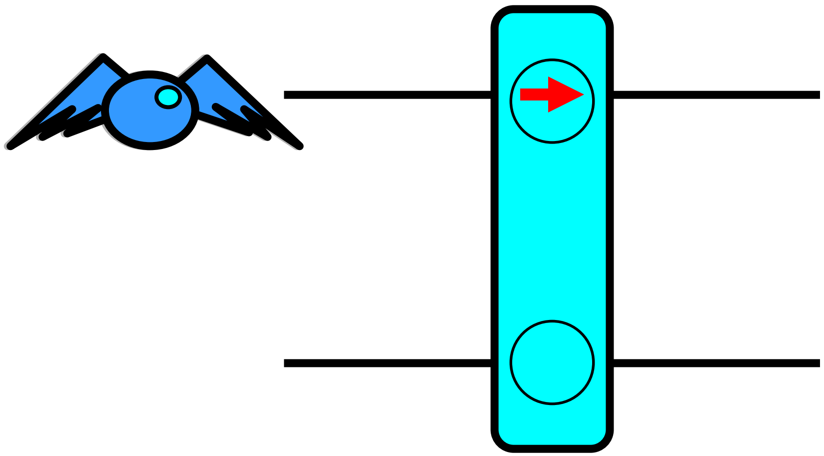
Conclusions

- Combining Trees
  - Work well under high contention
  - Sensitive to load fluctuations
  - Can be used for getAndMumble() ops
- Next
  - Counting networks
  - A different approach …

A Balancer

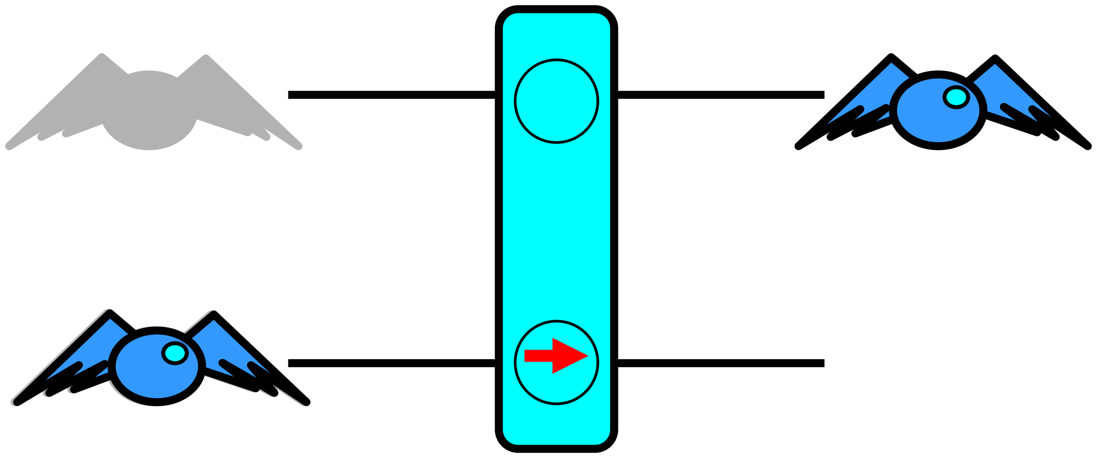


Tokens Traverse Balancers

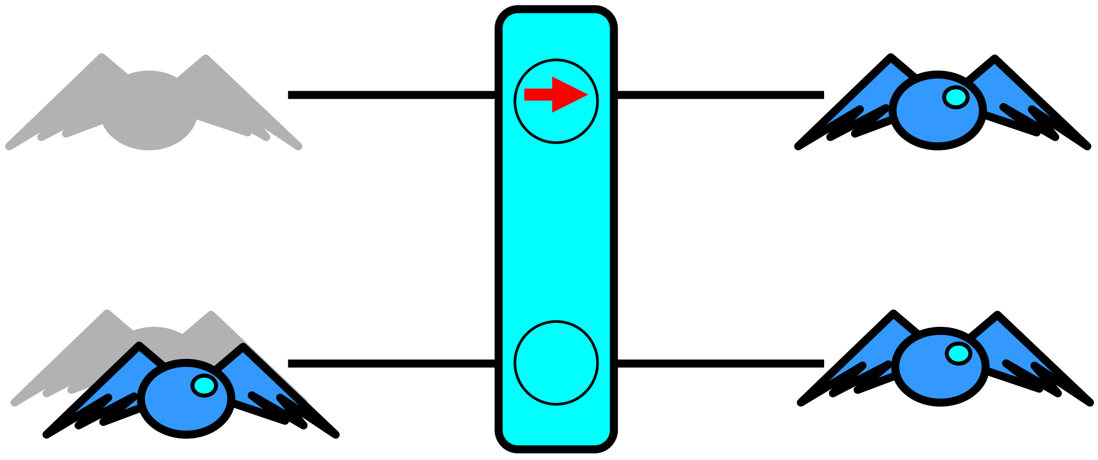


- Token i enters on any wire
- leaves on wire i mod (fan-out)

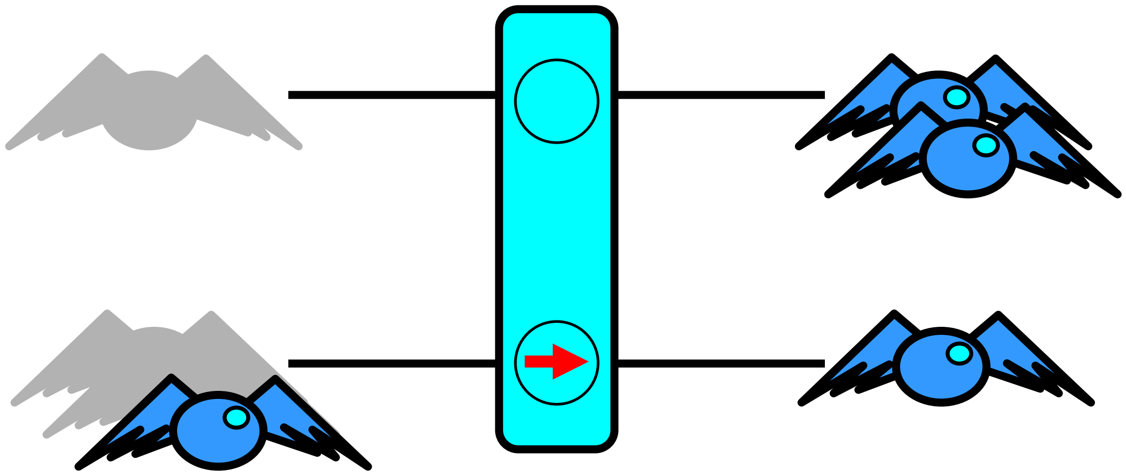
Tokens Traverse Balancers



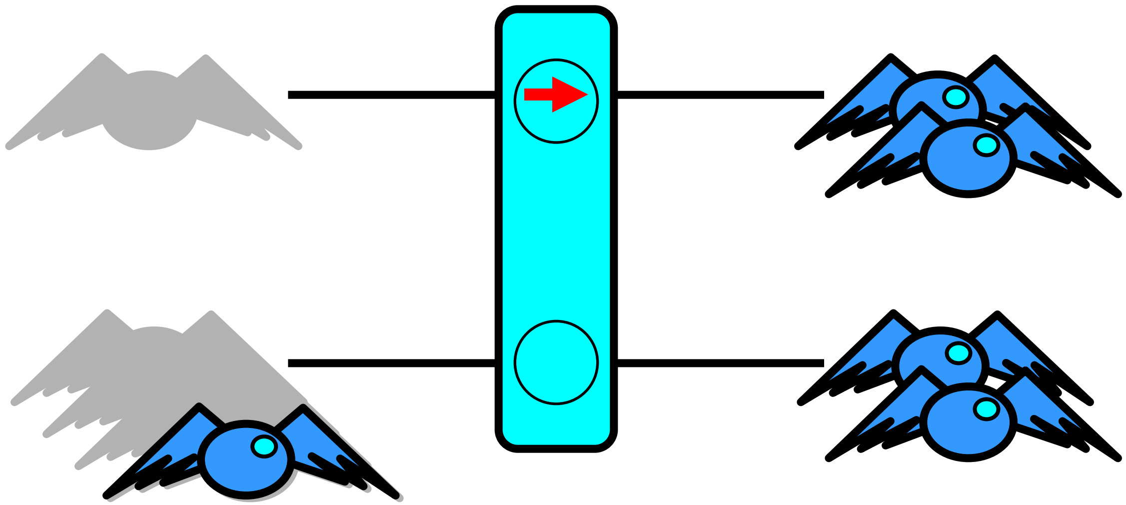
Tokens Traverse Balancers



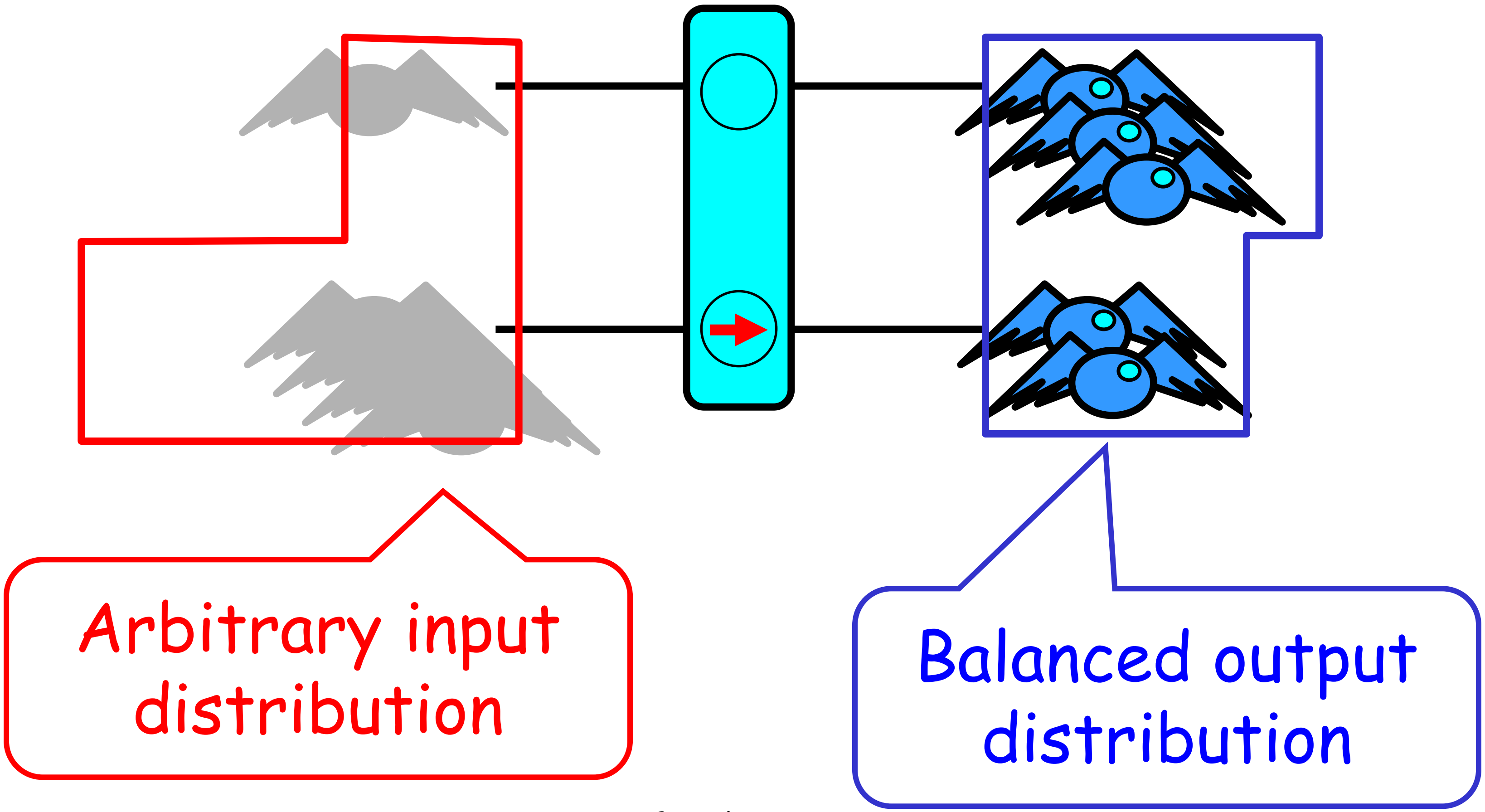
Tokens Traverse Balancers



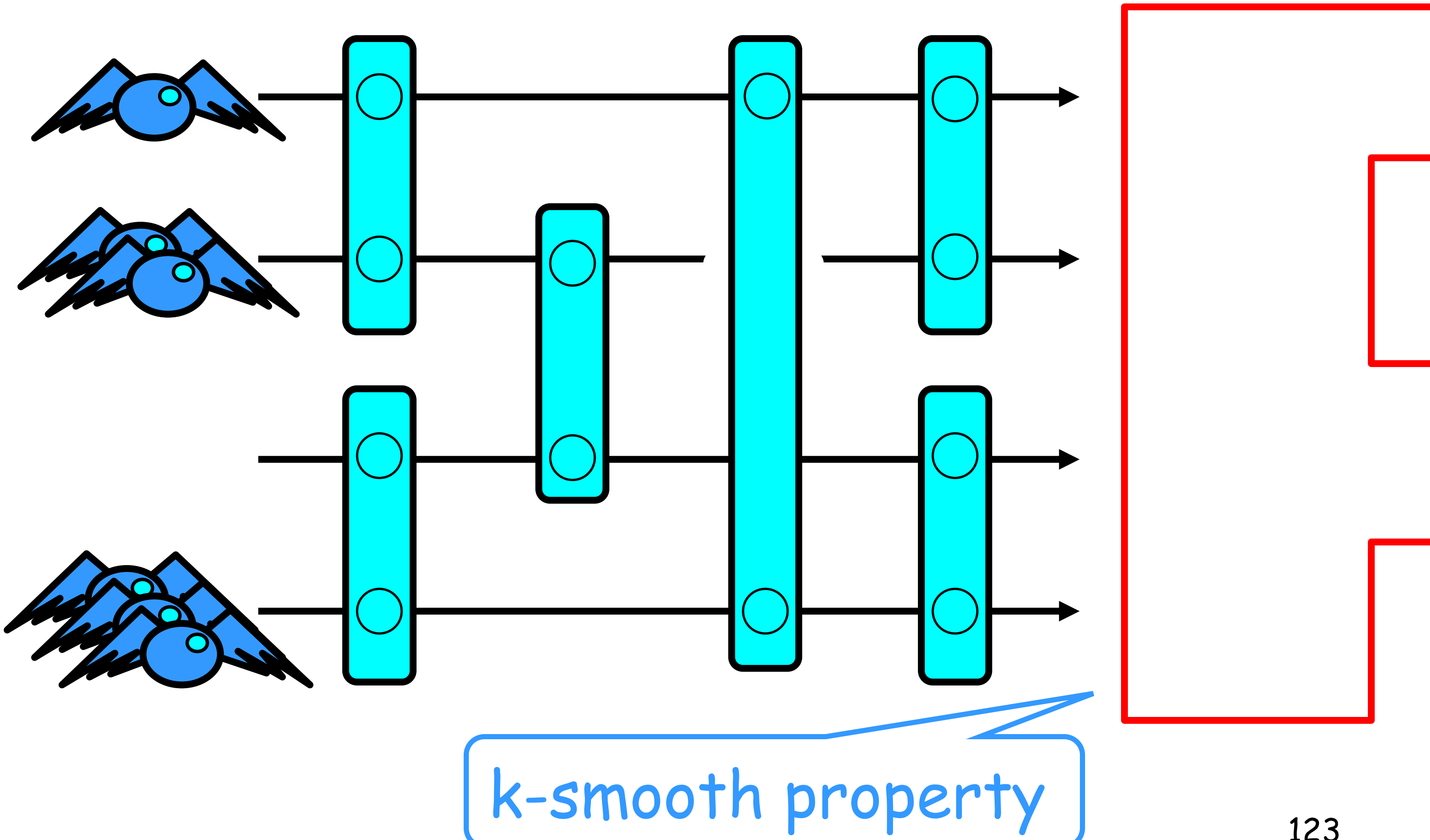
Tokens Traverse Balancers



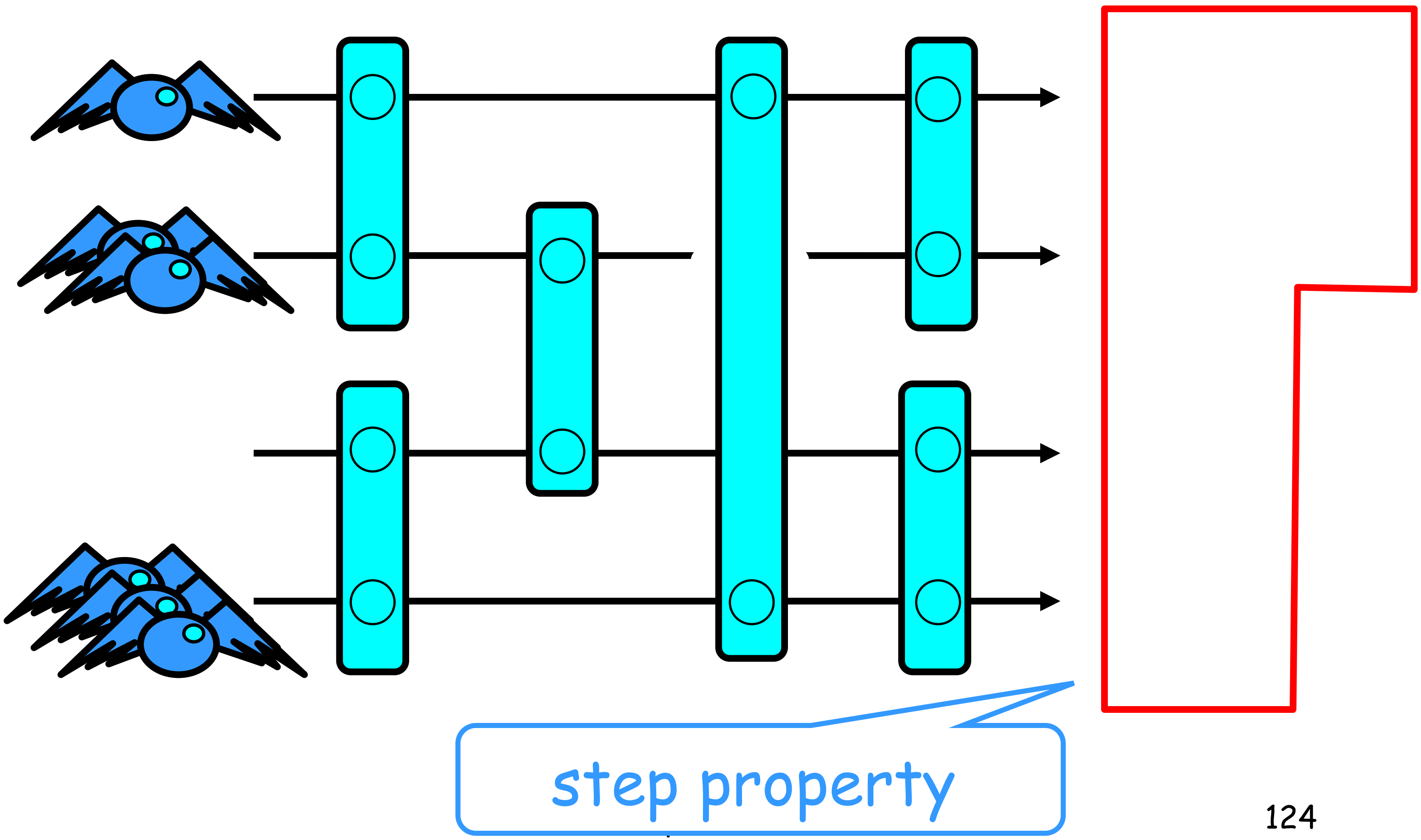
Tokens Traverse Balancers



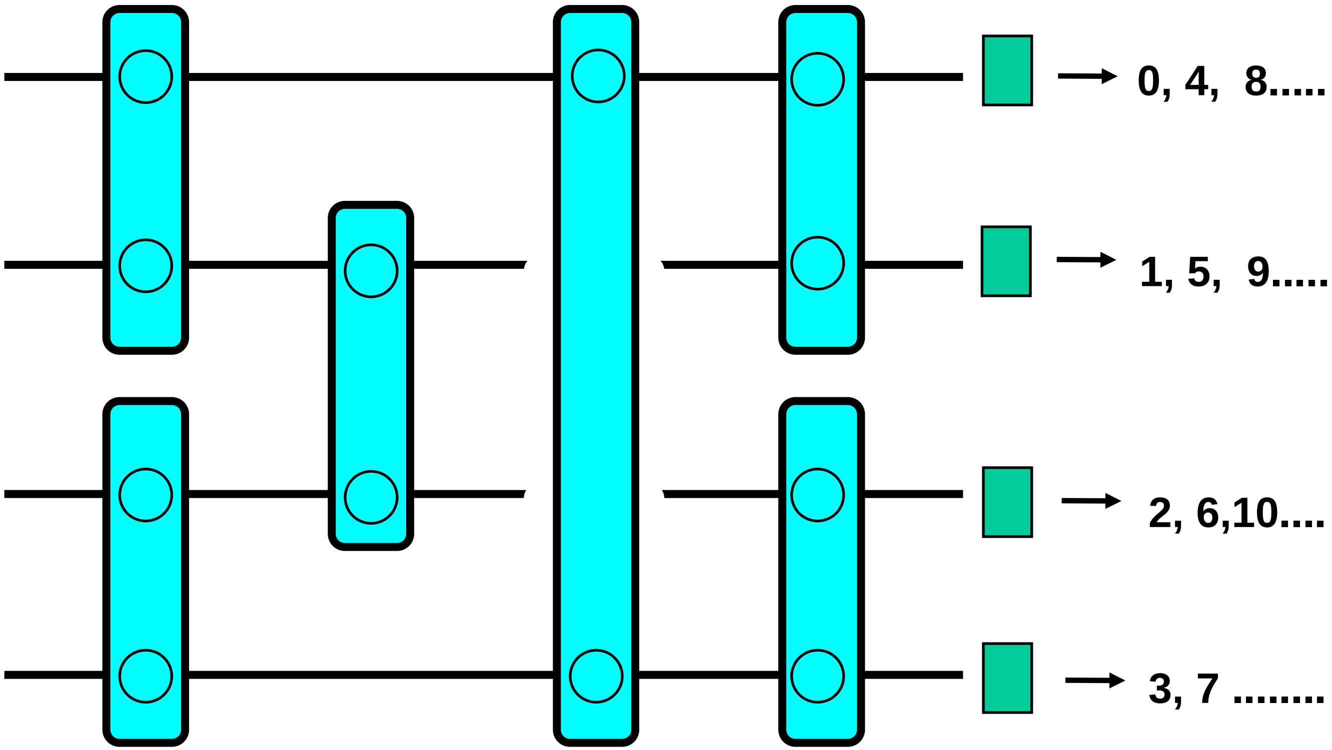
Smoothing Network



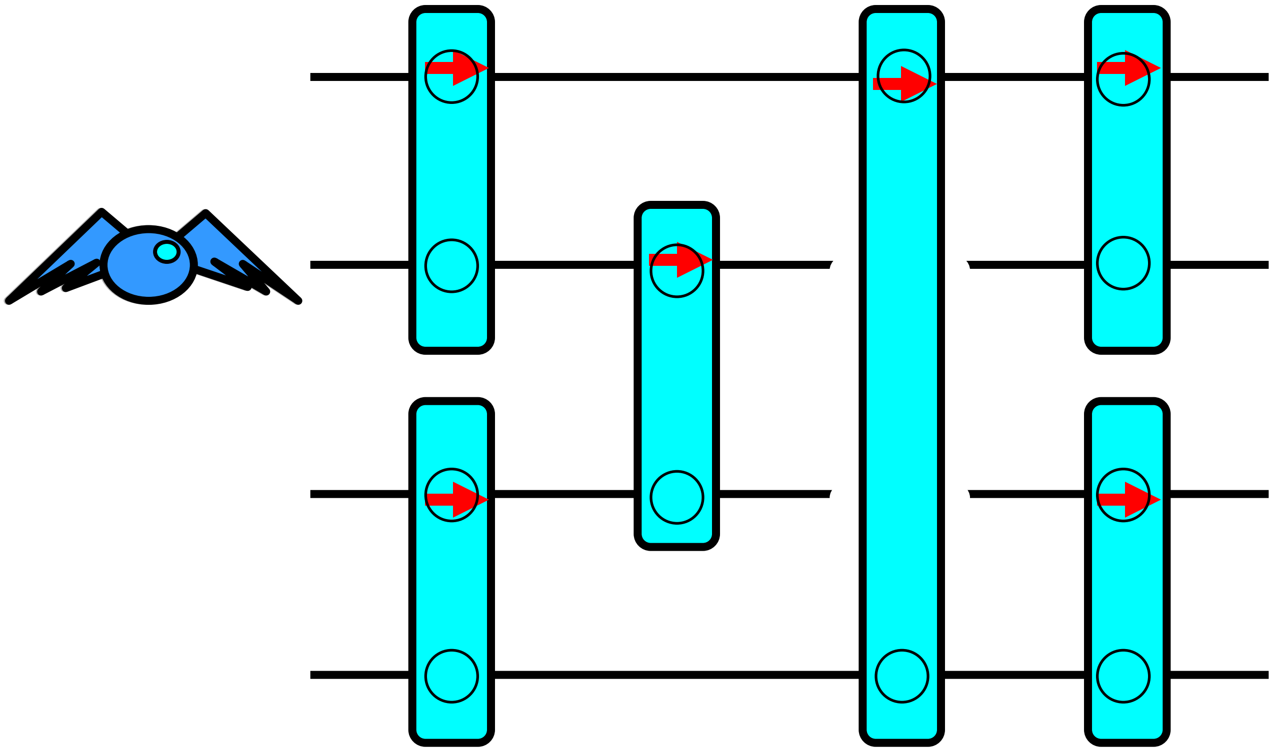
Counting Network



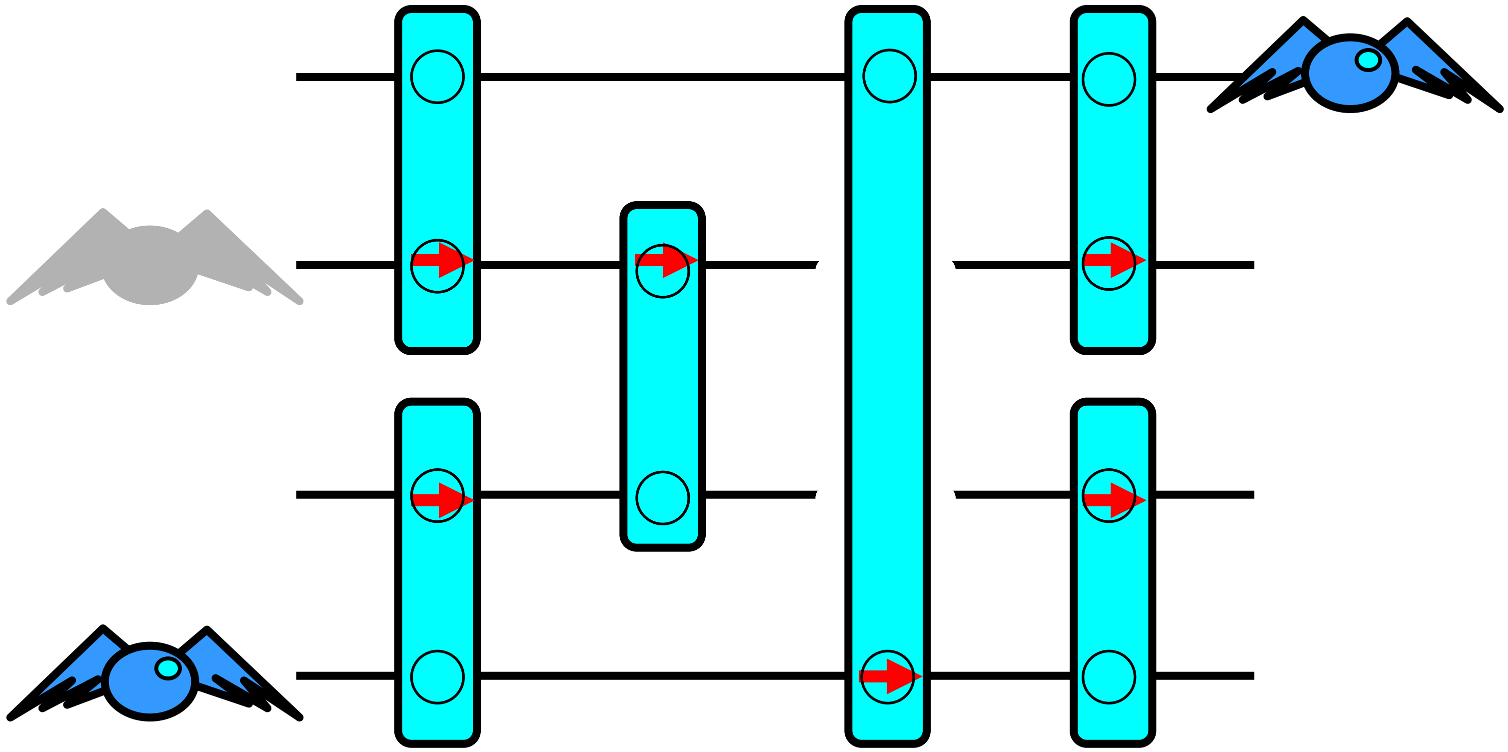
Counting Networks Count!



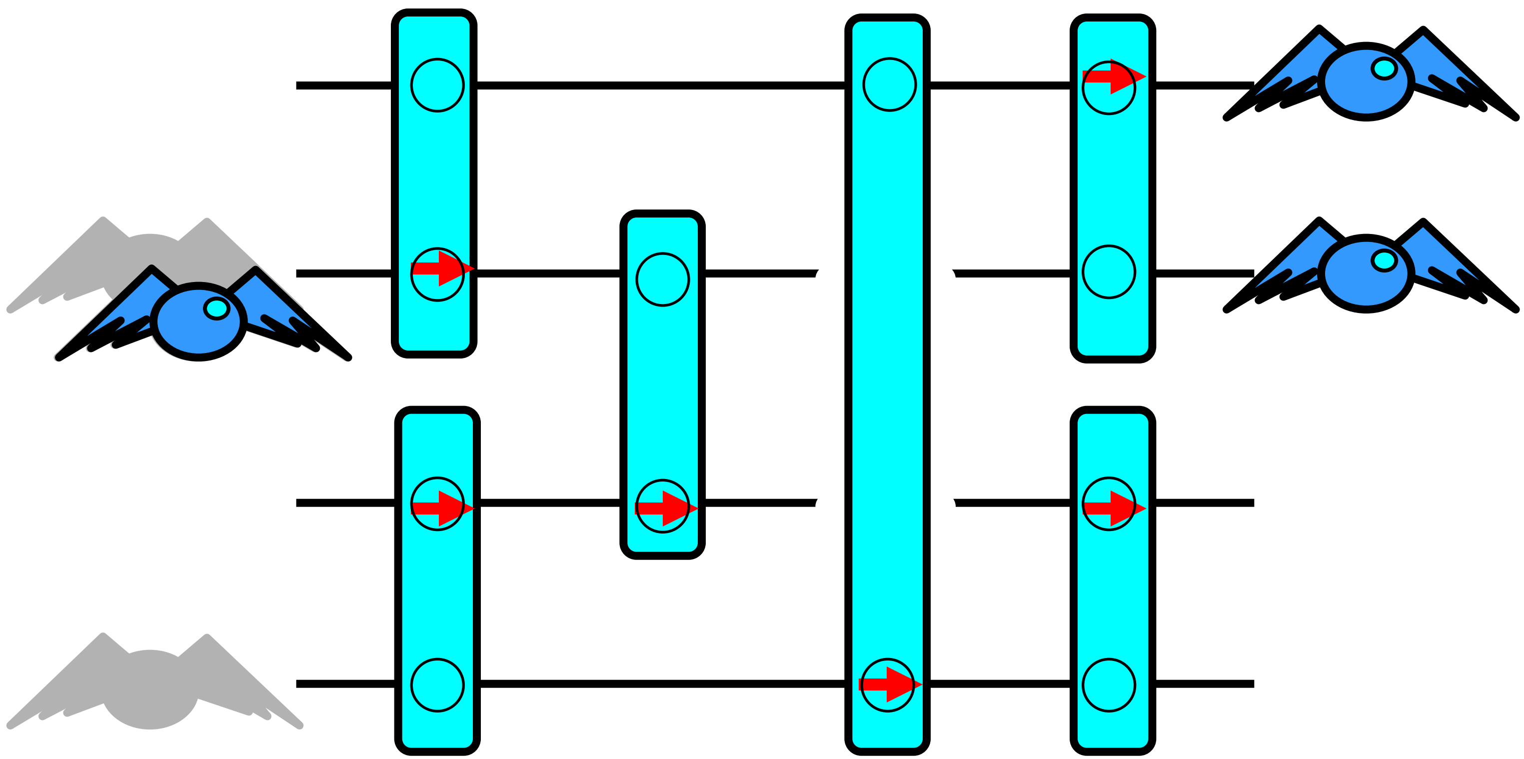
Bitonic[4]



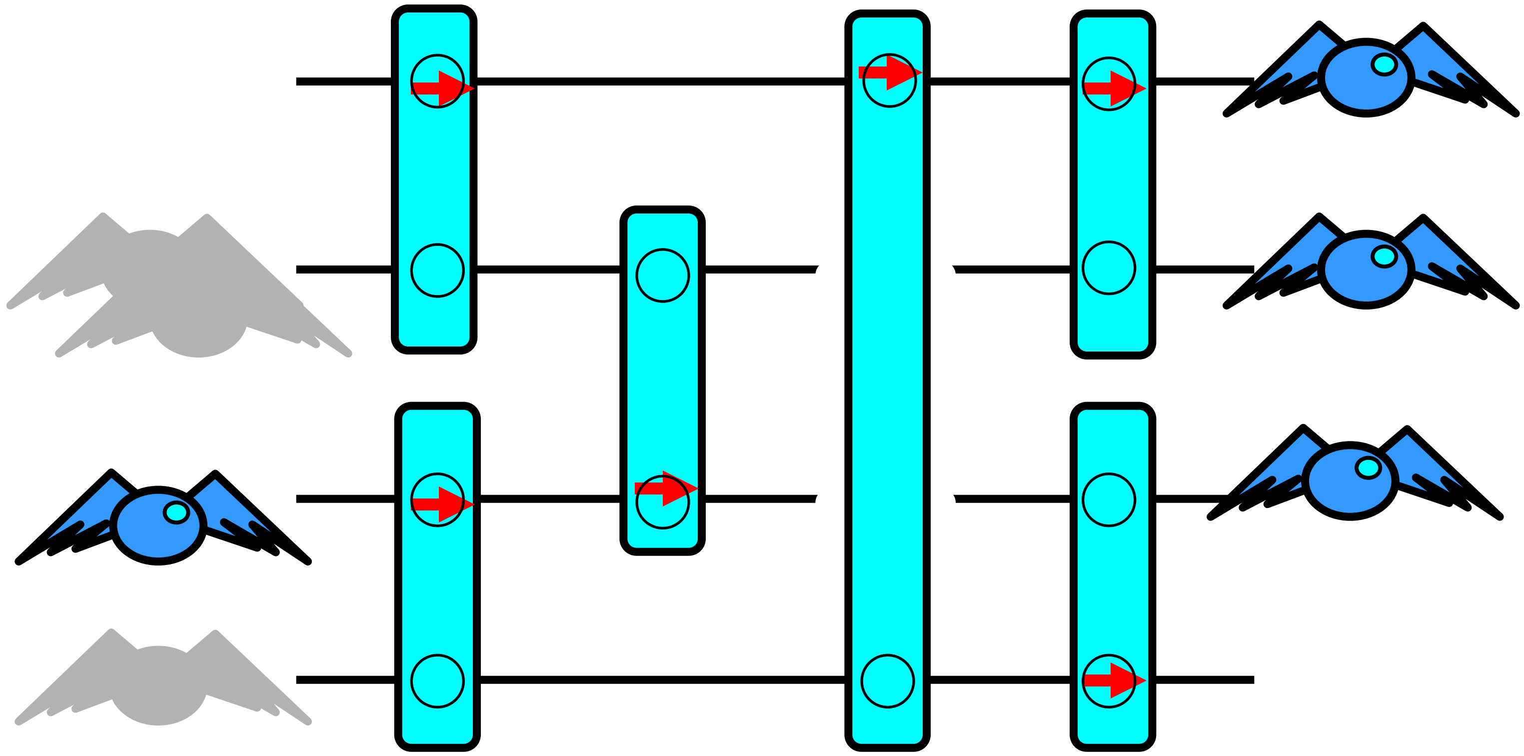
Bitonic[4]



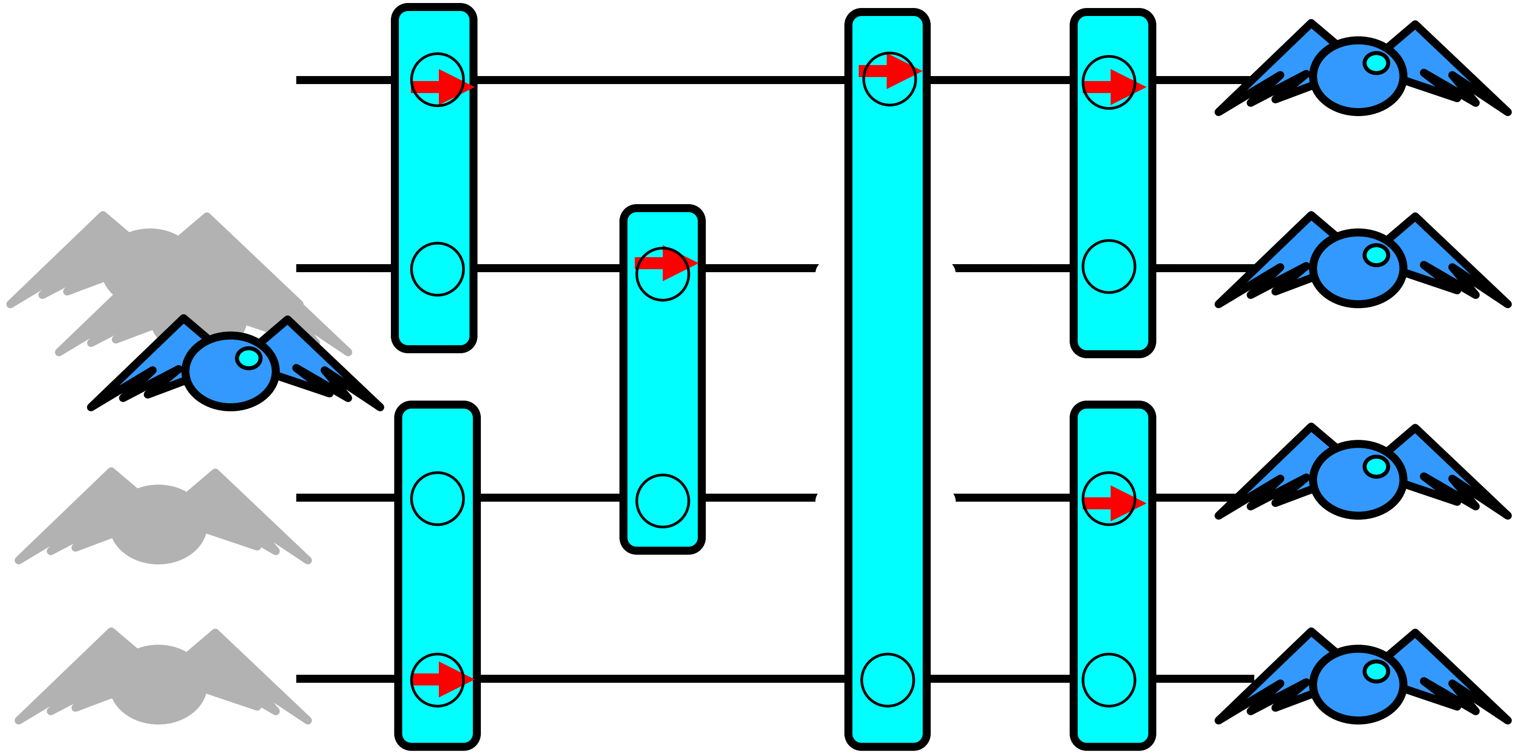
Bitonic[4]



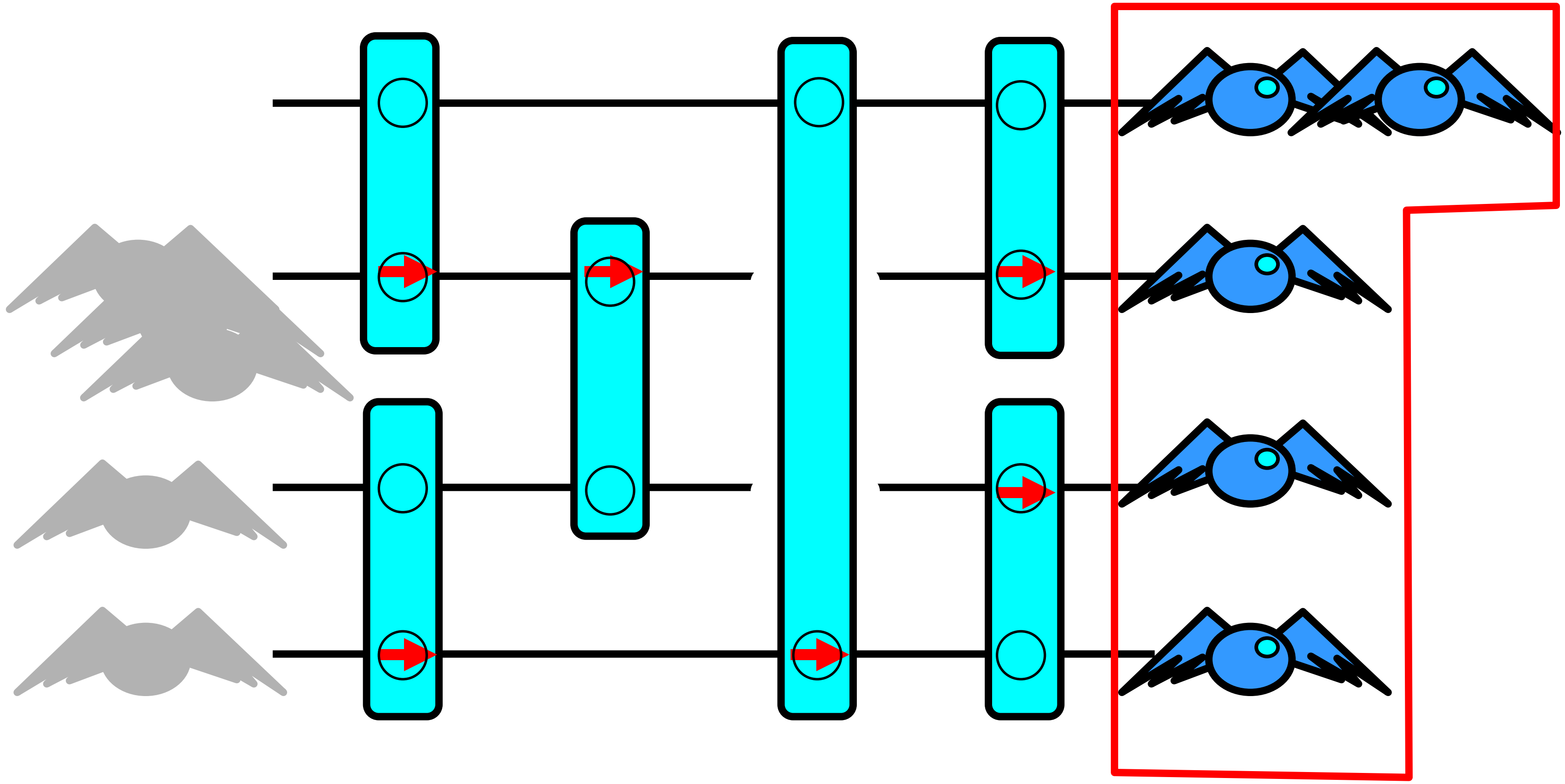
Bitonic[4]



Bitonic[4]



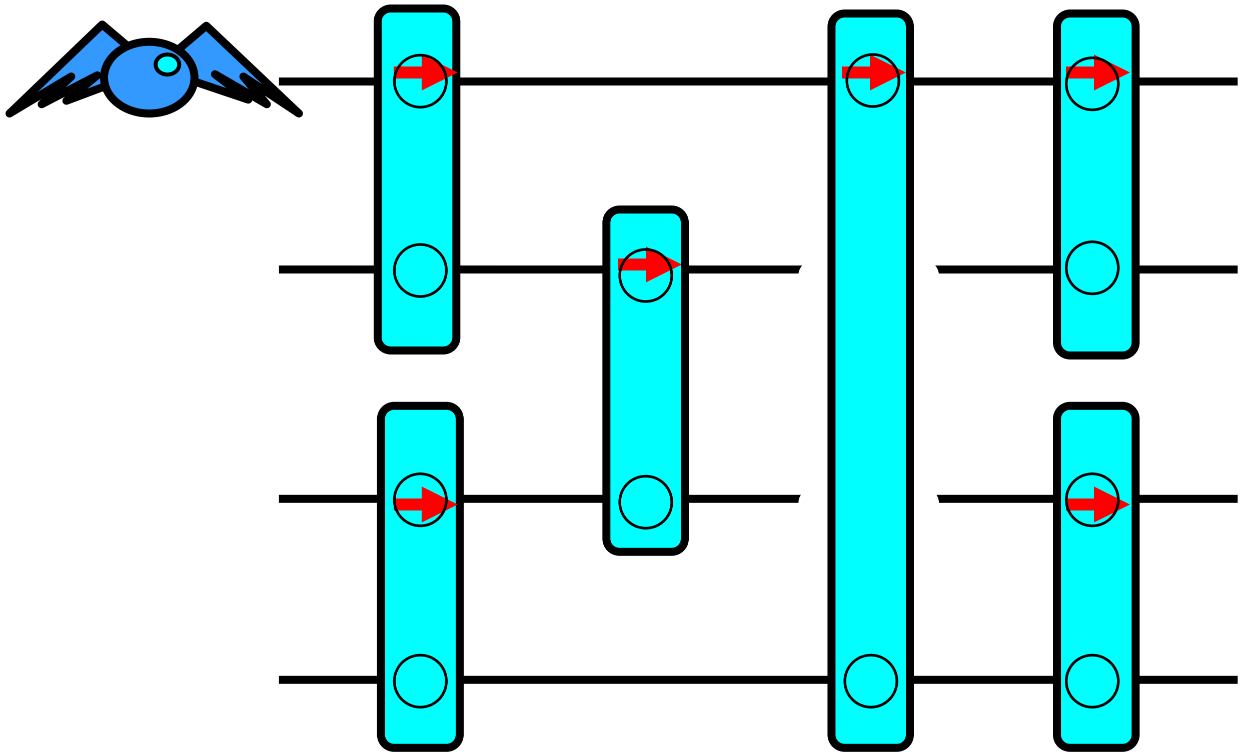
Bitonic[4]



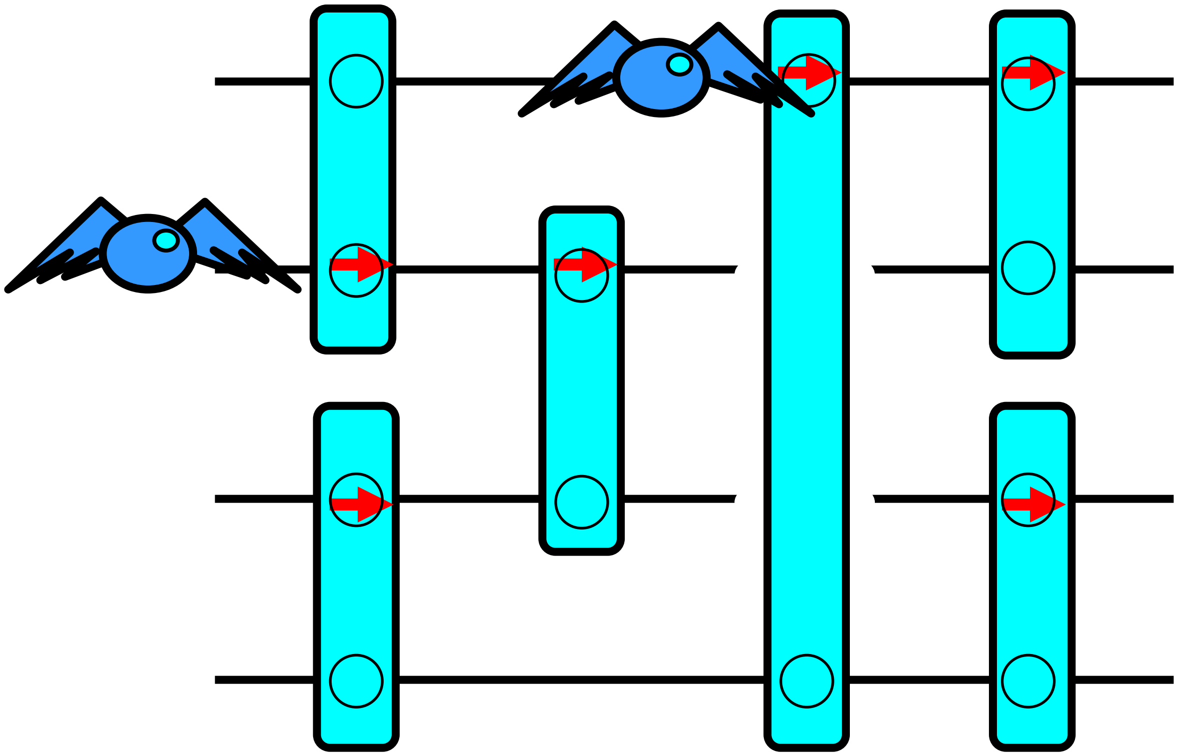
Counting Networks

- Good for counting number of tokens
- low contention
- no sequential bottleneck
- high throughput
- practical networks depth $\log^2 n$

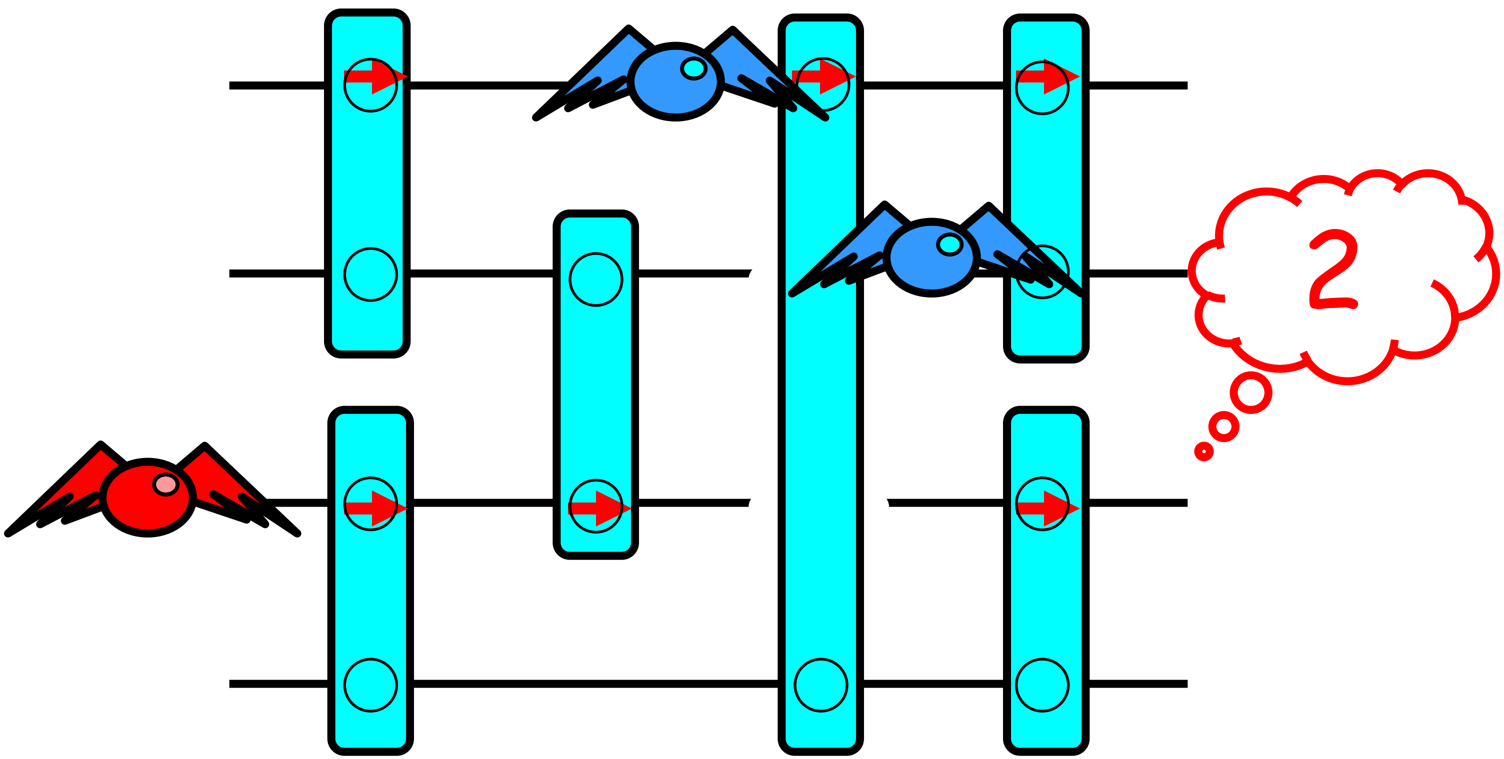
Bitonic[k] is not Linearizable



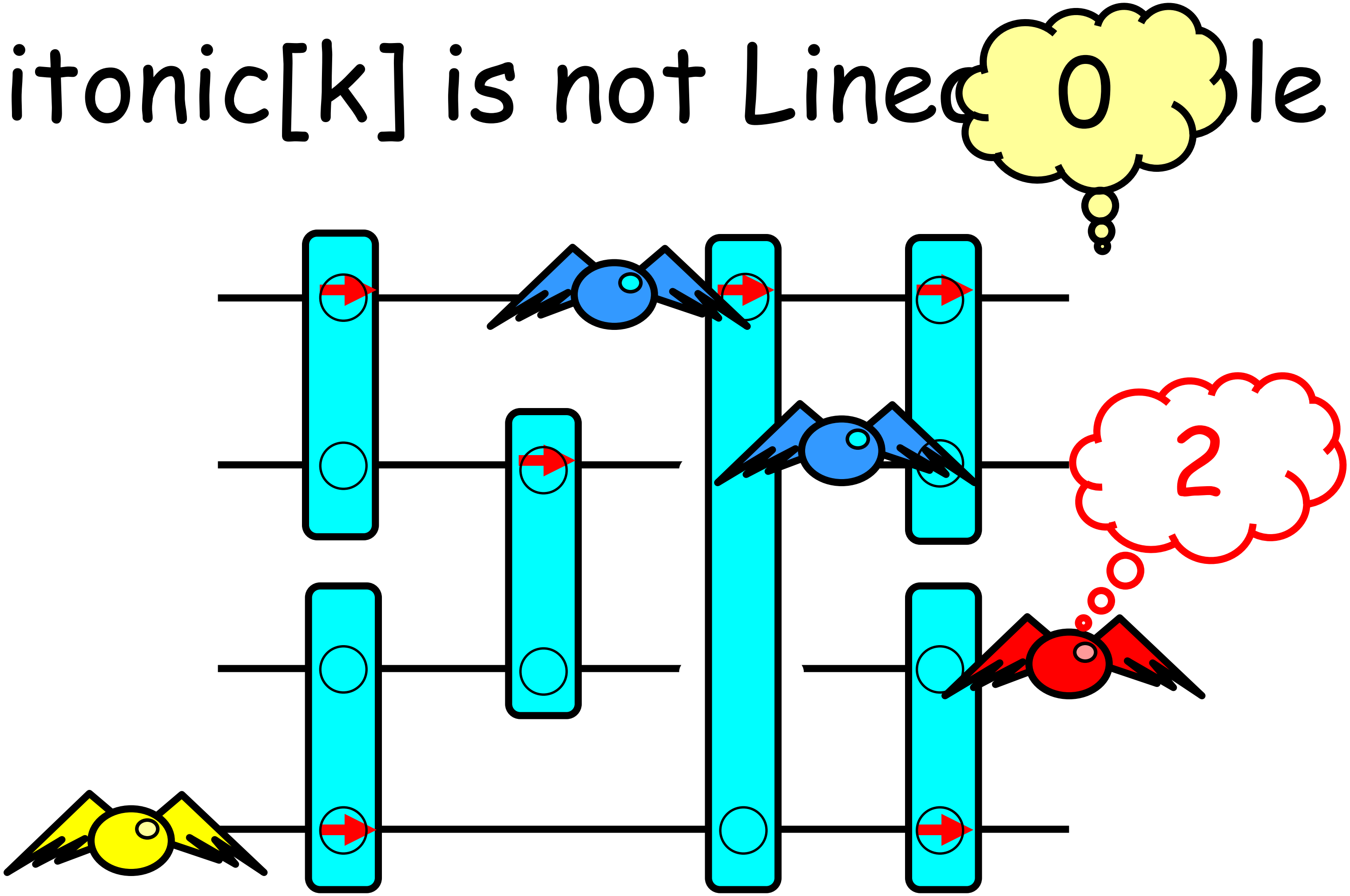
Bitonic[k] is not Linearizable



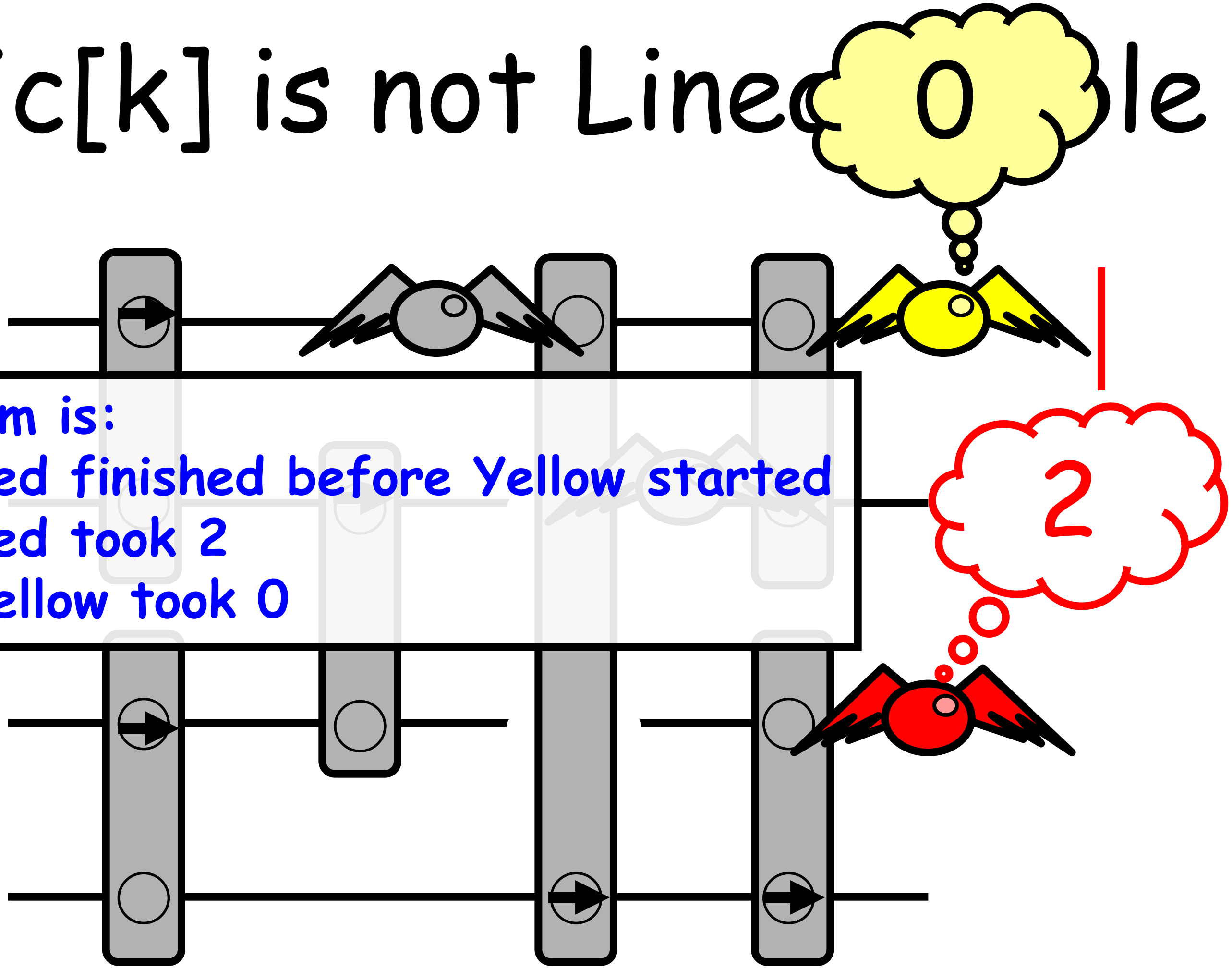
Bitonic[k] is not Linearizable



Bitonic[k] is not Linearizable



Bitonic[k] is not Linearizable



Problem is:
- Red finished before Yellow started
- Red took 2
- Yellow took 0

Shared Memory Implementation

```
class balancer {
 boolean toggle;
 balancer[] next;

synchronized boolean flip() {
 boolean oldValue = this.toggle;
 this.toggle = !this.toggle;
 return oldValue;
}
```

Shared Memory Implementation

```
class balancer {
 boolean toggle;
 balancer[] next;

synchronized boolean flip() {
 boolean oldValue = this.toggle;
 this.toggle = !this.toggle;
 return oldValue;
}
```

state

Shared Memory Implementation

```
class balancer {
 boolean toggle;
 balancer[] next;

synchronized boolean flip() {
 boolean oldValue = this.toggle;
 this.toggle = !this.toggle;
 return oldValue;
}
```

Output connections to balancers

Shared Memory Implementation

```
class balancer {
 boolean toggle;
 balancer[] next;

synchronized boolean flip() {
 boolean oldValue = this.toggle;
 this.toggle = !this.toggle;
 return oldValue;
}
```

Get-and-complement

Shared Memory Implementation

```
Balancer traverse (Balancer b) {
 while(!b.isLeaf()) {
  boolean toggle = b.flip();
  if (toggle)
    b = b.next[0]
  else
    b = b.next[1]
  return b;
}
```

Shared Memory Implementation

```
Balancer traverse (Balancer b) {
  while(!b.isLeaf()) {
    boolean toggle = b.flip();
    if (toggle)
      b = b.next[0]
    else
      b = b.next[1]
    return b;
}
```

Stop when we get to the end

Shared Memory Implementation

```
Balancer traverse (Balancer b) {
 while(!b.isLeaf()) {
  boolean toggle = b.flip();
  if (toggle)
    b = b.next[0]
  else
    b = b.next[1]
  return b;
}
```

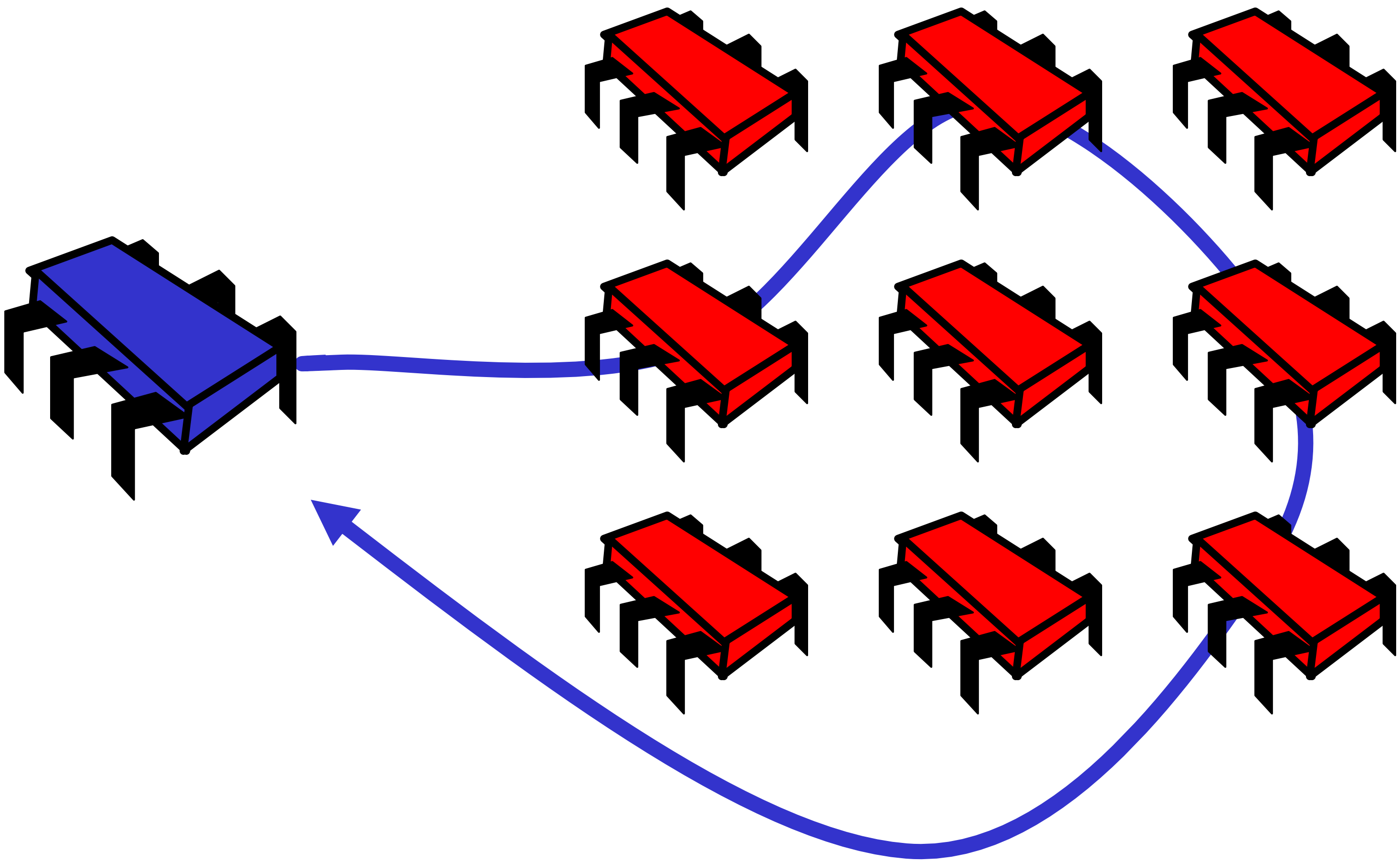
Flip state

Shared Memory Implementation

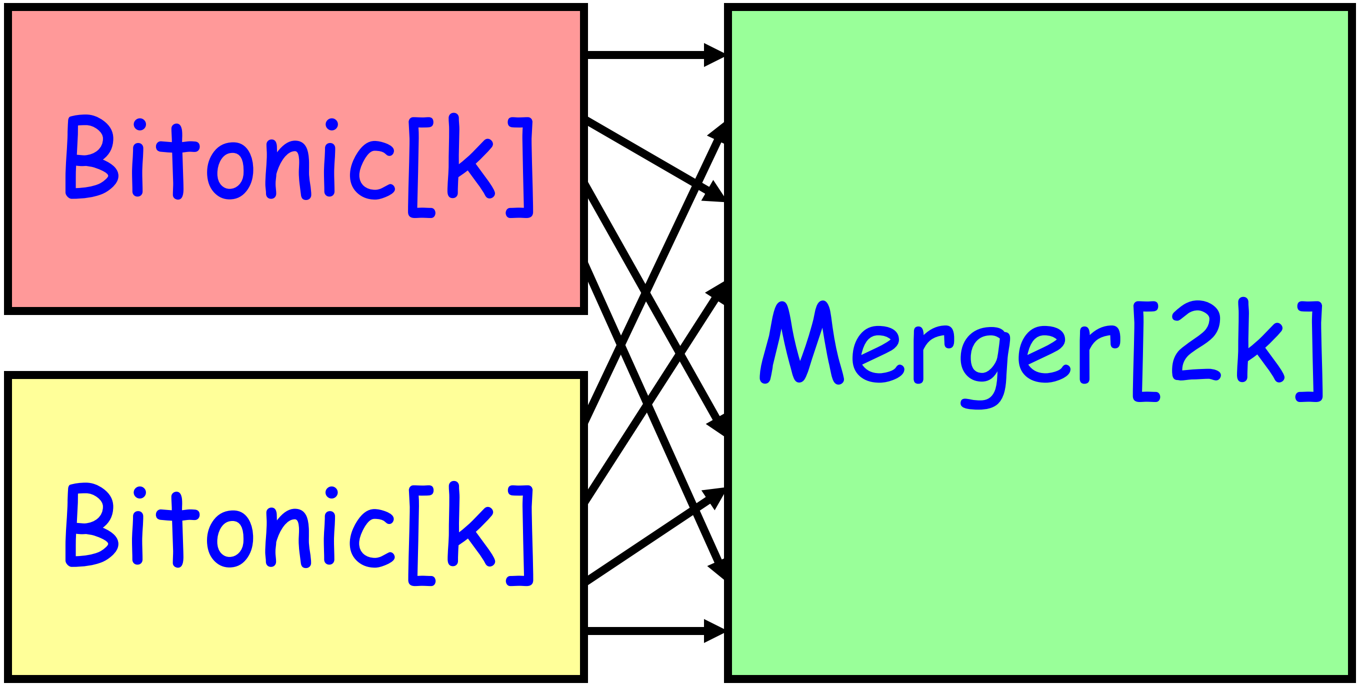
```
Balancer traverse (Balancer b) {
 while(!b.isLeaf()) {
  boolean toggle = b.flip();
  if (toggle)
    b = b.next[0]
  else
    b = b.next[1]
  return b;
}
```

Exit on wire

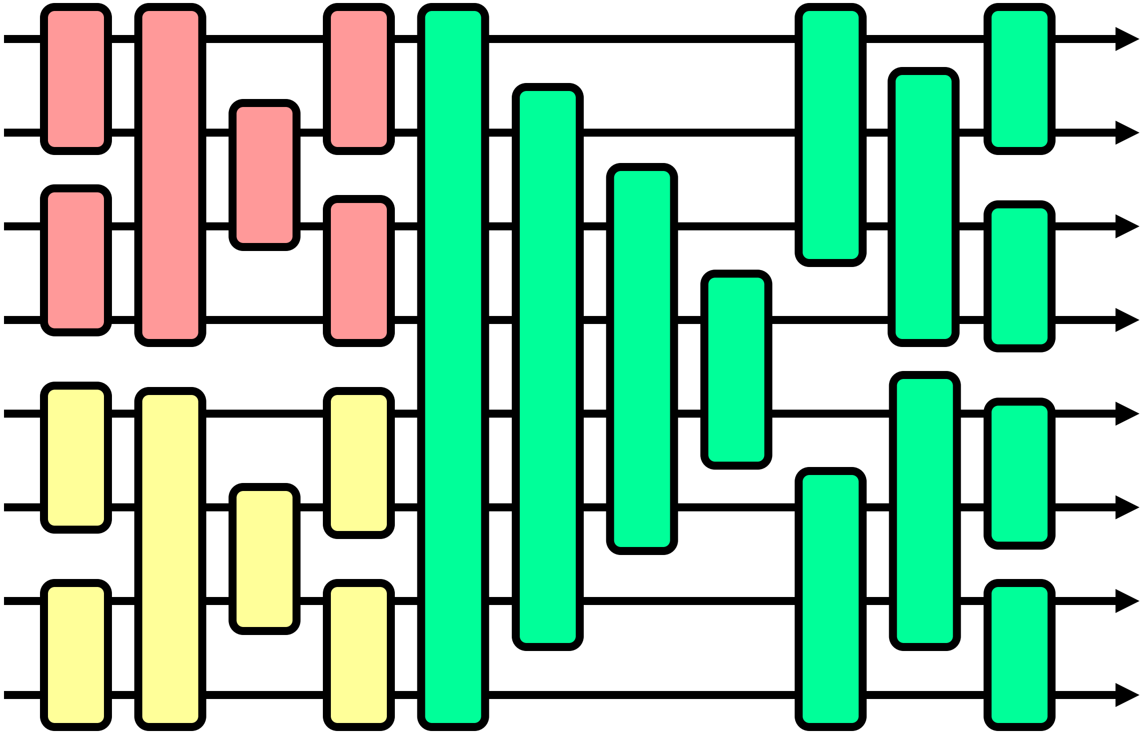
Alternative Implementation: Message-Passing



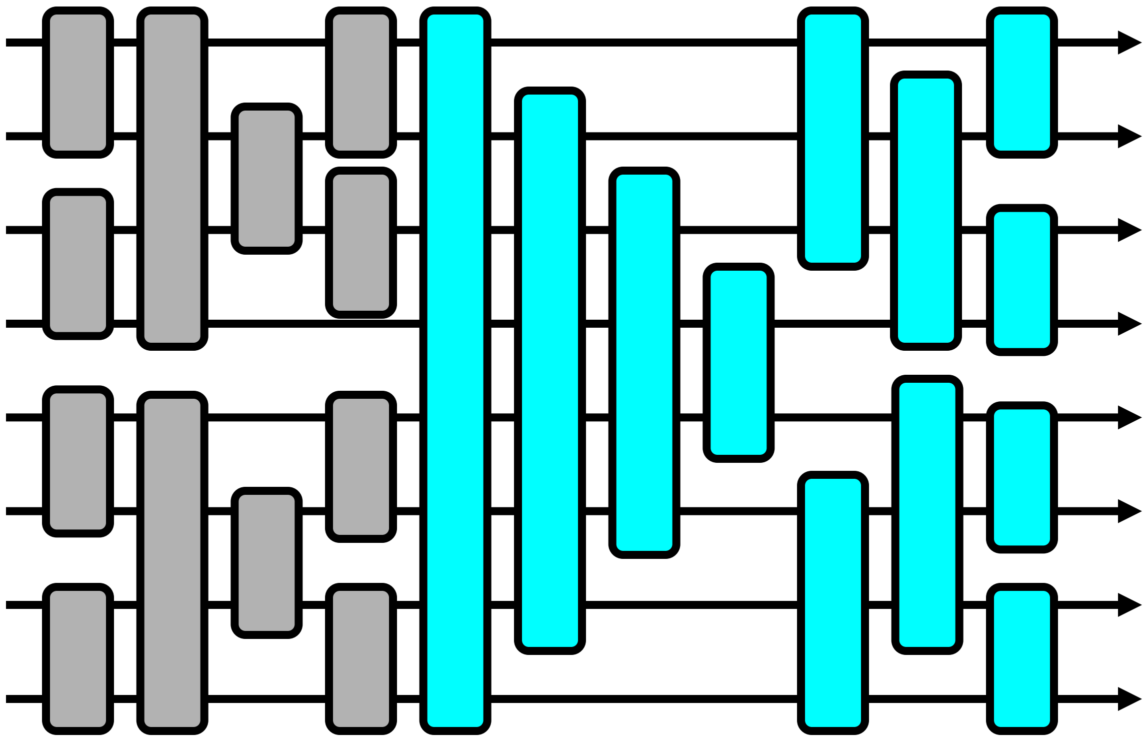
Bitonic[2k] Schematic



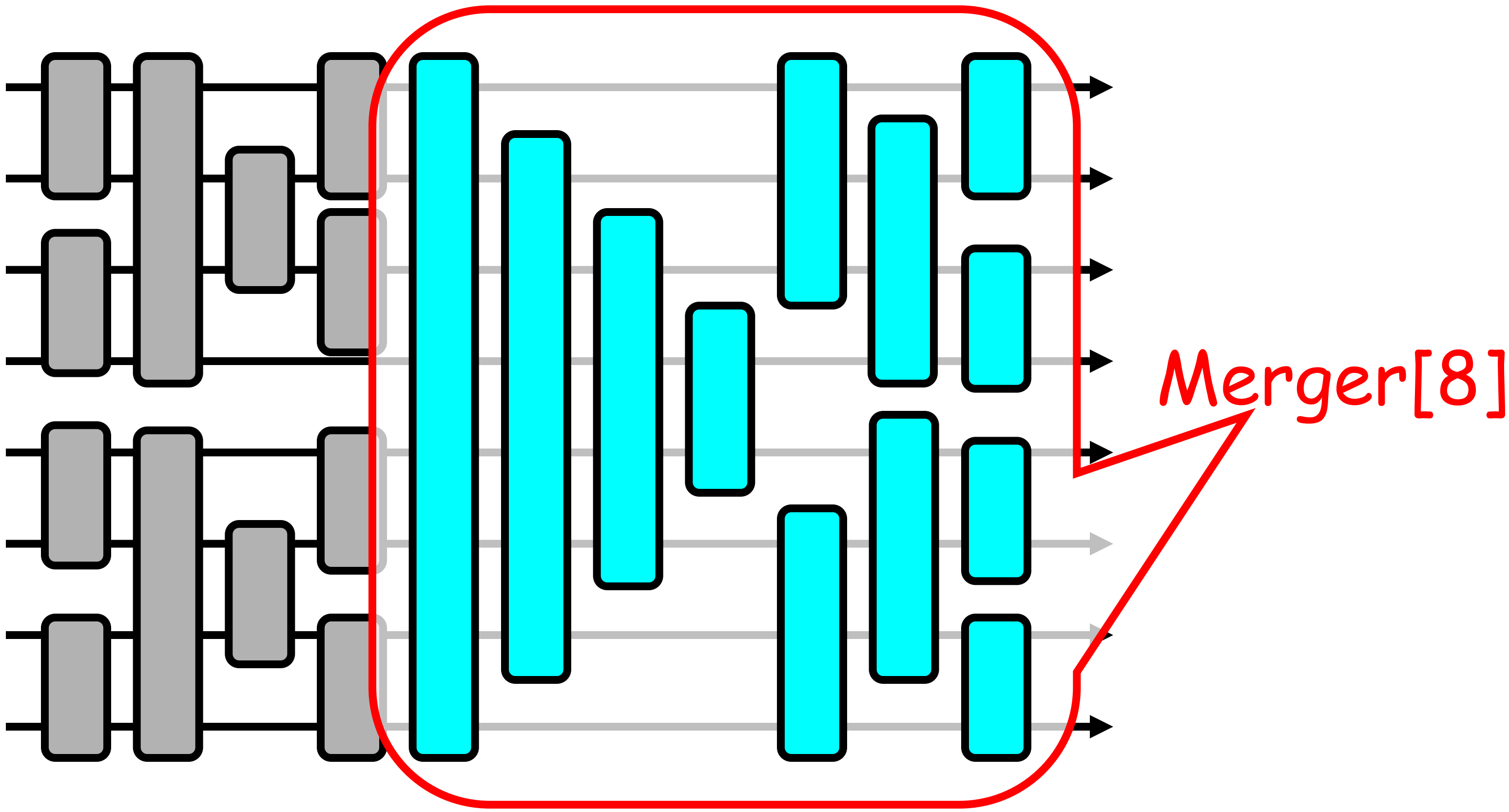
Bitonic[2k] Layout



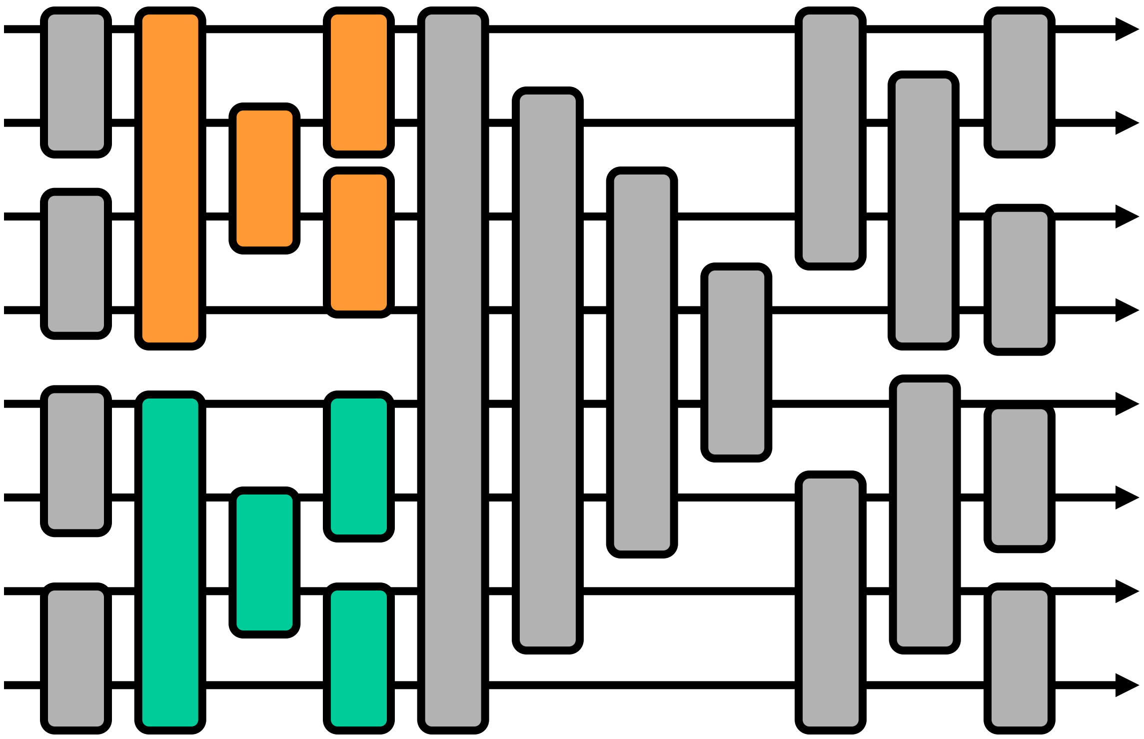
Unfolded Bitonic Network



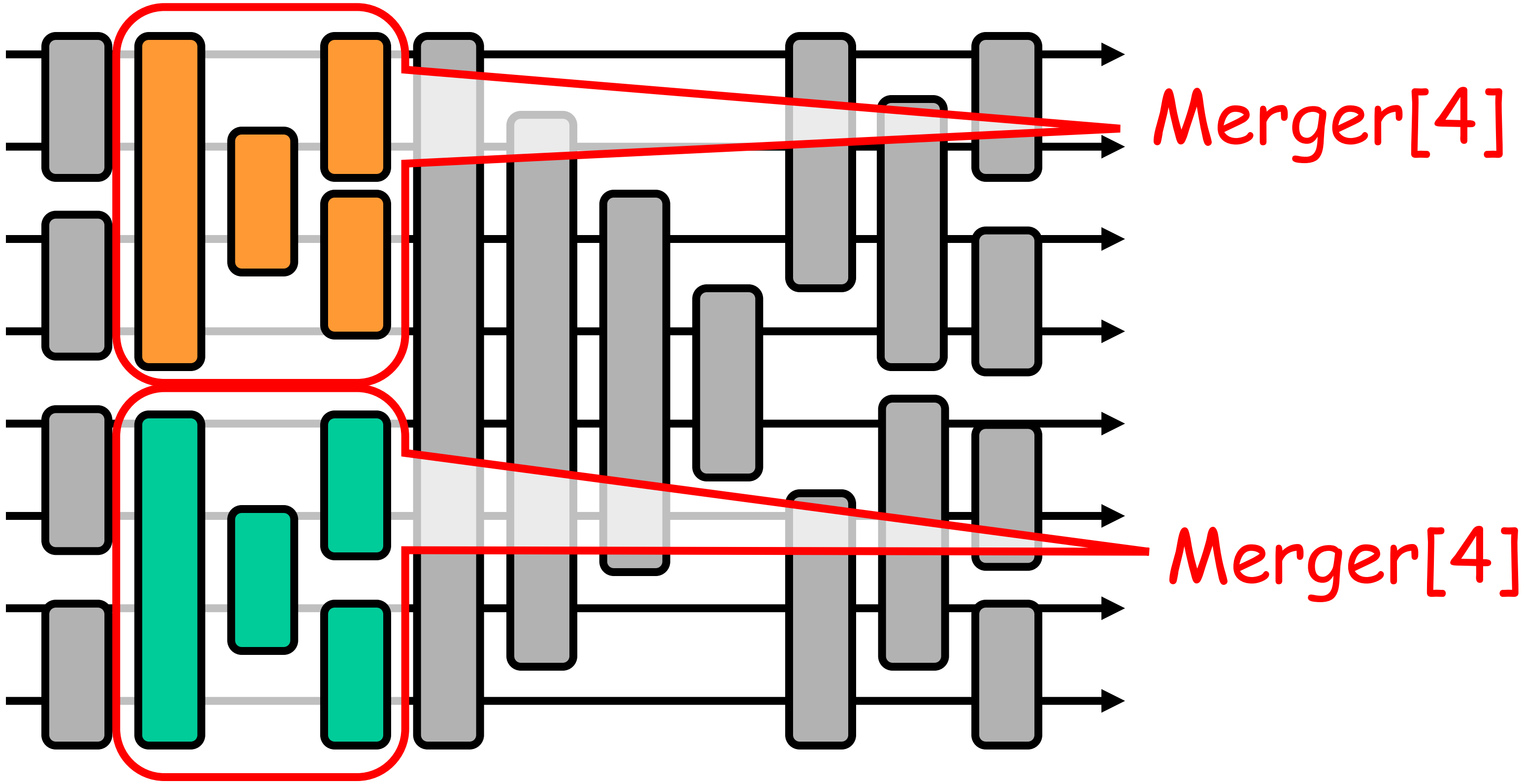
Unfolded Bitonic Network



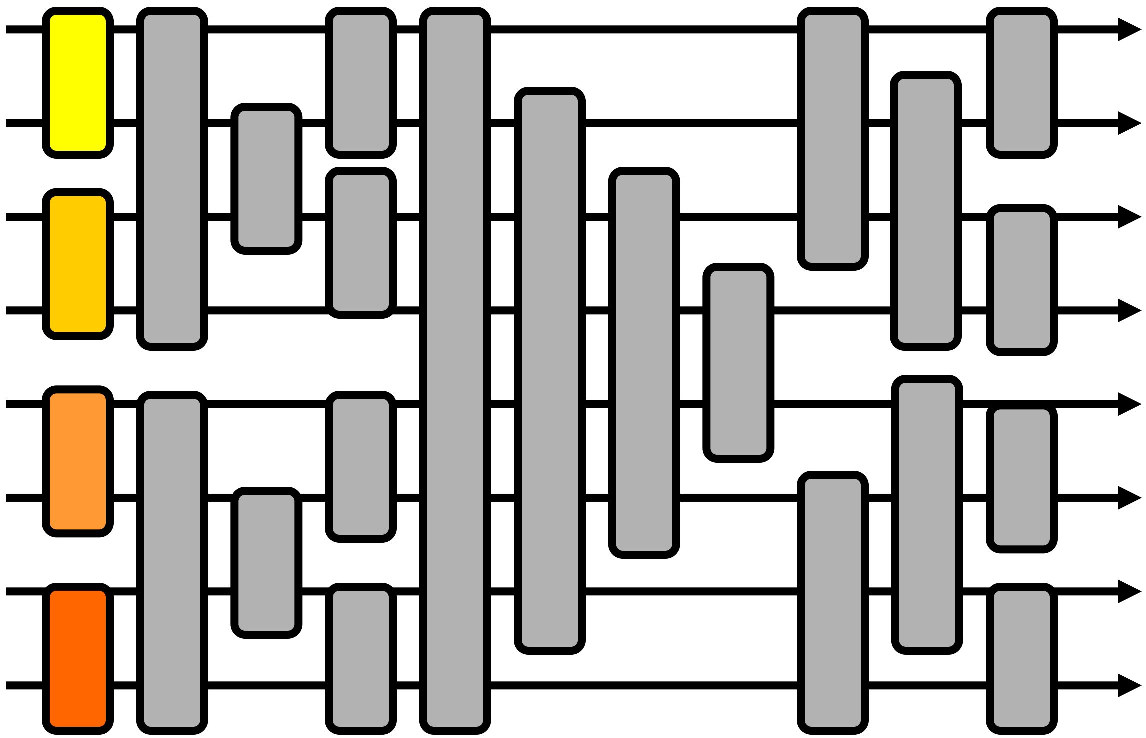
Unfolded Bitonic Network



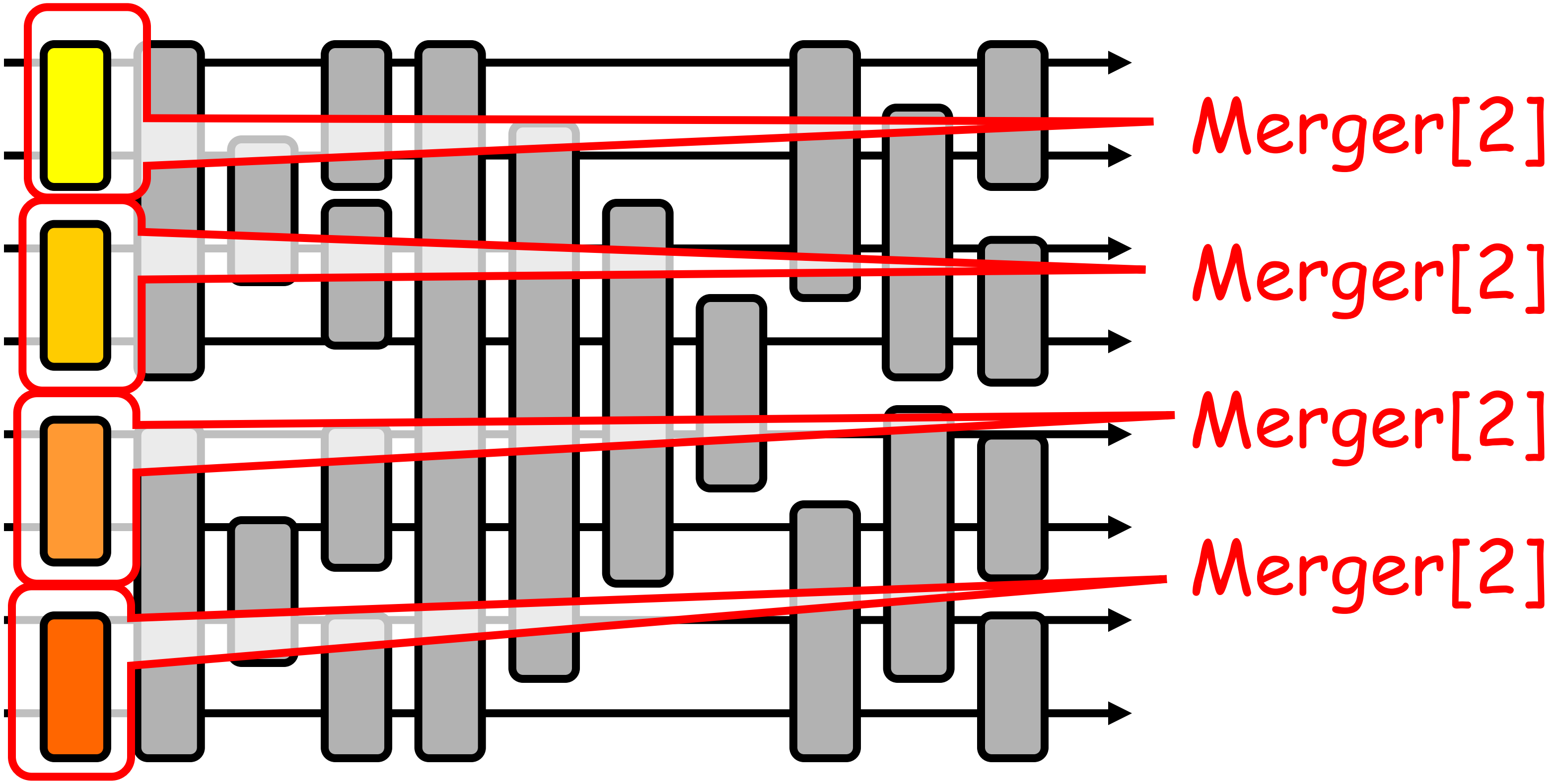
Unfolded Bitonic Network



Unfolded Bitonic Network



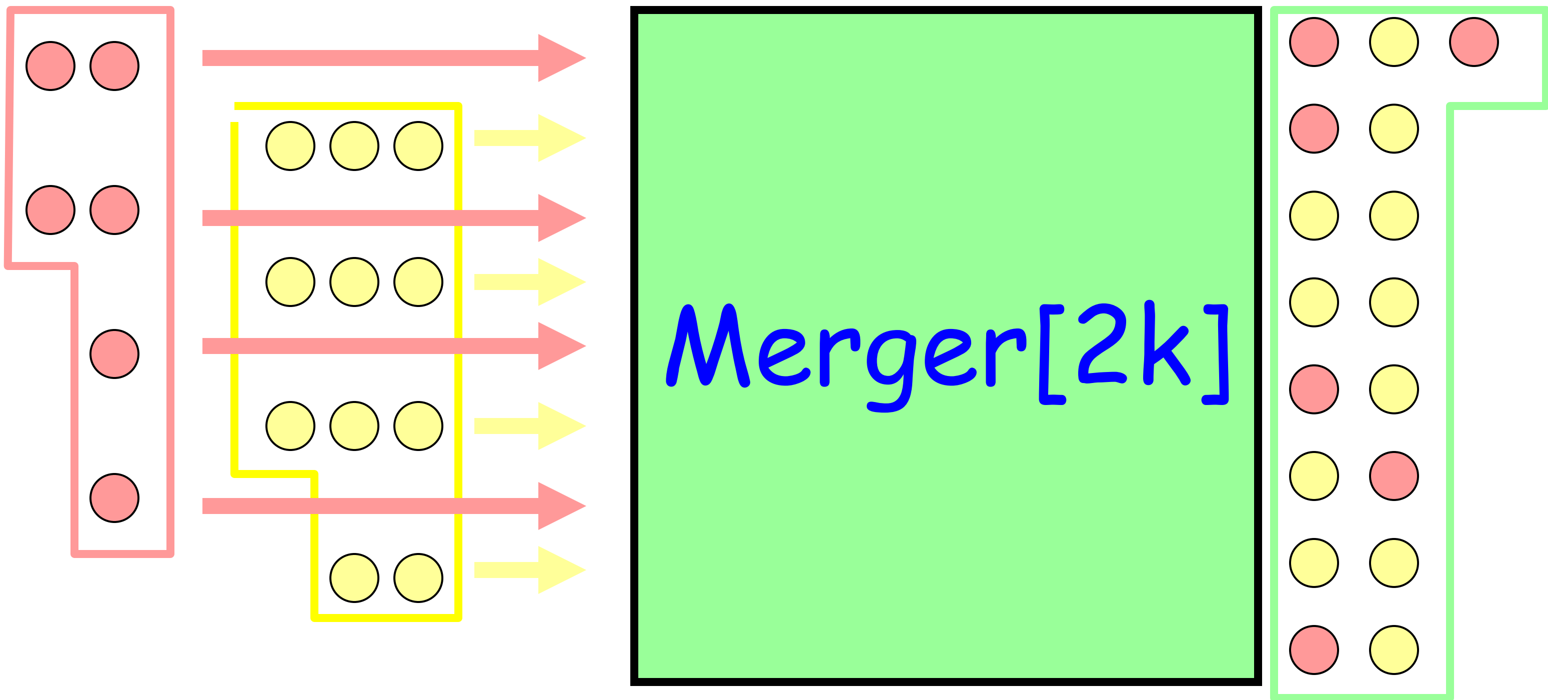
Unfolded Bitonic Network



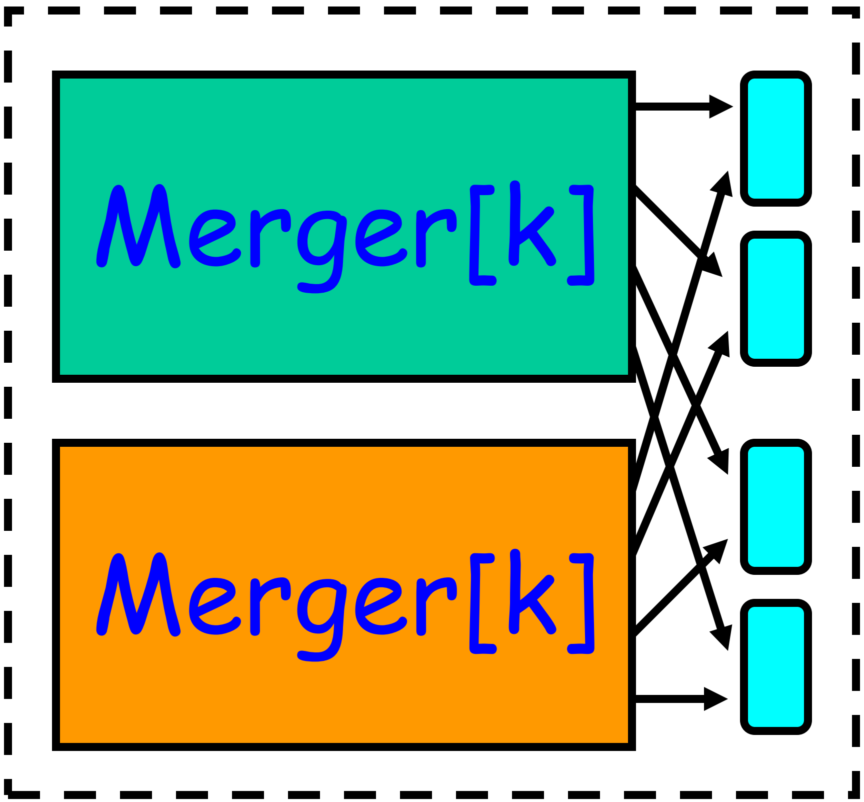
Bitonic[k] Depth

- Width k
- Depth is $(\log_2 k)(\log_2 k + 1)/2$

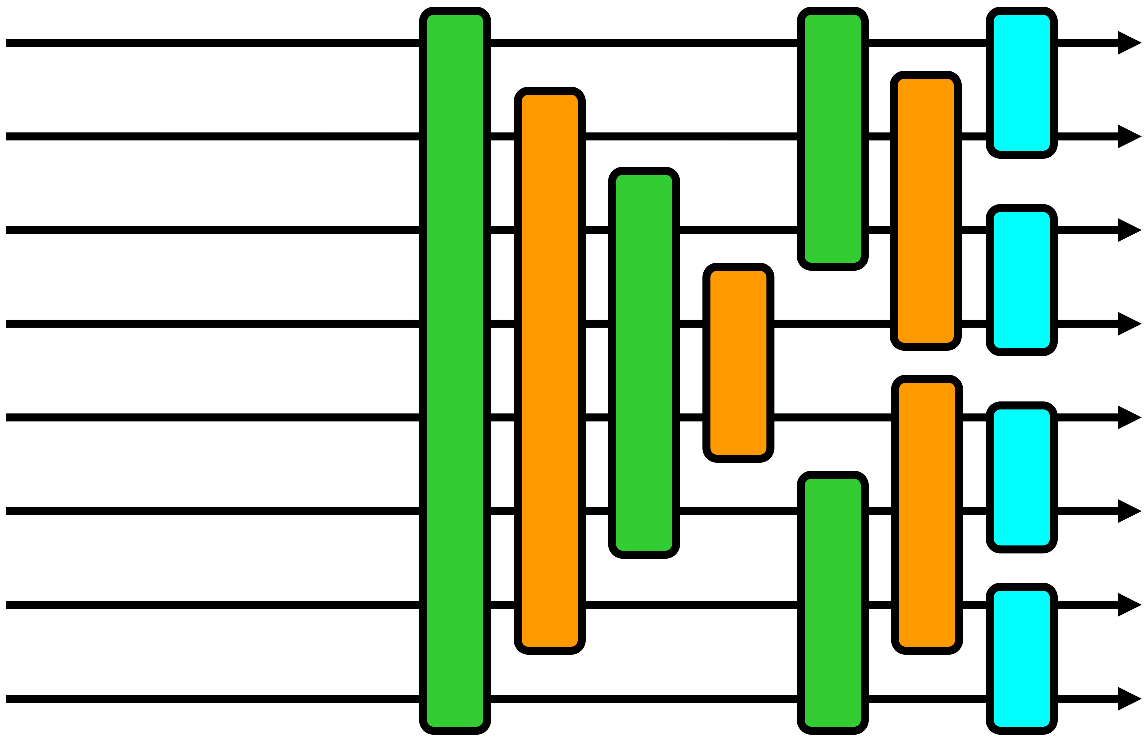
Merger[2k]



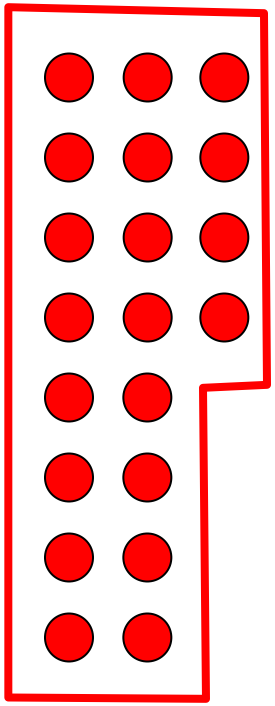
Merger[2k] Schematic



Merger[2k] Layout

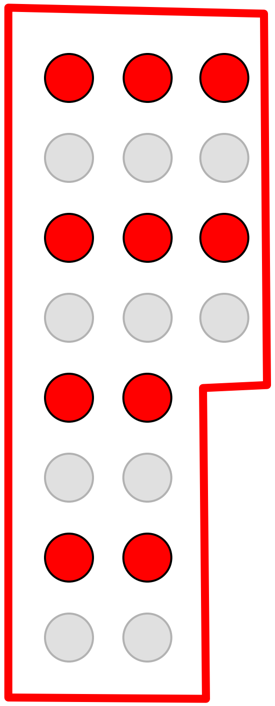


Lemma



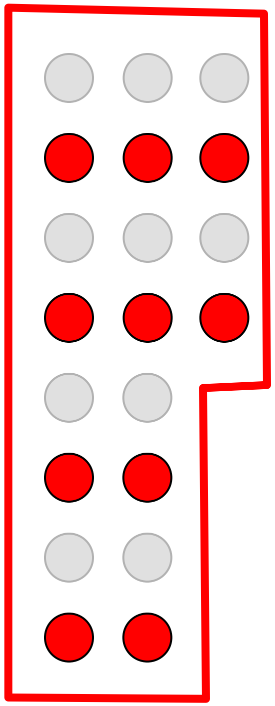
If a sequence has the step property …

Lemma



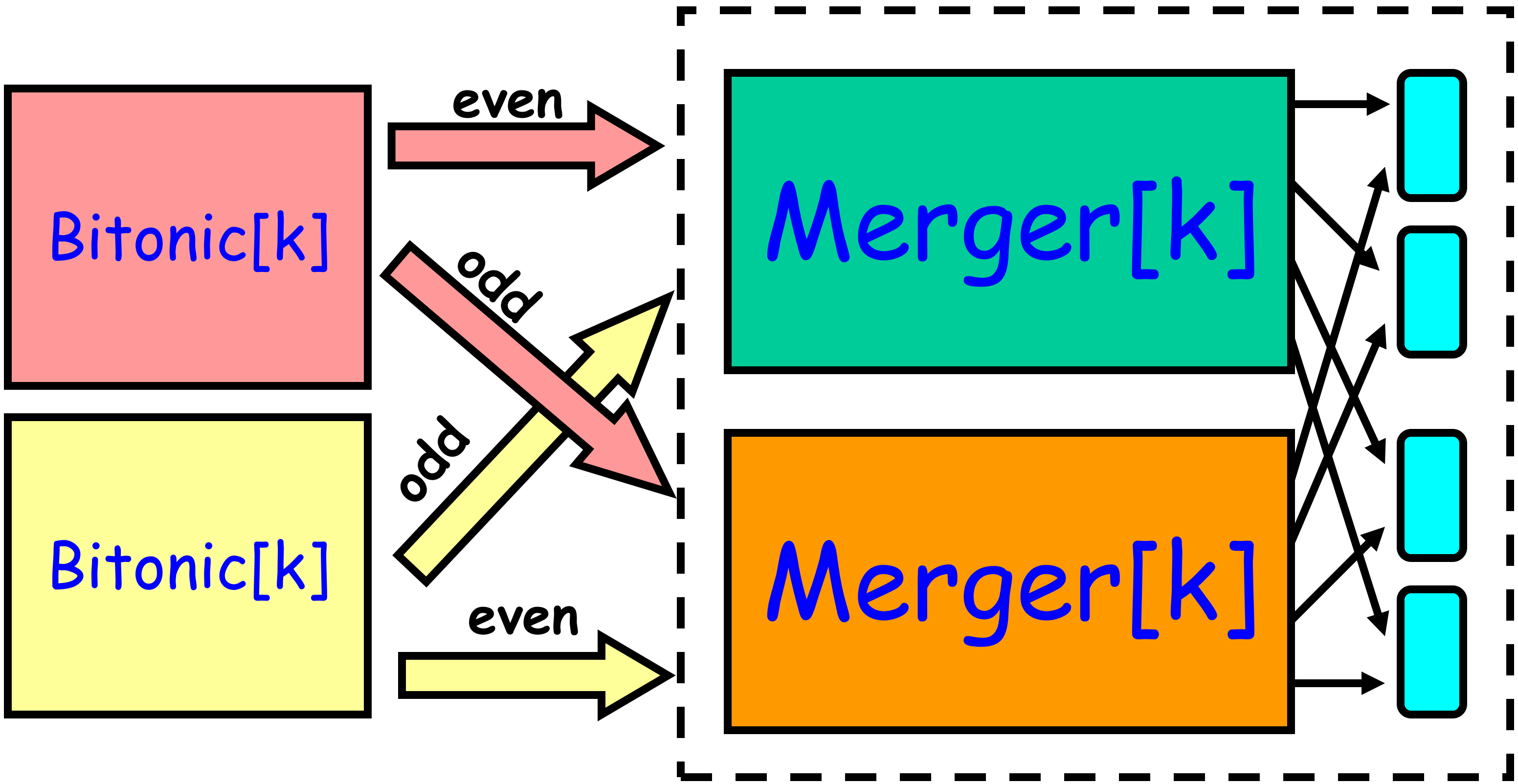
So does its even subsequence

Lemma

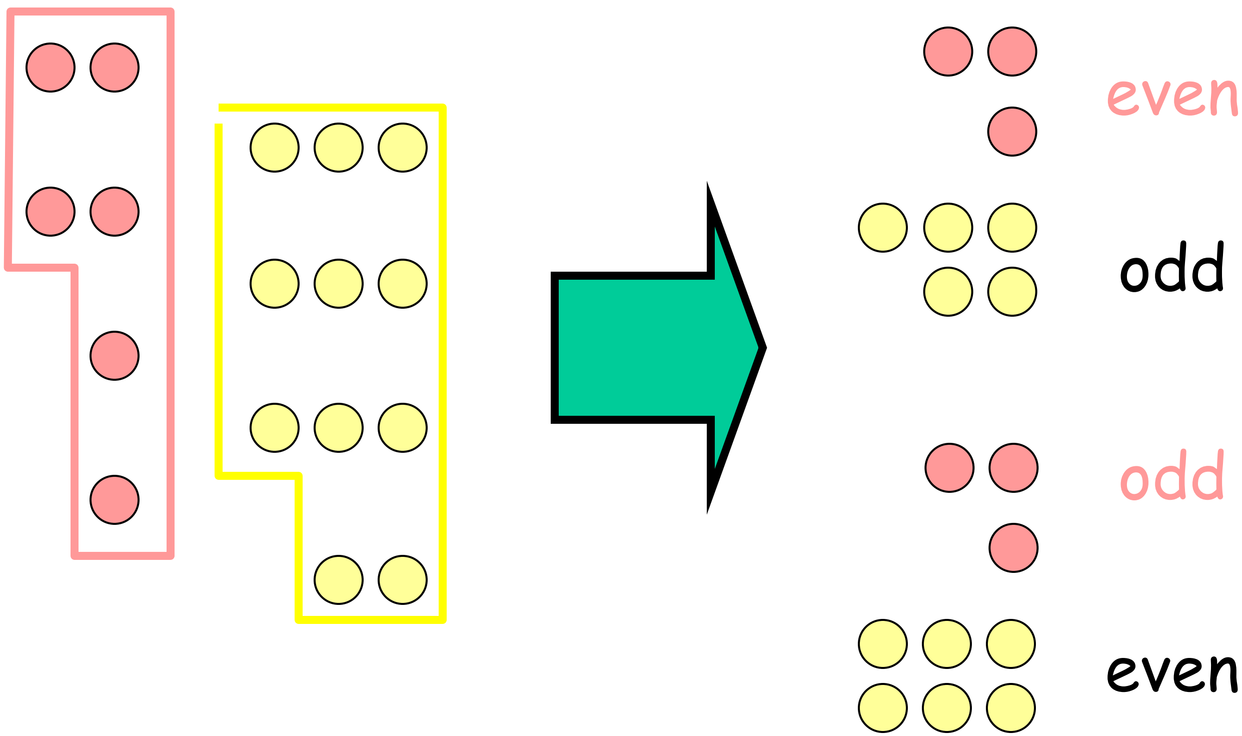


And its odd subsequence

Merger[2k] Schematic



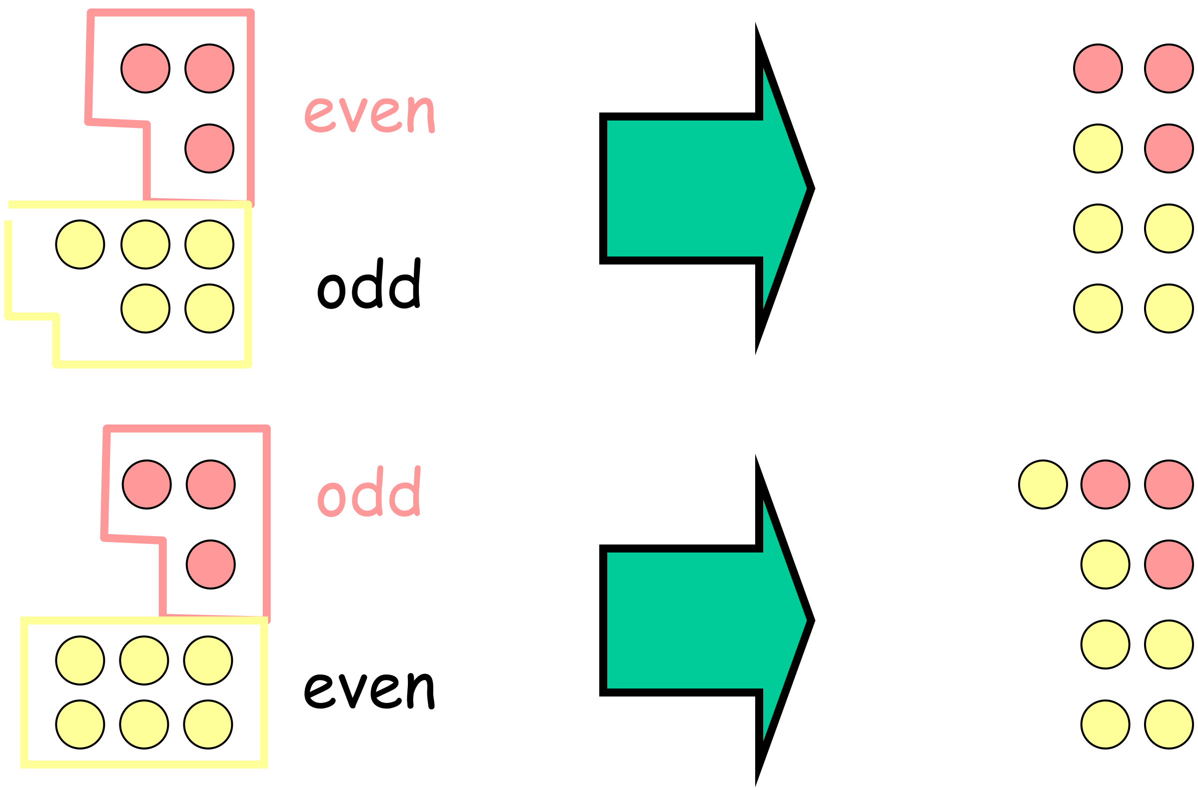
Proof Outline



Outputs from Bitonic[k]

Inputs to Merger[k]

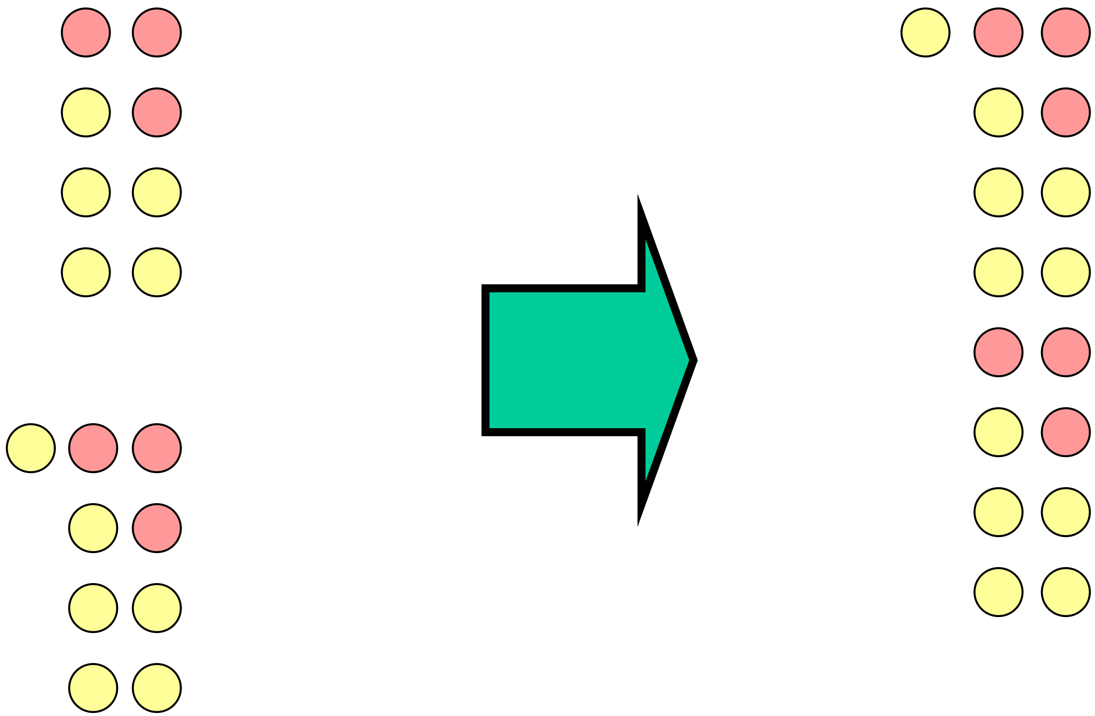
Proof Outline



Inputs to Merger[k]

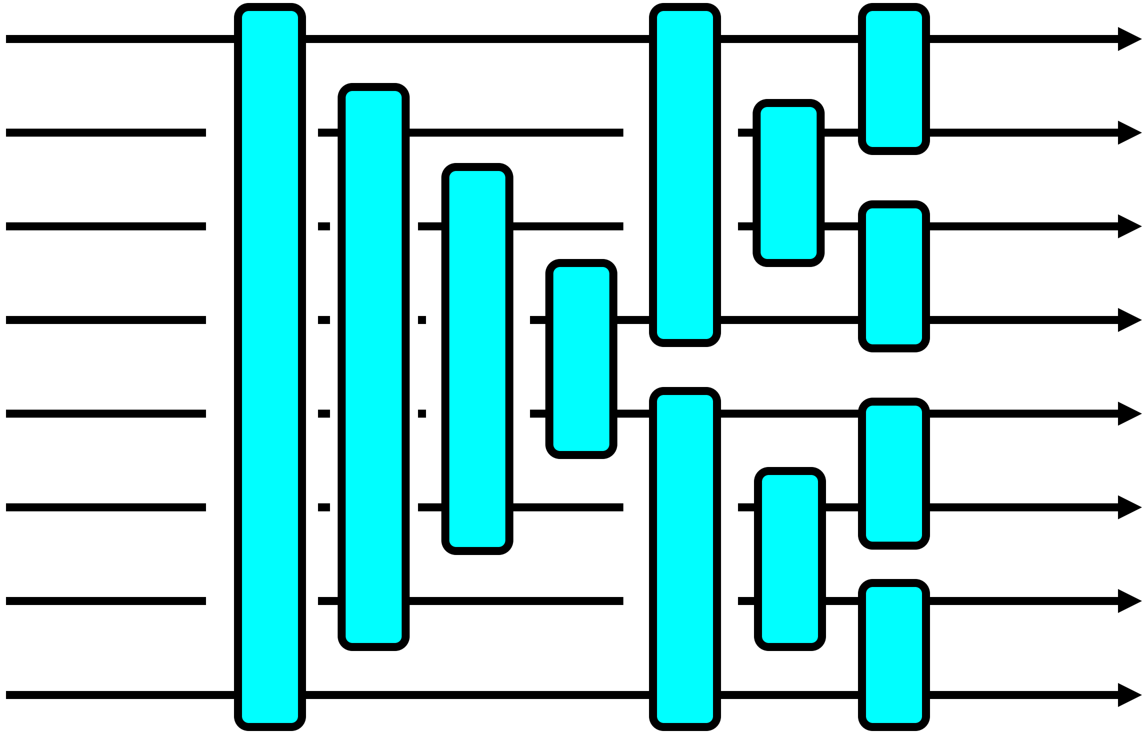
Outputs of Merger[k]

Proof Outline

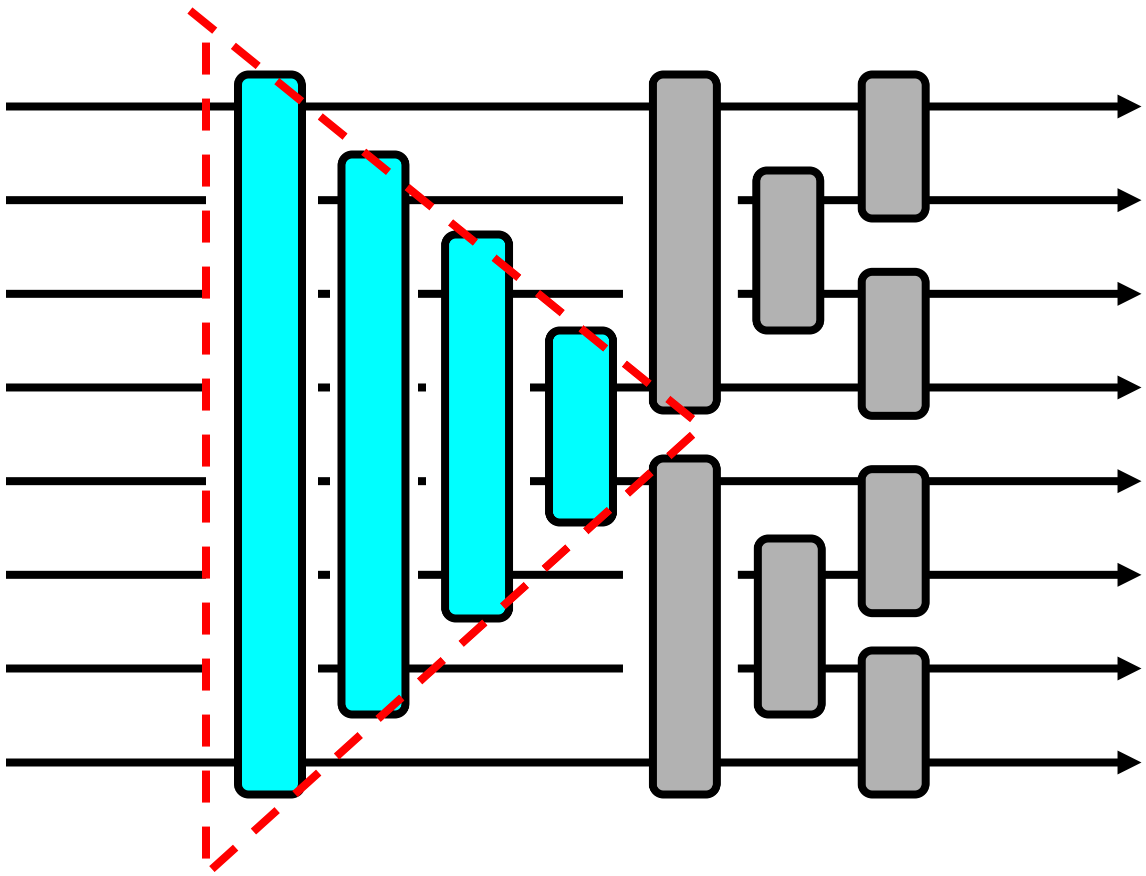


Outputs of Merger[k]   Outputs of last layer

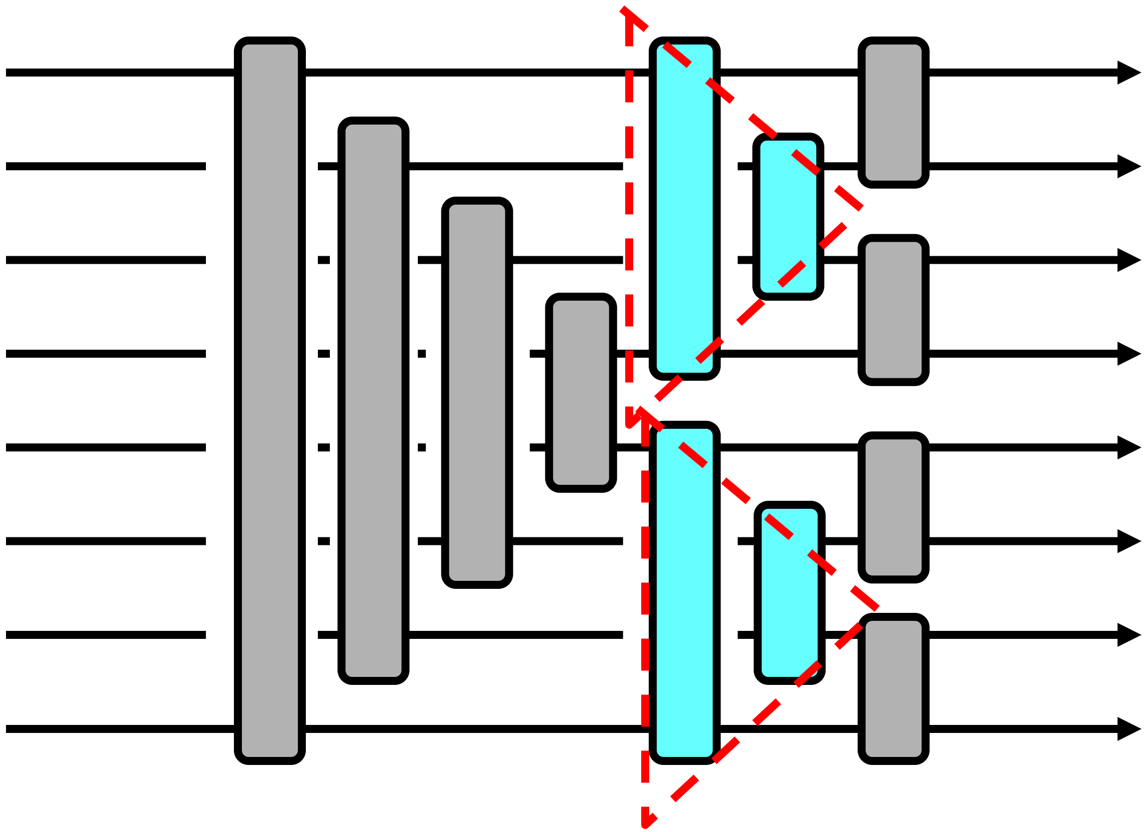
Periodic Network Block



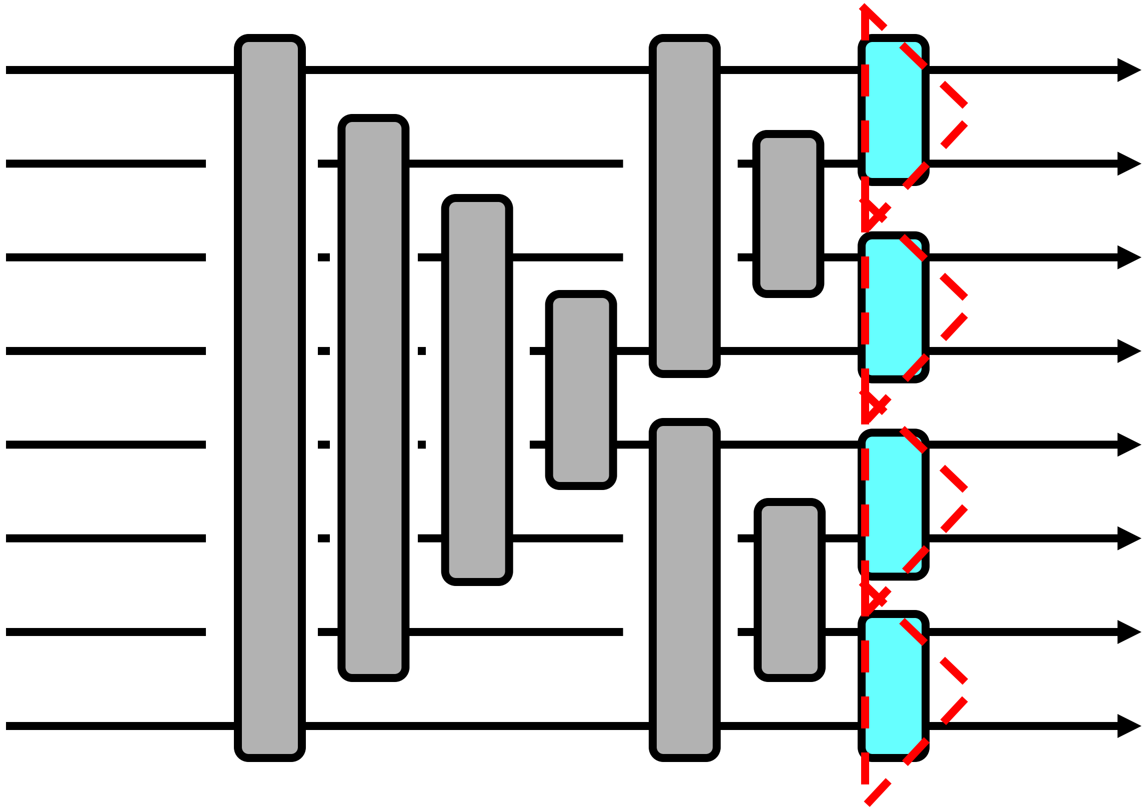
Periodic Network Block



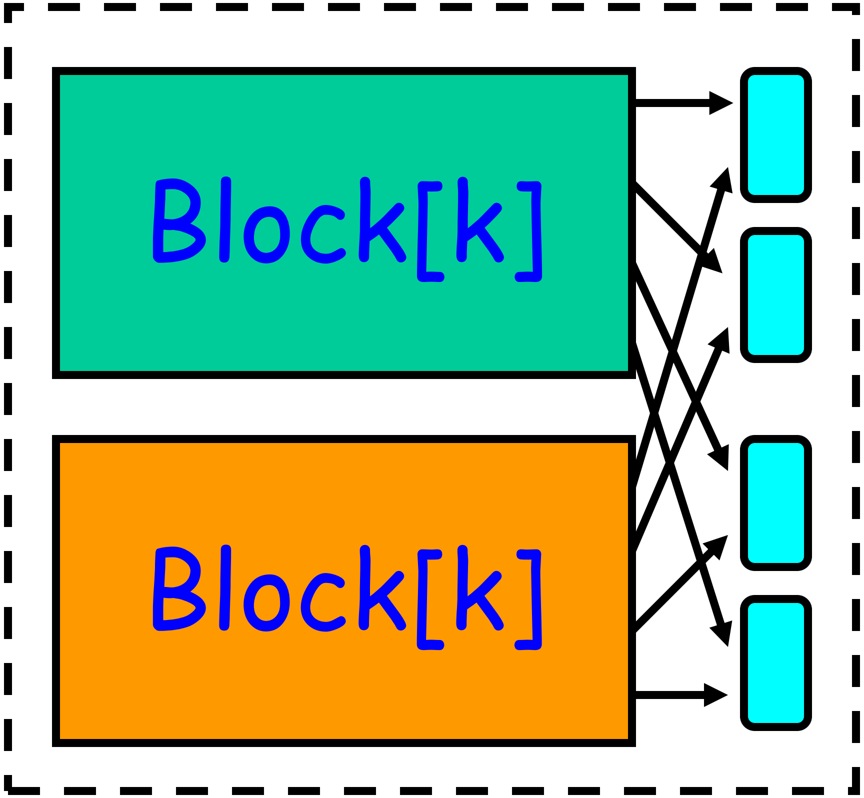
Periodic Network Block



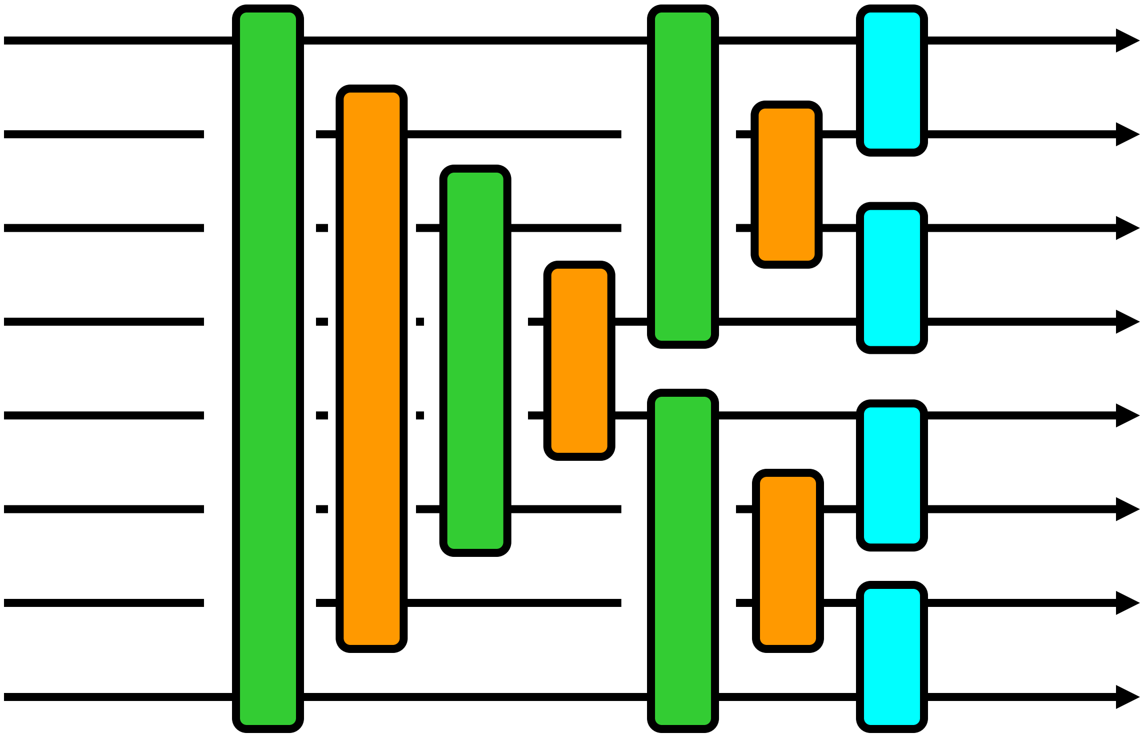
Periodic Network Block



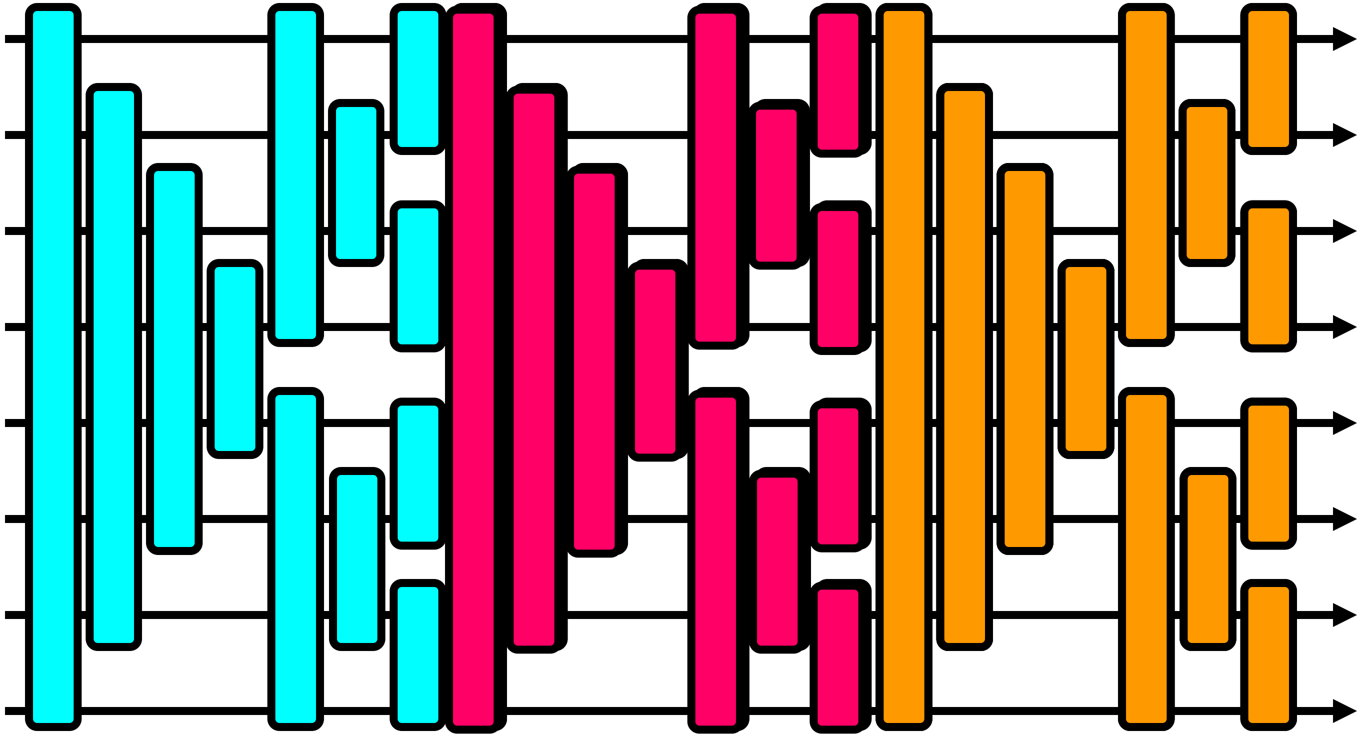
Block[2k] Schematic



Block[2k] Layout



Periodic[8]



Network Depth

- Each block[k] has depth $\log_2 k$
- Need $\log_2 k$ blocks
- Grand total of $(\log_2 k)^2$

Lower Bound on Depth

Theorem: The depth of any width w counting network is at least $\Omega(\log w)$.

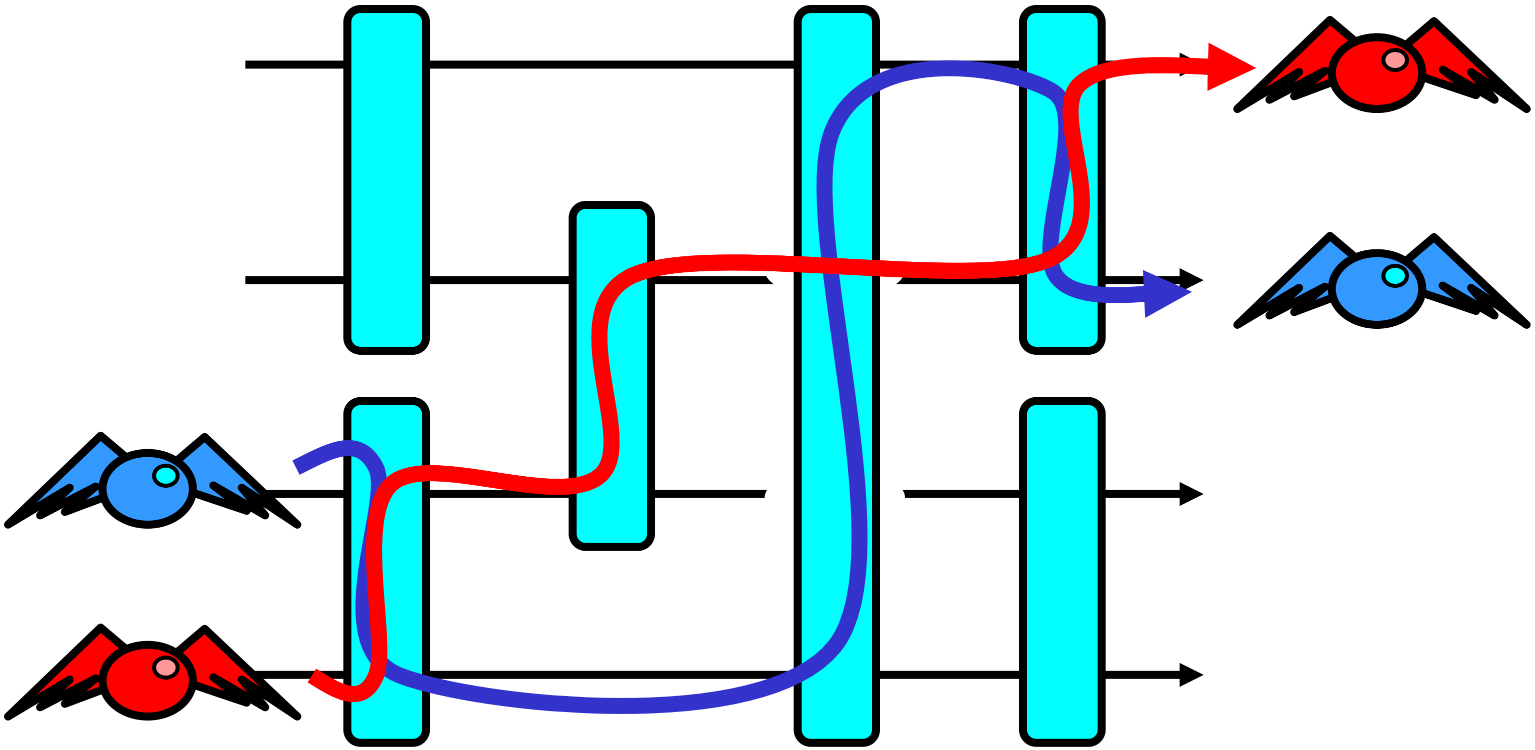
Theorem: there exists a counting network of $\theta(\log w)$ depth.

Unfortunately, proof is non-constructive and constants in the 1000s.

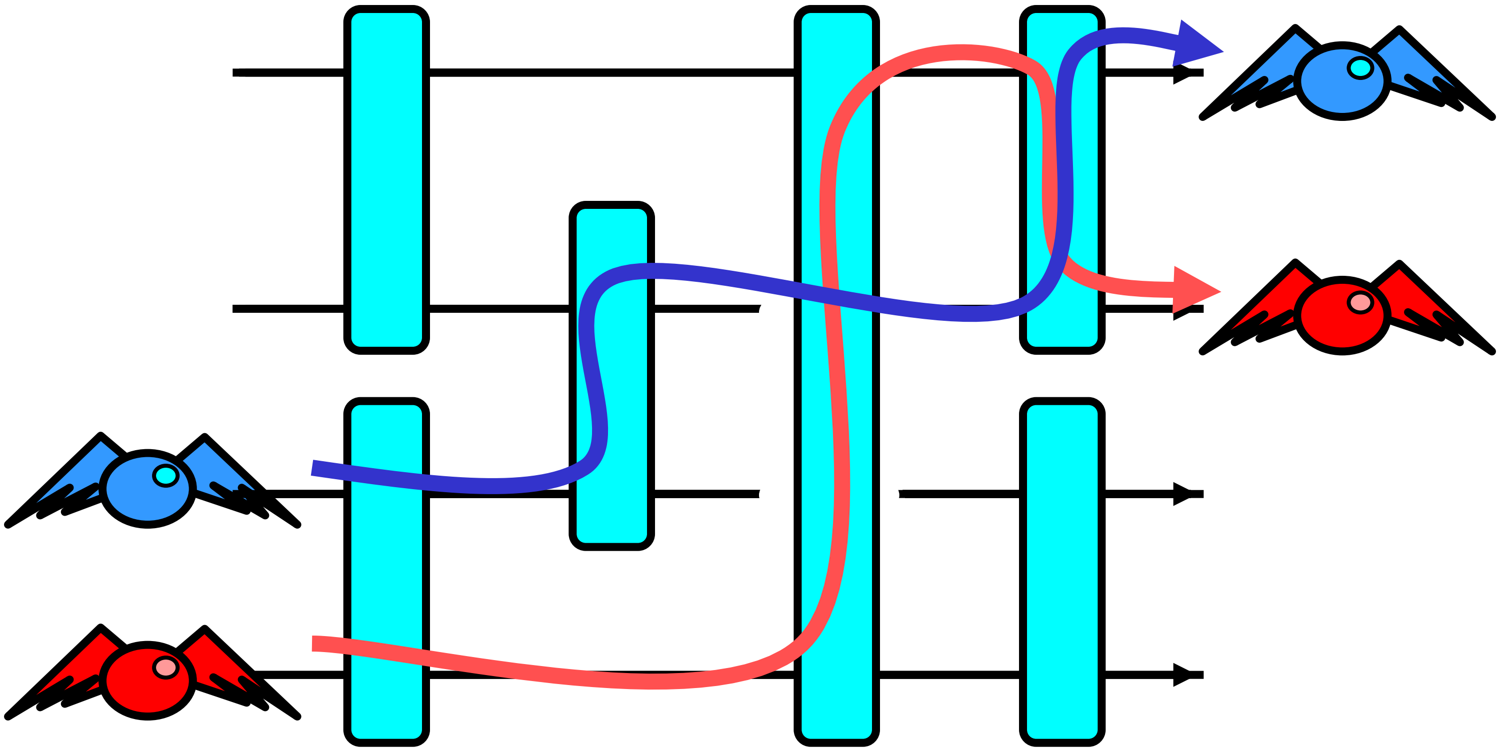
Sequential Theorem

- If a balancing network counts
  - Sequentially, meaning that
  - Tokens traverse one at a time
- Then it counts
  - Even if tokens traverse concurrently

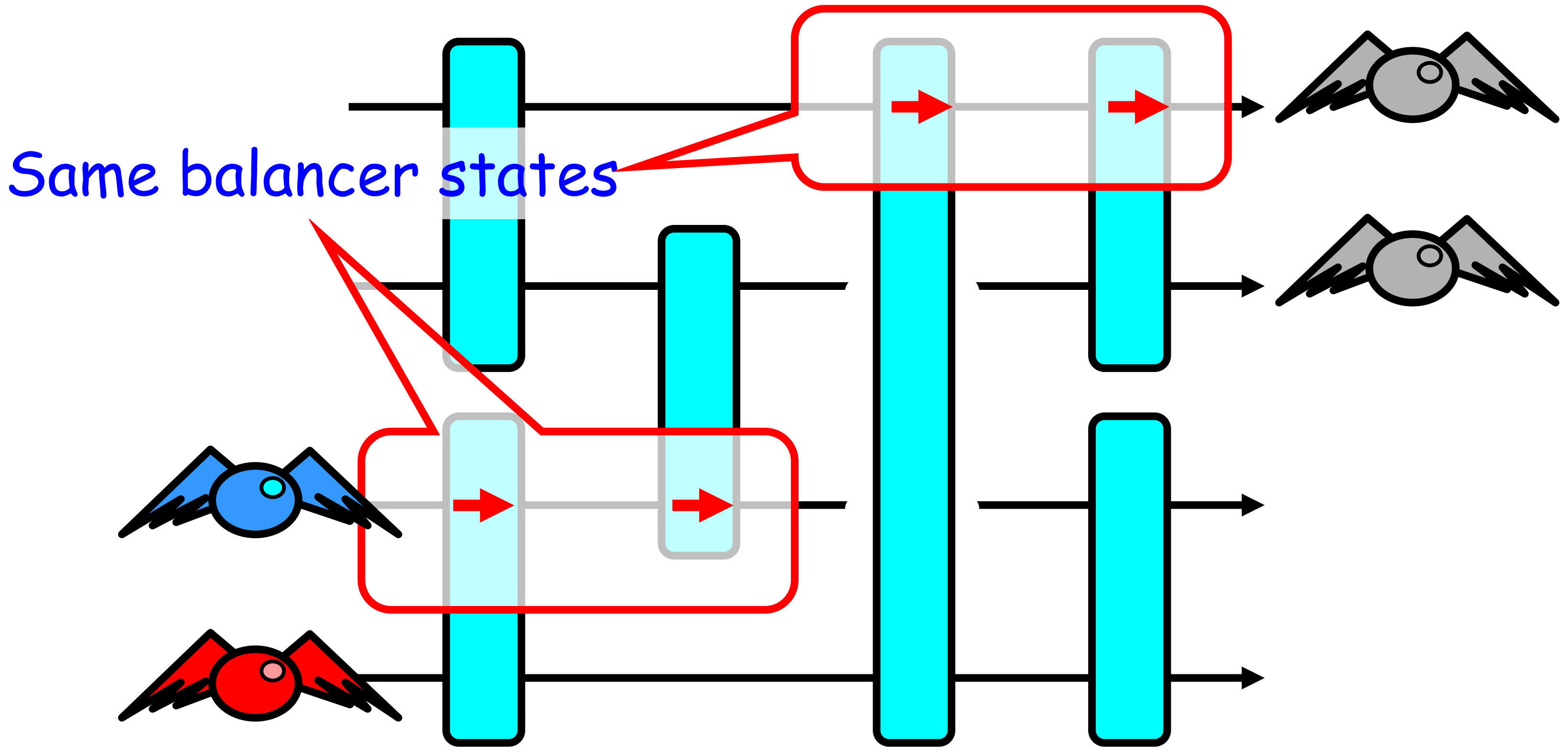
Red First, Blue Second



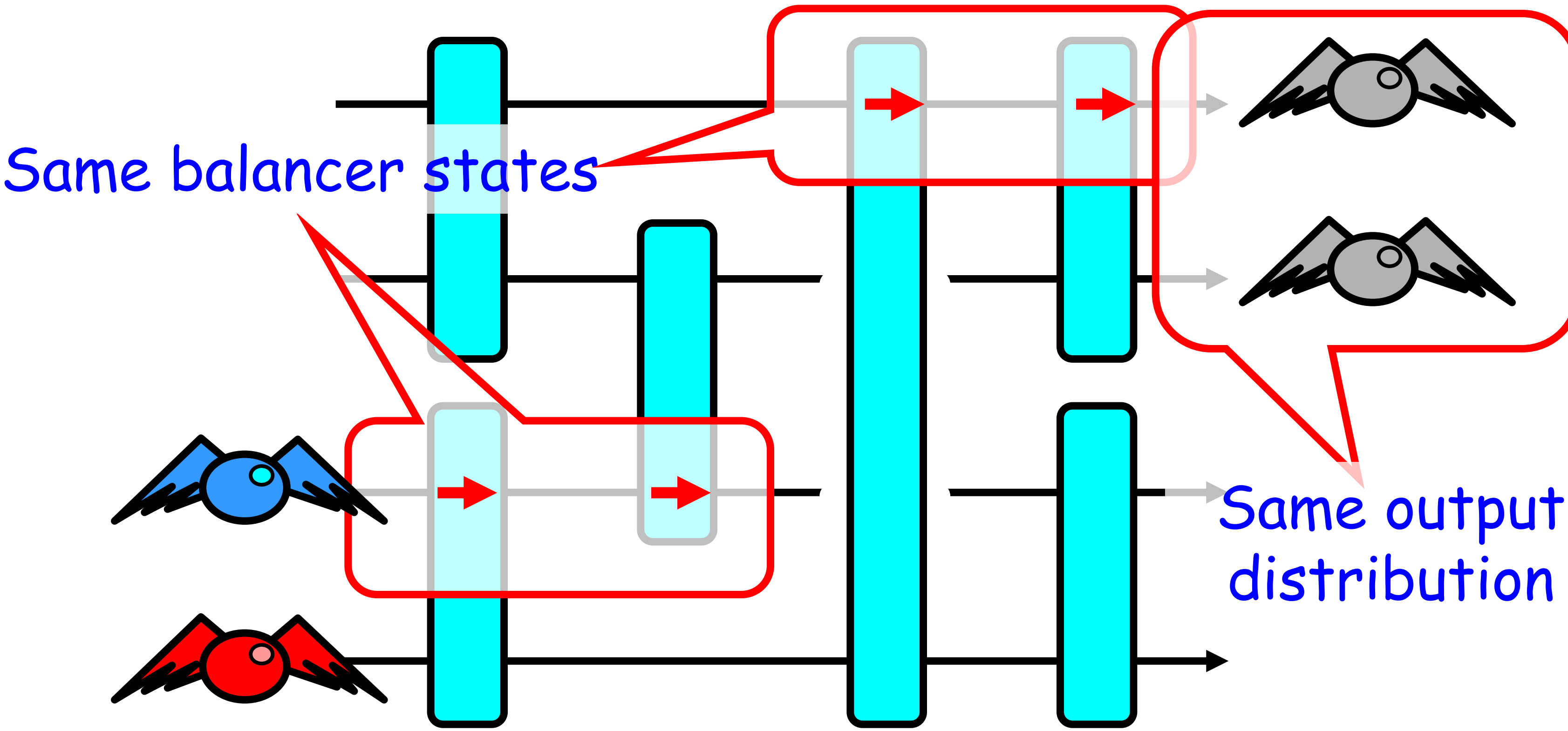
Blue First, Red Second



Either Way



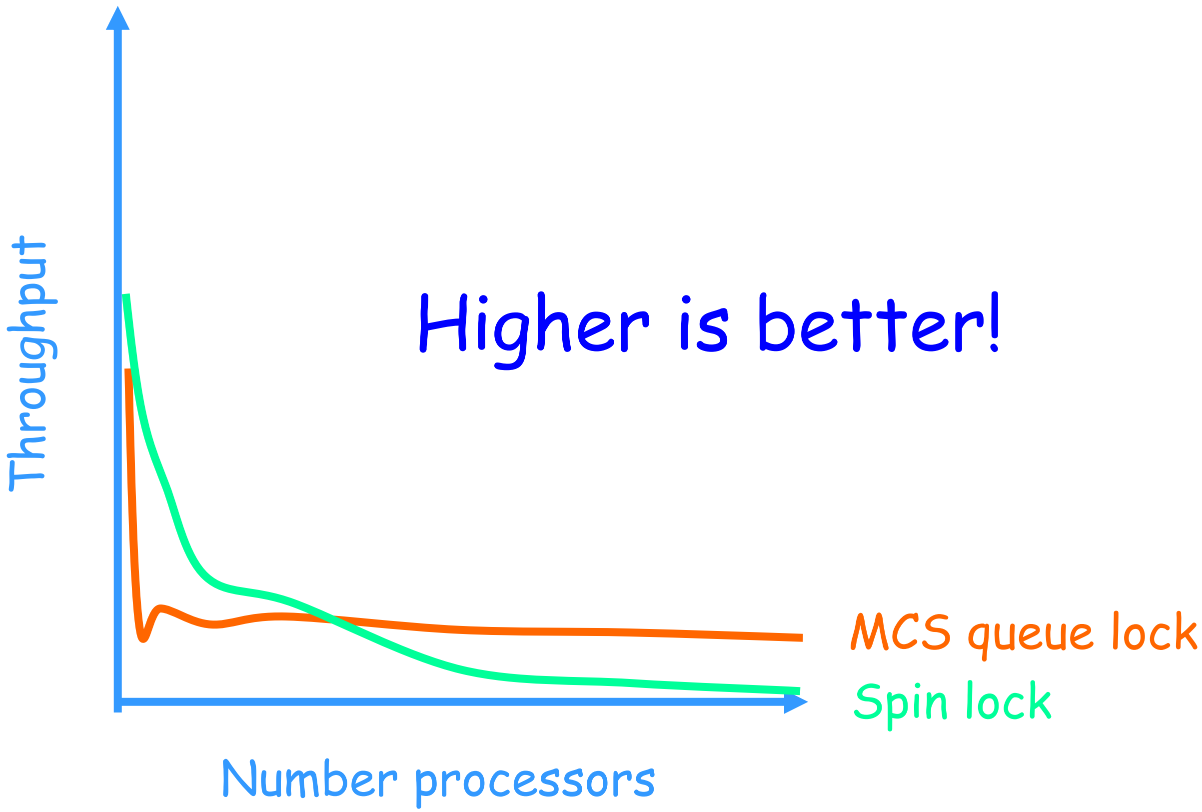
Order Doesn't Matter



Index Distribution Benchmark

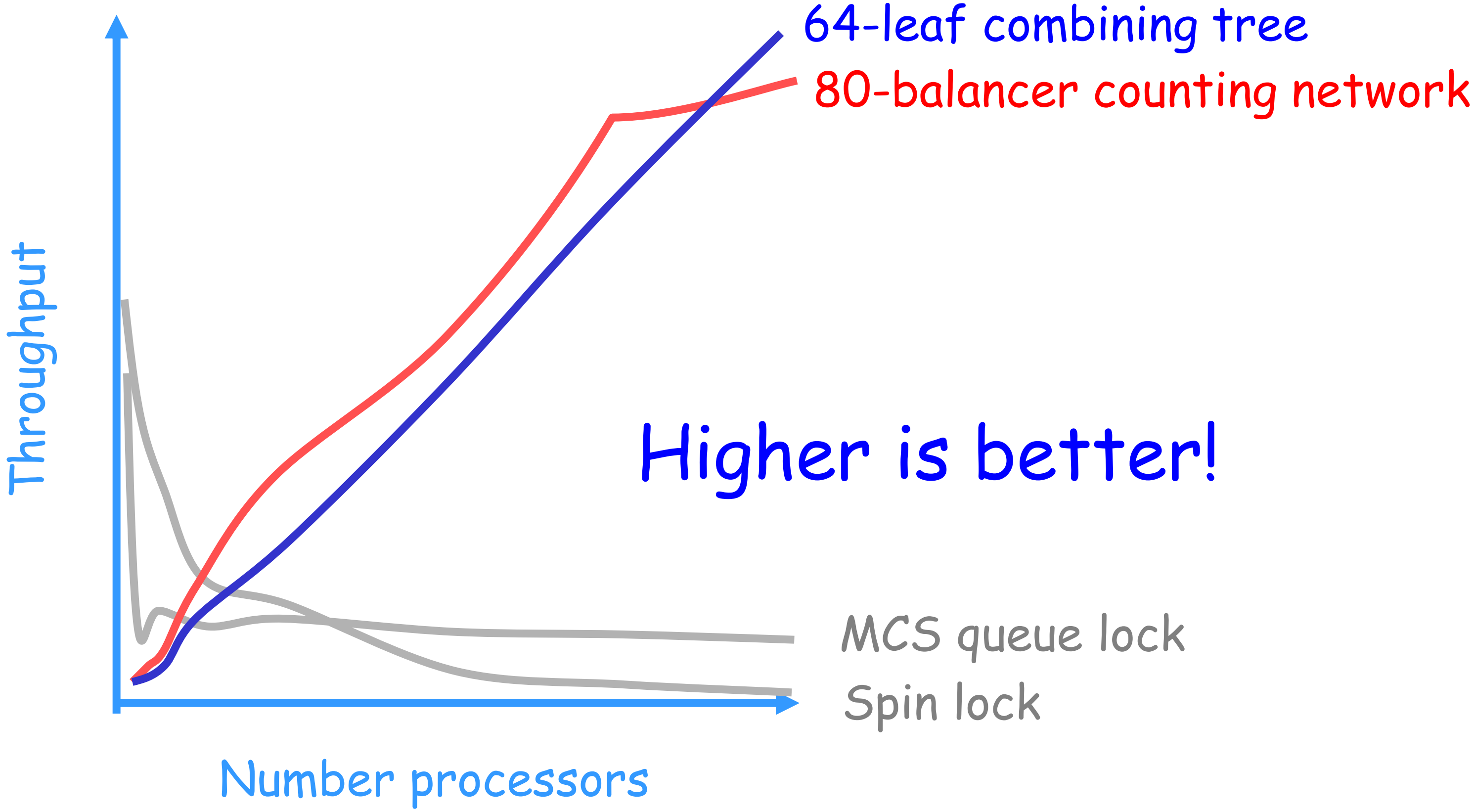
```
void indexBench(int iters, int work) {
 while (int i = 0 < iters) {
  i = fetch&inc();
  Thread.sleep(random() % work);
 }
}
```

Performance (Simulated)



* All graphs taken from Herlihy,Lim,Shavit, copyright ACM.

Performance (Simulated)



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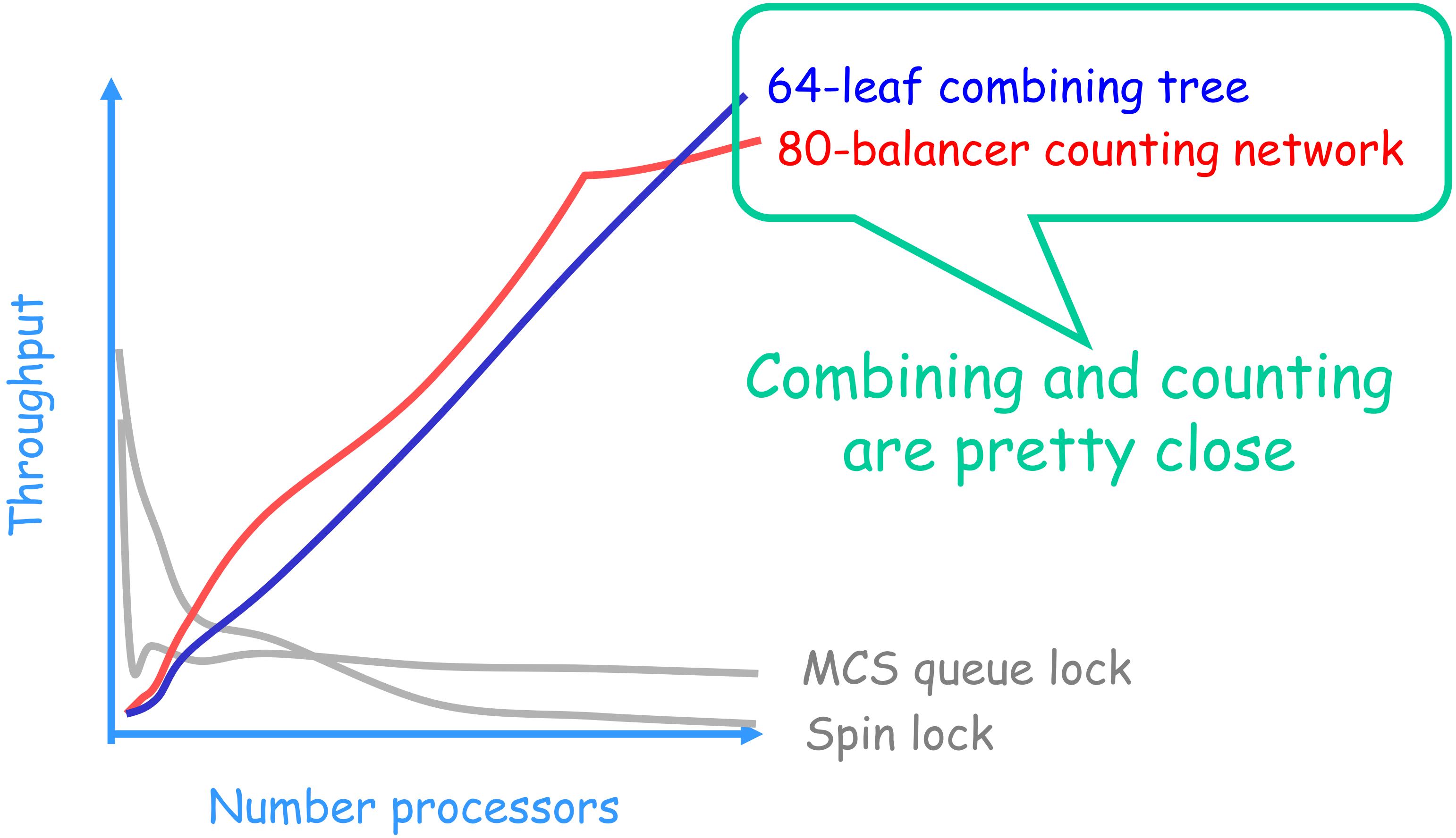
Performance (Simulated)



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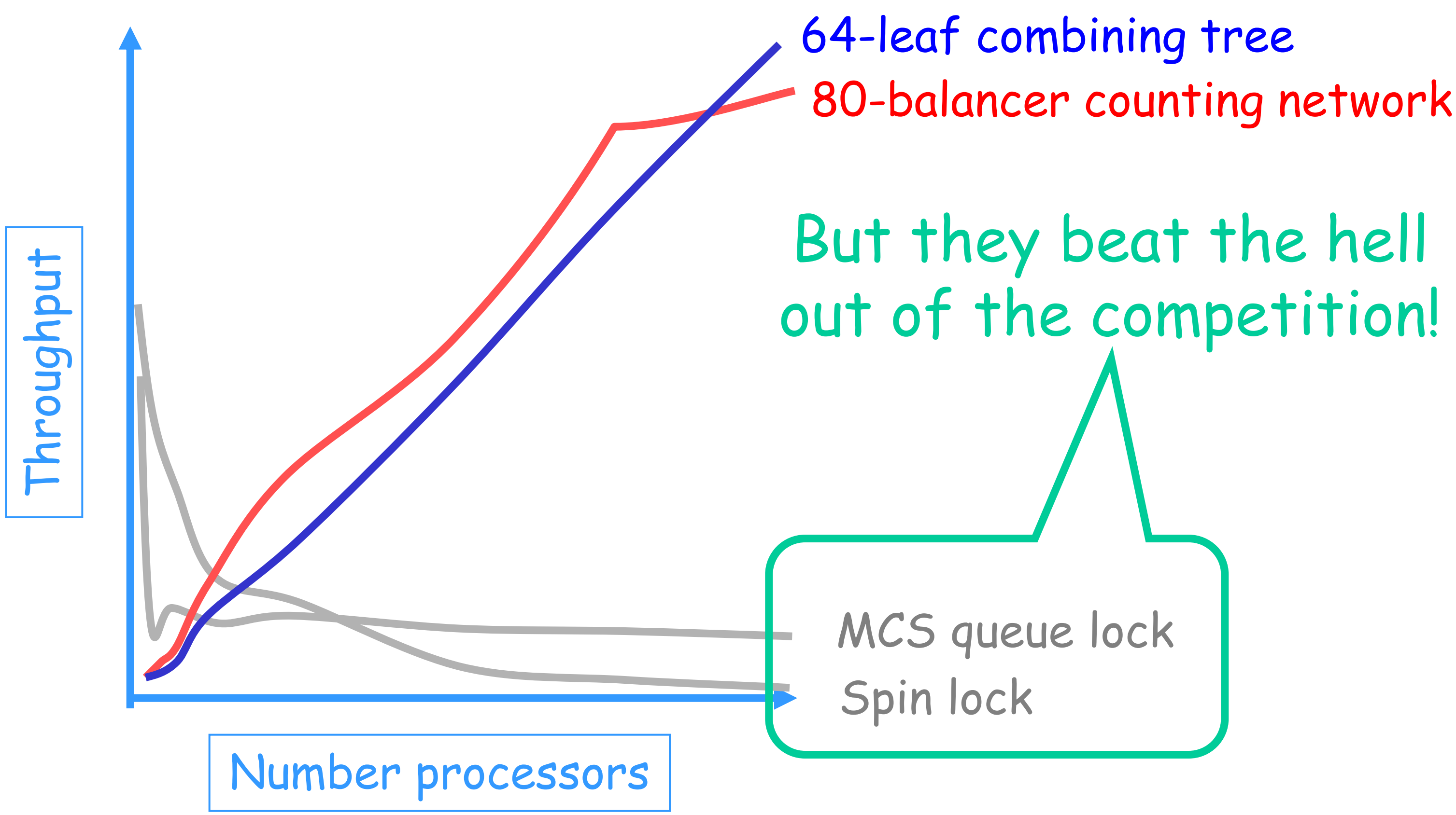
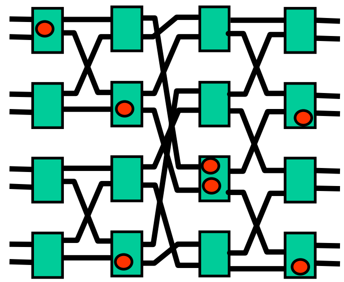
Performance (Simulated)

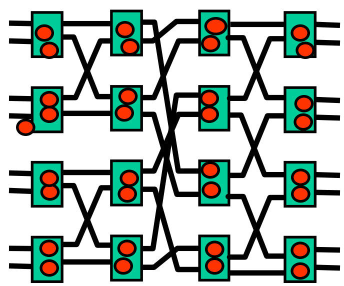


* All graphs taken from Herlihy,Lim,Shavit, copyright ACM.

Saturation and Performance

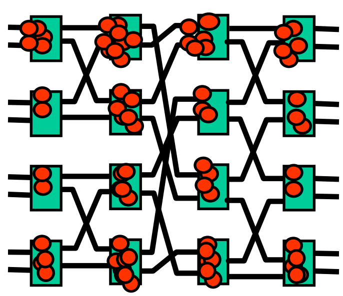


Undersaturated $P < w \log w$



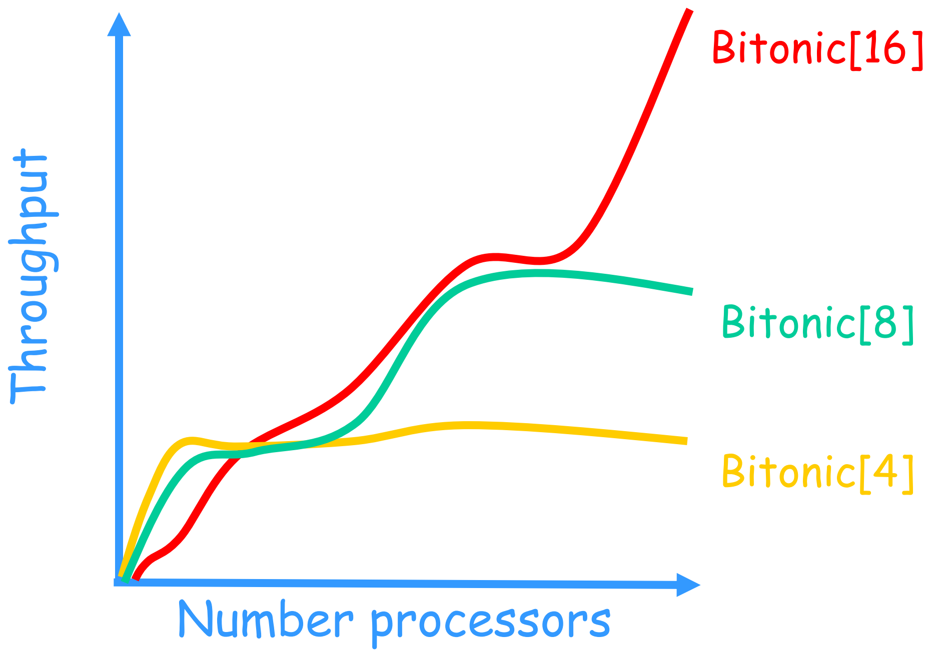
Optimal performance

Saturated $P = w \log w$

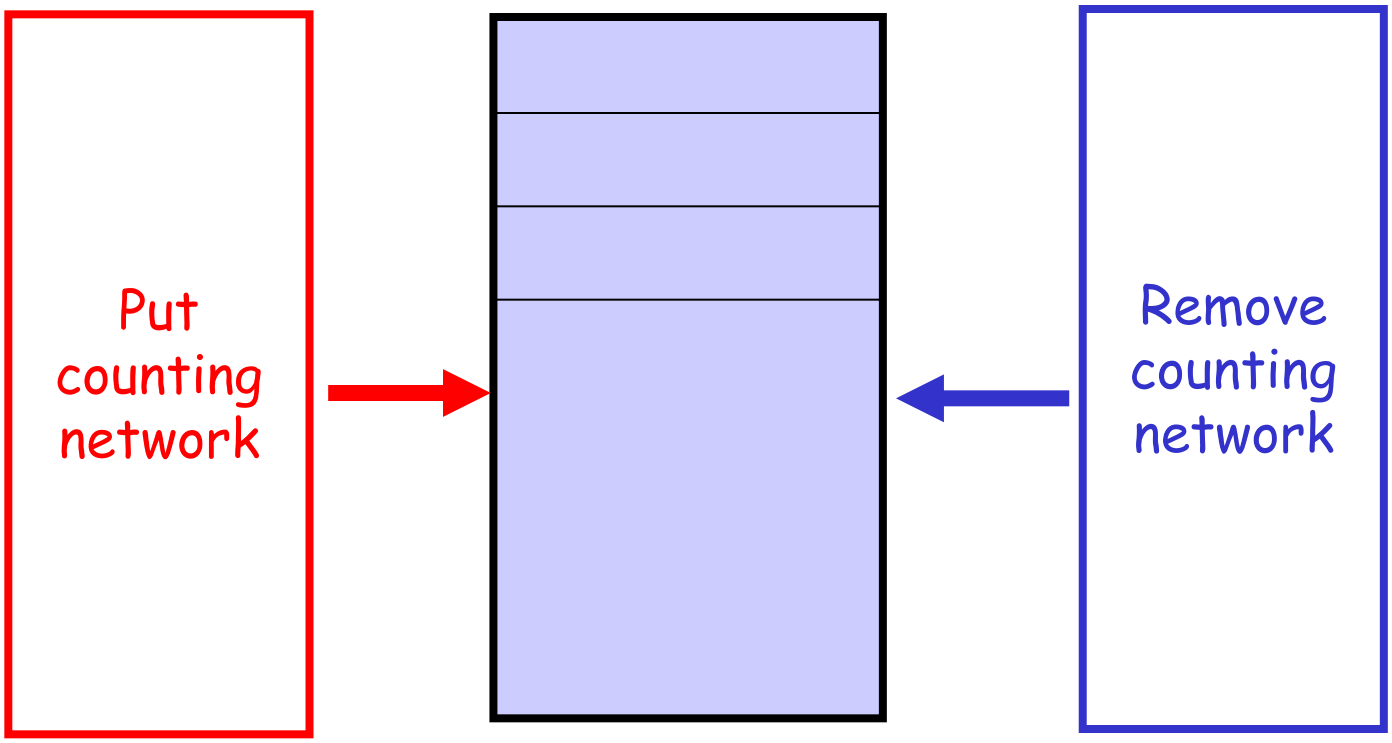


Oversaturated $P > w \log w$

Throughput vs. Size



Shared Pool



Put/Remove Network

- Guarantees never:
  - Put waiting for item, while
  - Get has deposited item
- Otherwise OK to wait
  - Put delayed while pool slot is full
  - Get delayed while pool slot is empty

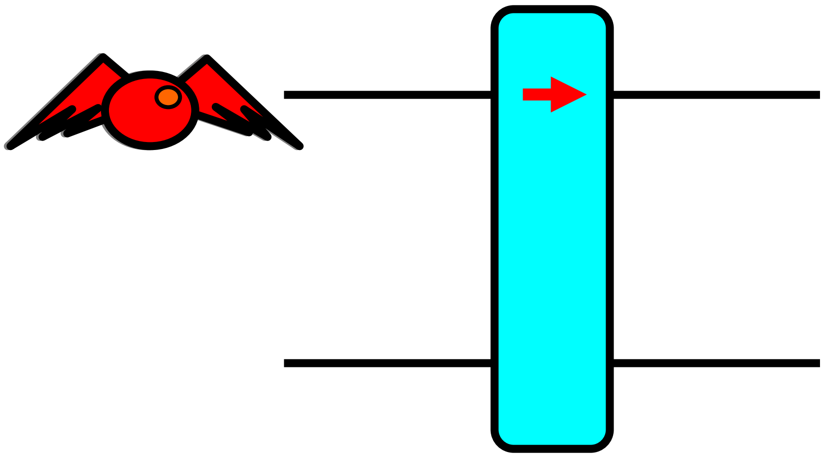
What About

- Decrements
- Adding arbitrary values
- Other operations
  - Multiplication
  - Vector addition
  - Horoscope casting …

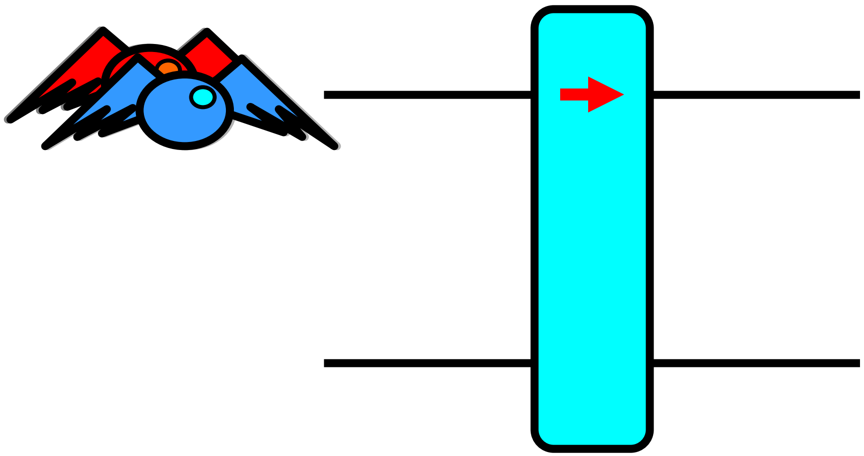
First Step

- Can we decrement as well as increment?
- What goes up, must come down …

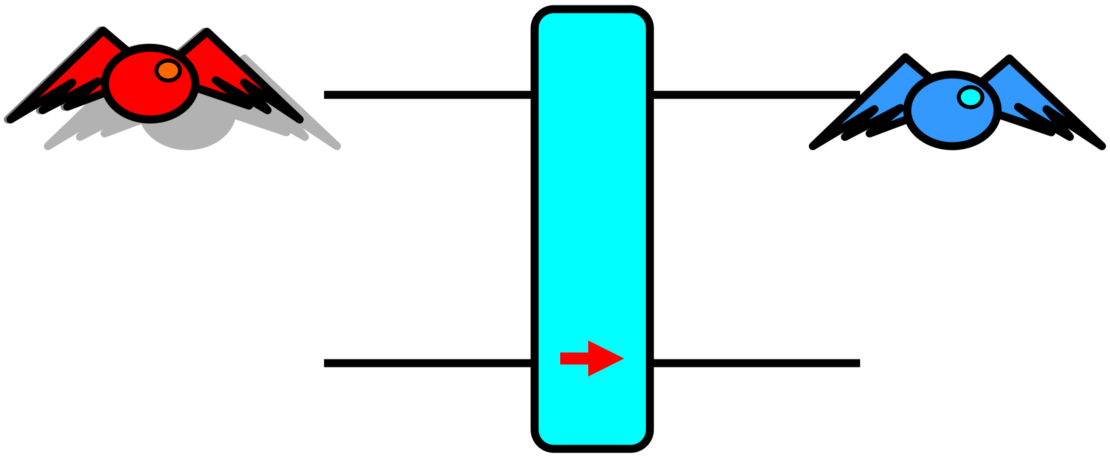
Anti-Tokens



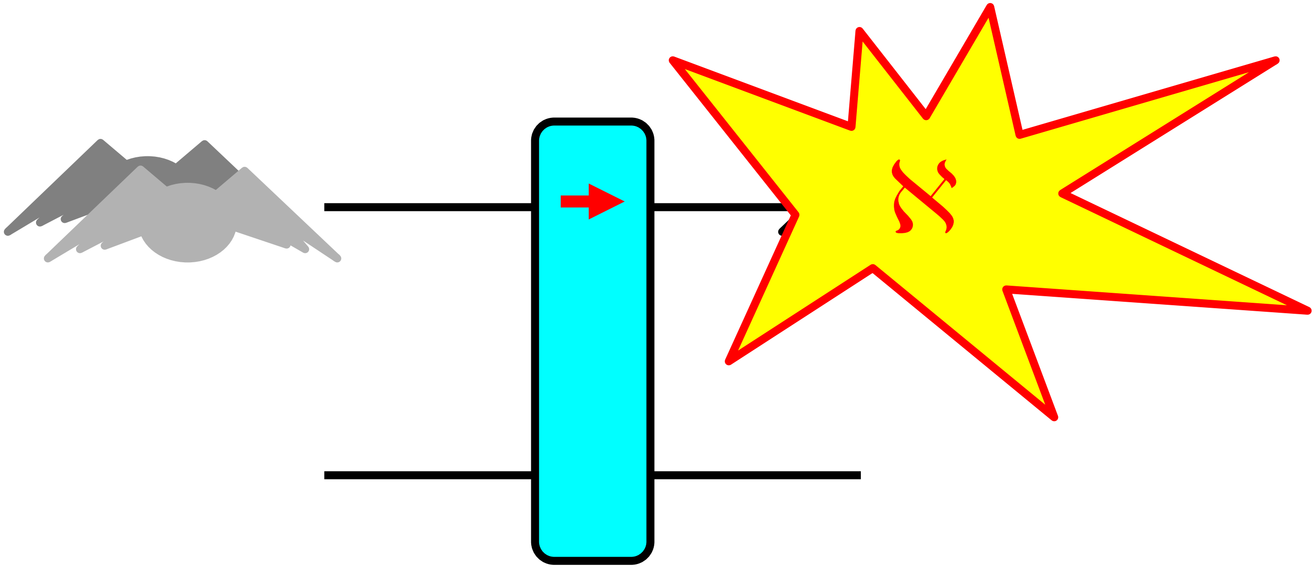
Tokens & Anti-Tokens Cancel



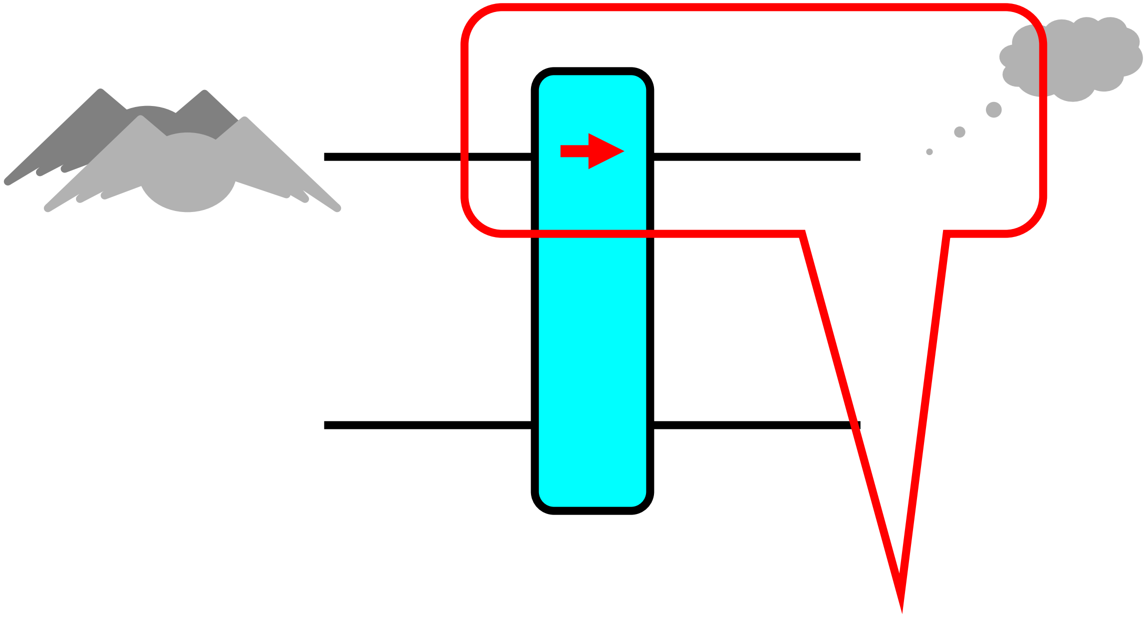
Tokens & Anti-Tokens Cancel



Tokens & Anti-Tokens Cancel



Tokens & Anti-Tokens Cancel

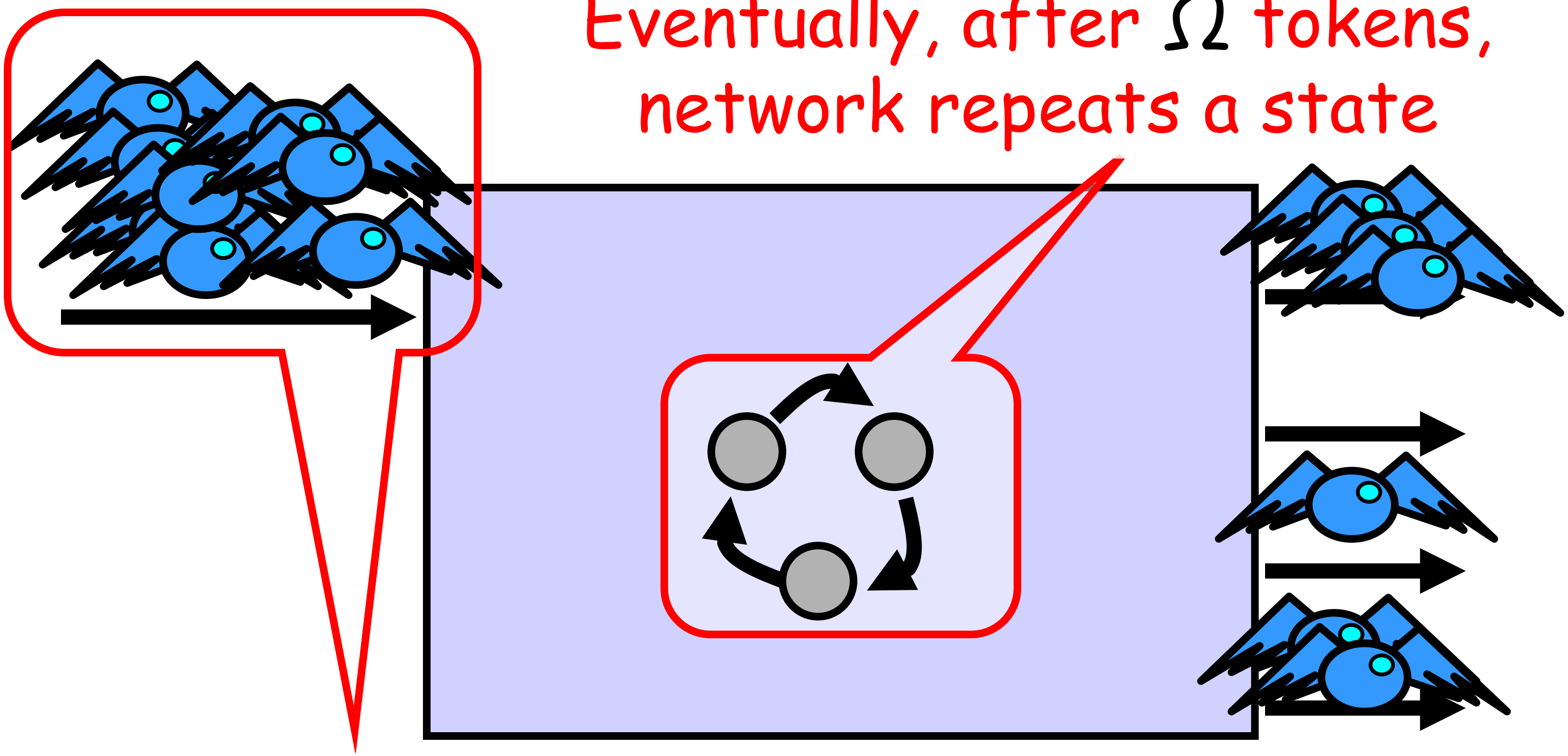


Tokens vs Antitokens

- Tokens
  - read balancer
  - flip
  - proceed
- Antitokens
  - flip balancer
  - read
  - proceed

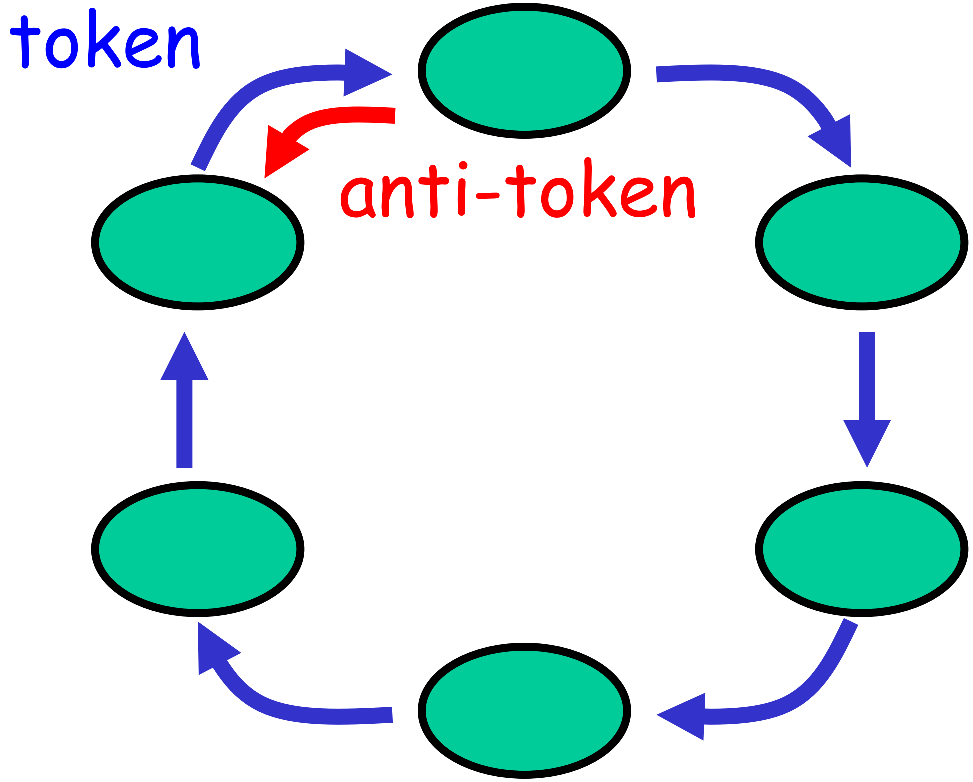
Pumping Lemma

Eventually, after Ω tokens, network repeats a state



Keep pumping tokens through one wire

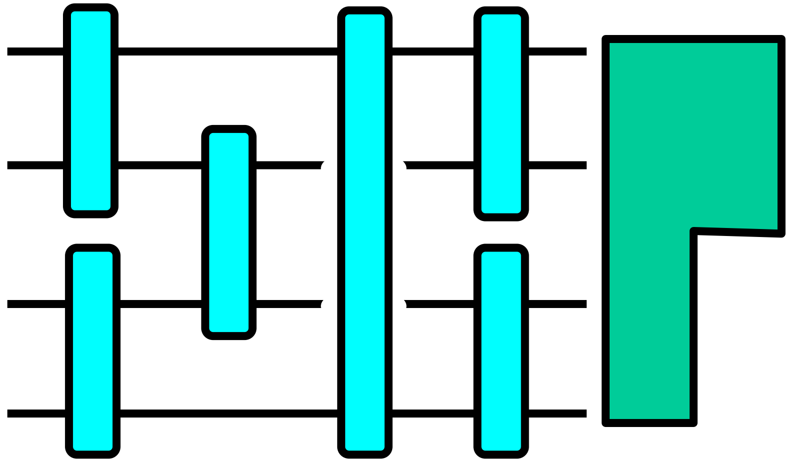
Anti-Token Effect



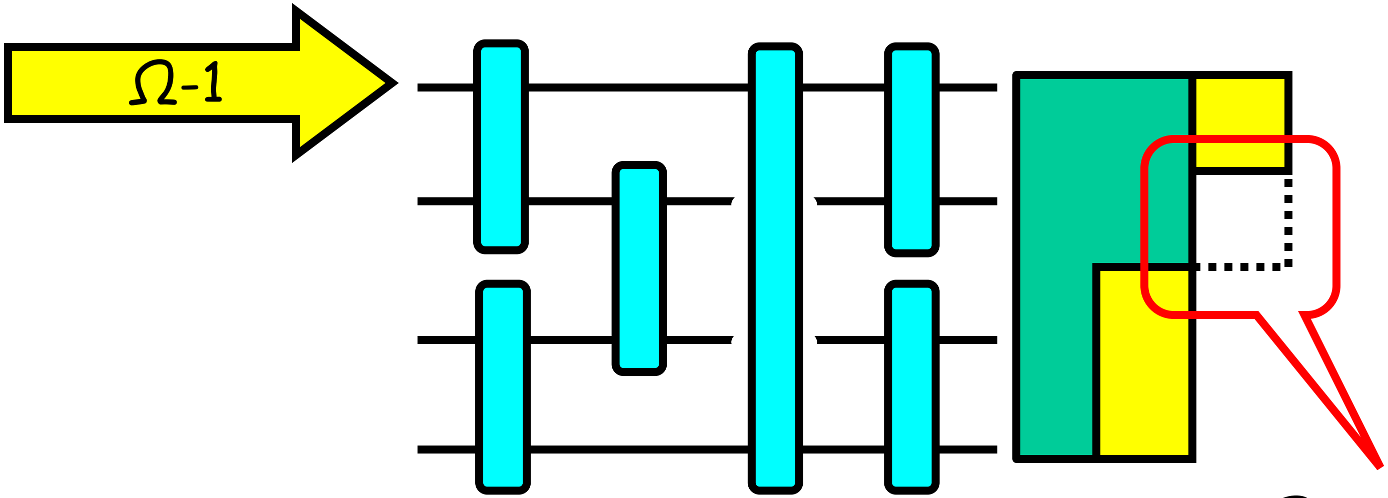
Observation

- Each anti-token on wire i
  - Has same effect as Ω-1 tokens on wire i
  - So network still in legal state
- Moreover, network width w divides Ω
  - So Ω-1 tokens

Before Antitoken

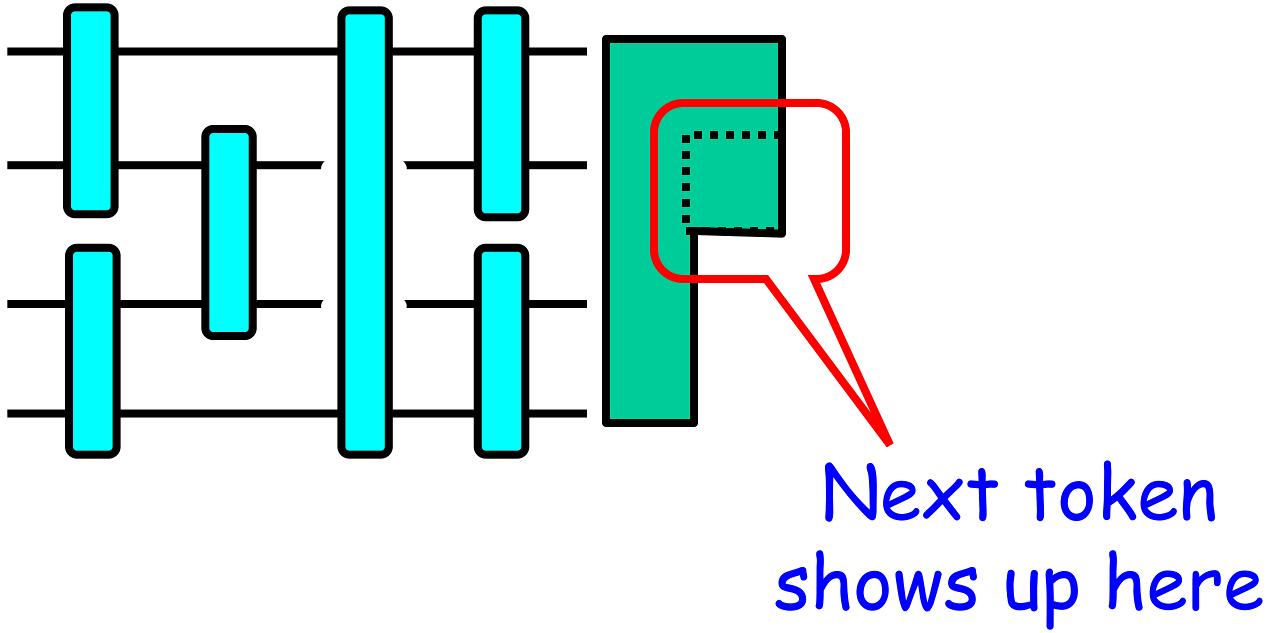


Balancer states as if …



Ω-1 is one brick shy of a load

Post Antitoken



Implication

- Counting networks with
  - Tokens (+1)
  - Anti-tokens (-1)
- Give
  - Highly concurrent
  - Low contention
- getAndIncrement + getAndDecrement methods

QED

Adding Networks

- Combining trees implement
  - Fetch&add
  - Add any number, not just 1
- What about counting networks?

Fetch-and-add

- Beyond getAndIncrement + getAndDecrement
- What about getAndAdd(x)?
  - Atomically returns prior value
  - And adds x to value?
- Not to mention
  - getAndMultiply
  - getAndFourierTransform?

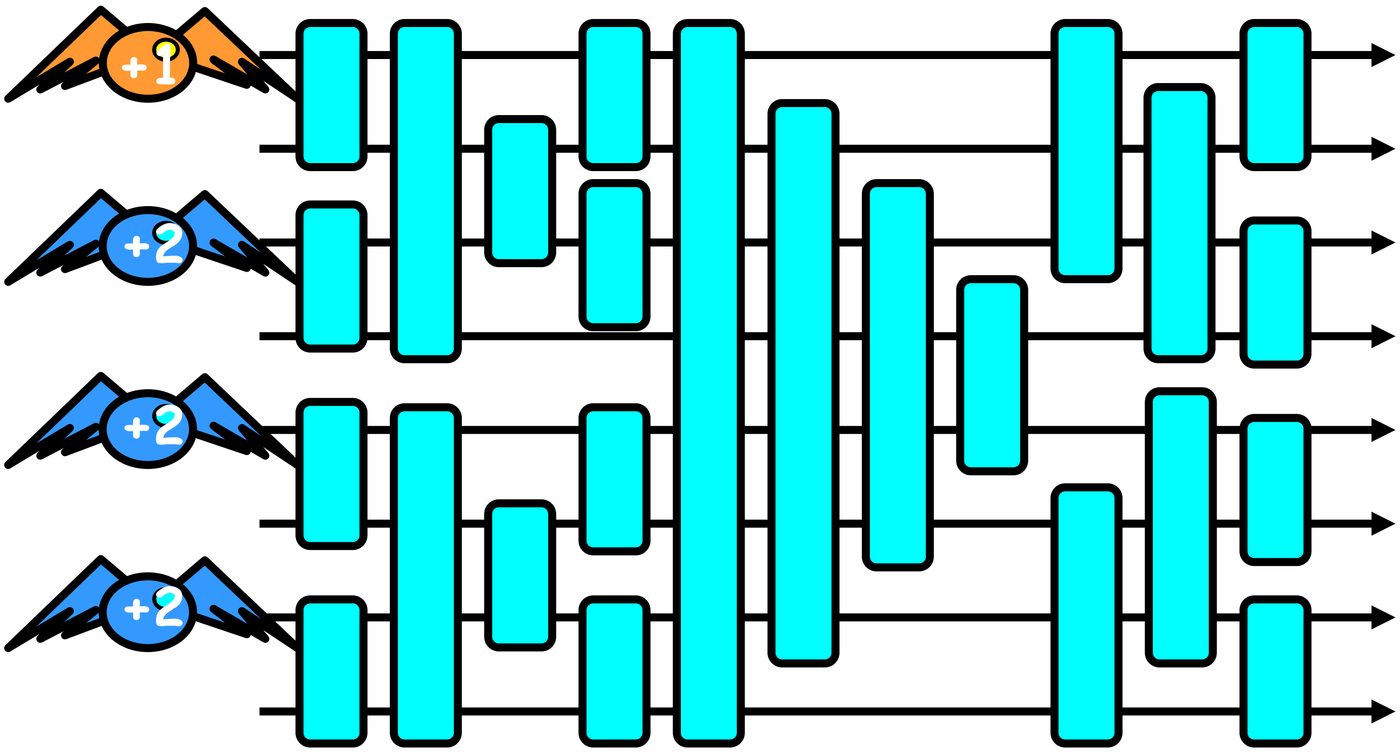
Bad News

- If an adding network
  - Supports n concurrent tokens
- Then every token must traverse
  - At least n-1 balancers
  - In sequential executions

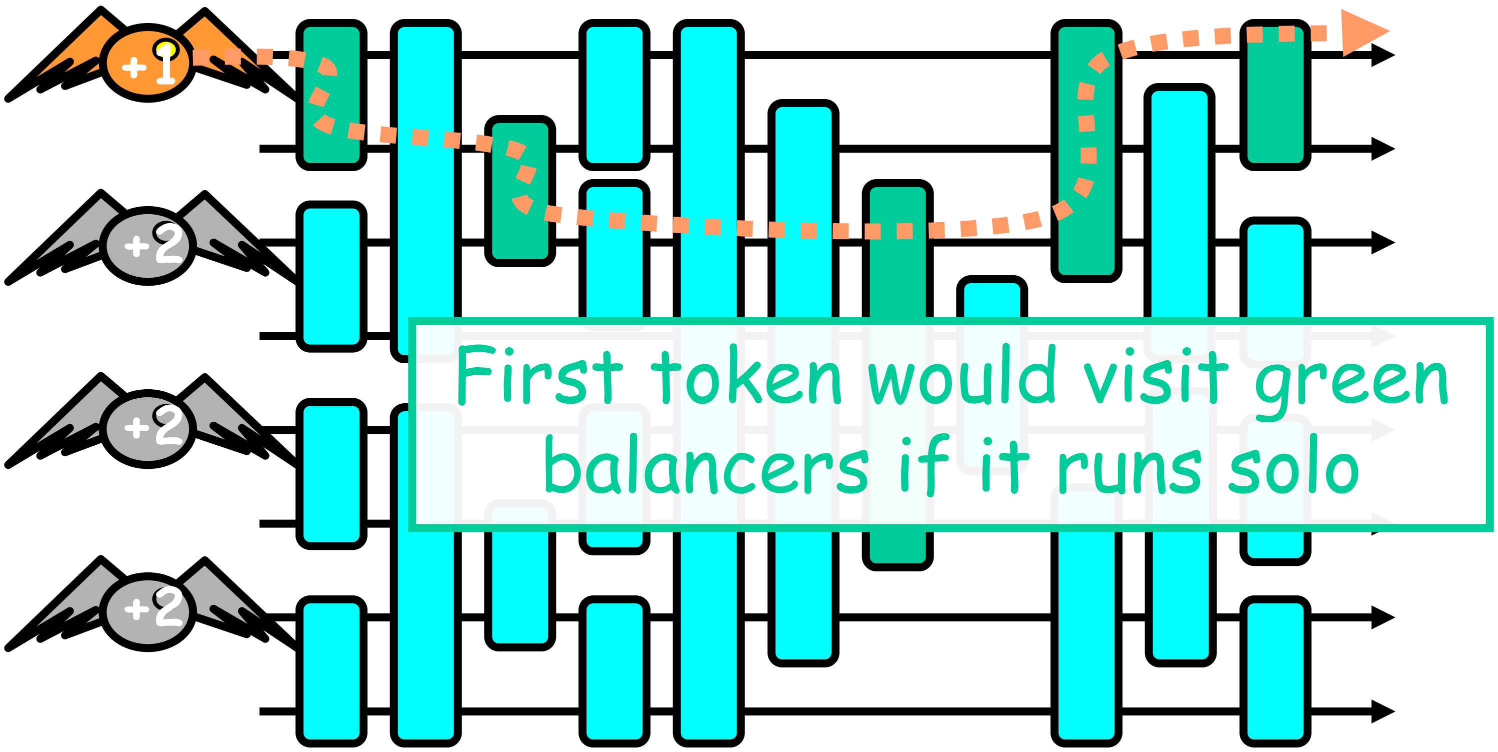
Uh-Oh

- Adding network size depends on n
  - Like combining trees
  - Unlike counting networks
- High latency
  - Depth linear in n
  - Not logarithmic in w

Generic Counting Network



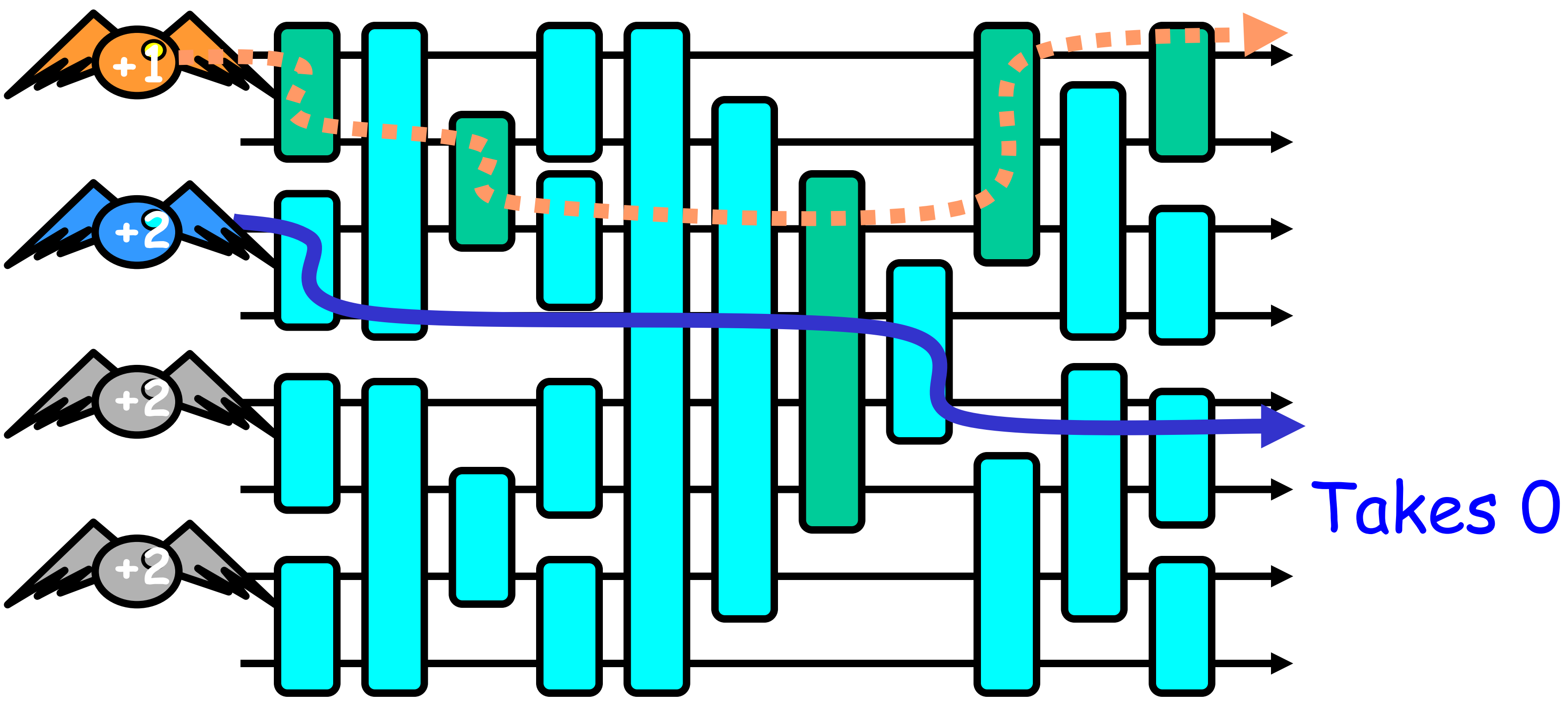
First Token


+1
+2
+2
+2
First token would visit green balancers if it runs solo

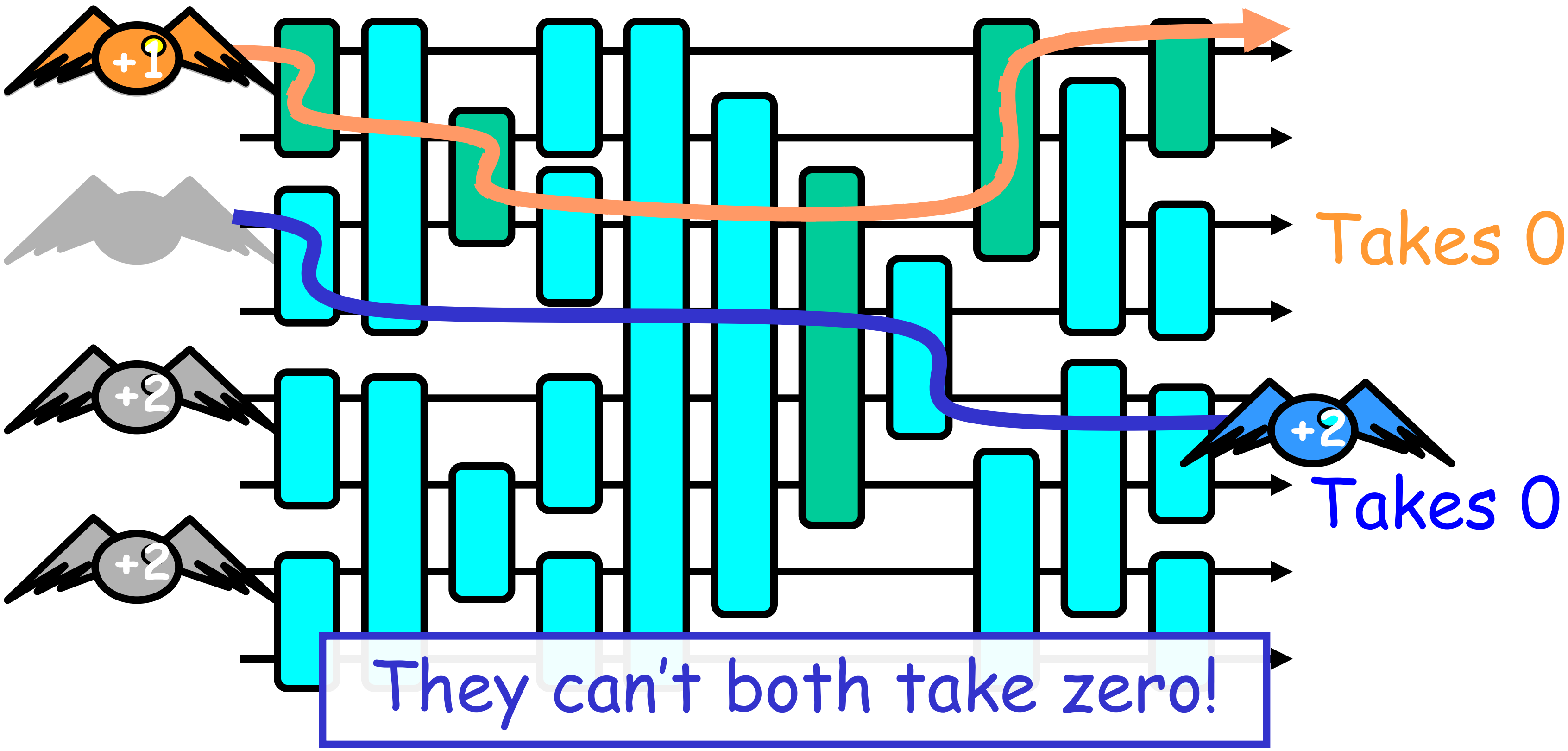
Claim

- Look at path of +1 token
- All other +2 tokens must visit some balancer on +1 token's path

Second Token



Second Token



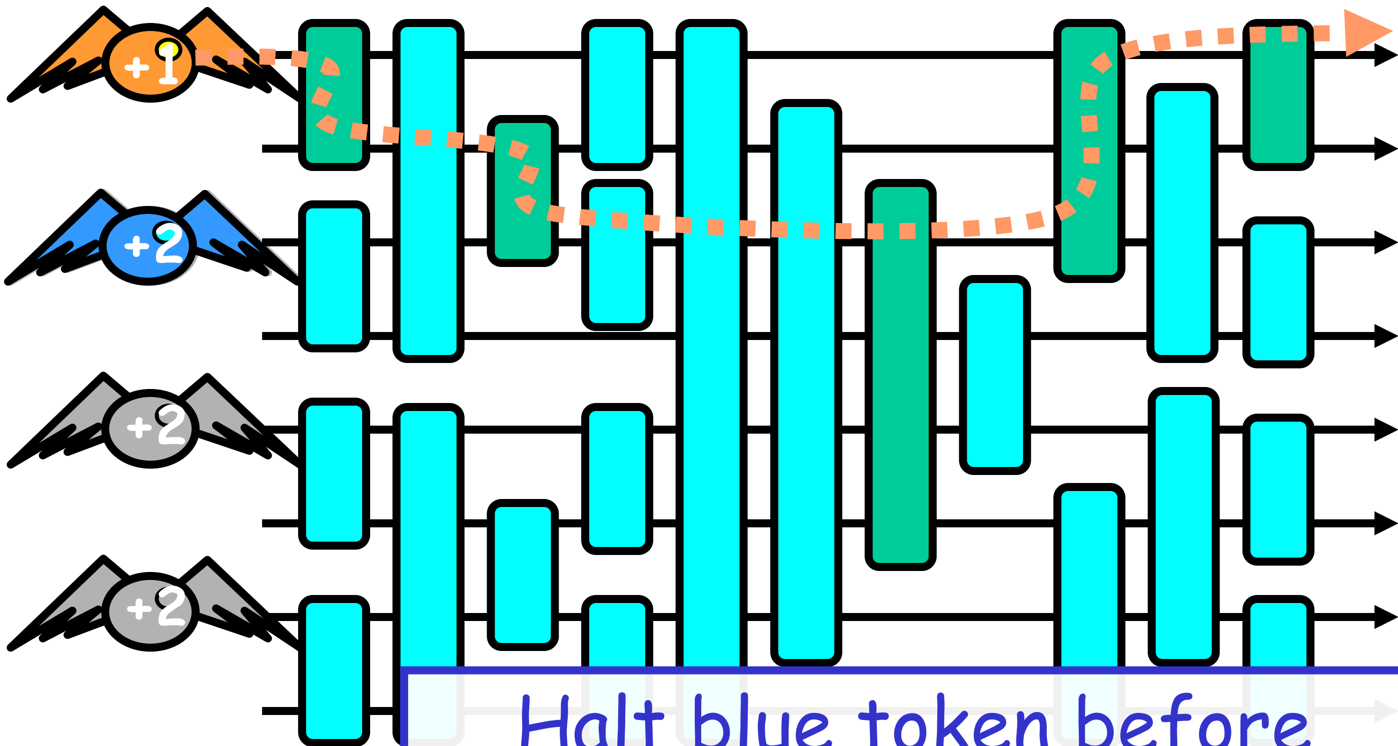
If Second avoids First's Path

- Second token
  - Doesn't observe first
  - First hasn't run
  - Chooses 0
- First token
  - Doesn't observe second
  - Disjoint paths
  - Chooses 0

If Second avoids First's Path

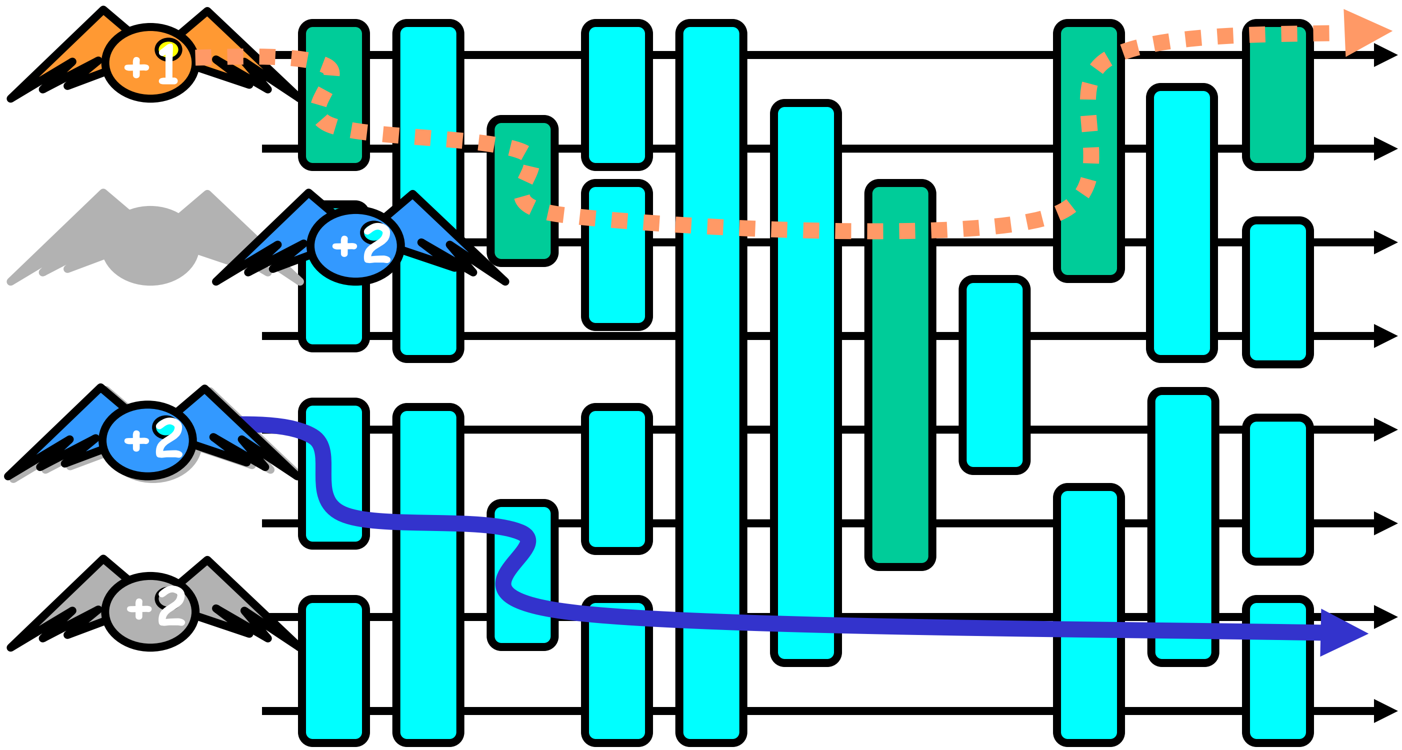
- Because +1 token chooses 0
  - It must be ordered first
  - So +2 token ordered second
  - So +2 token should return 1
- Something's wrong!

Second Token



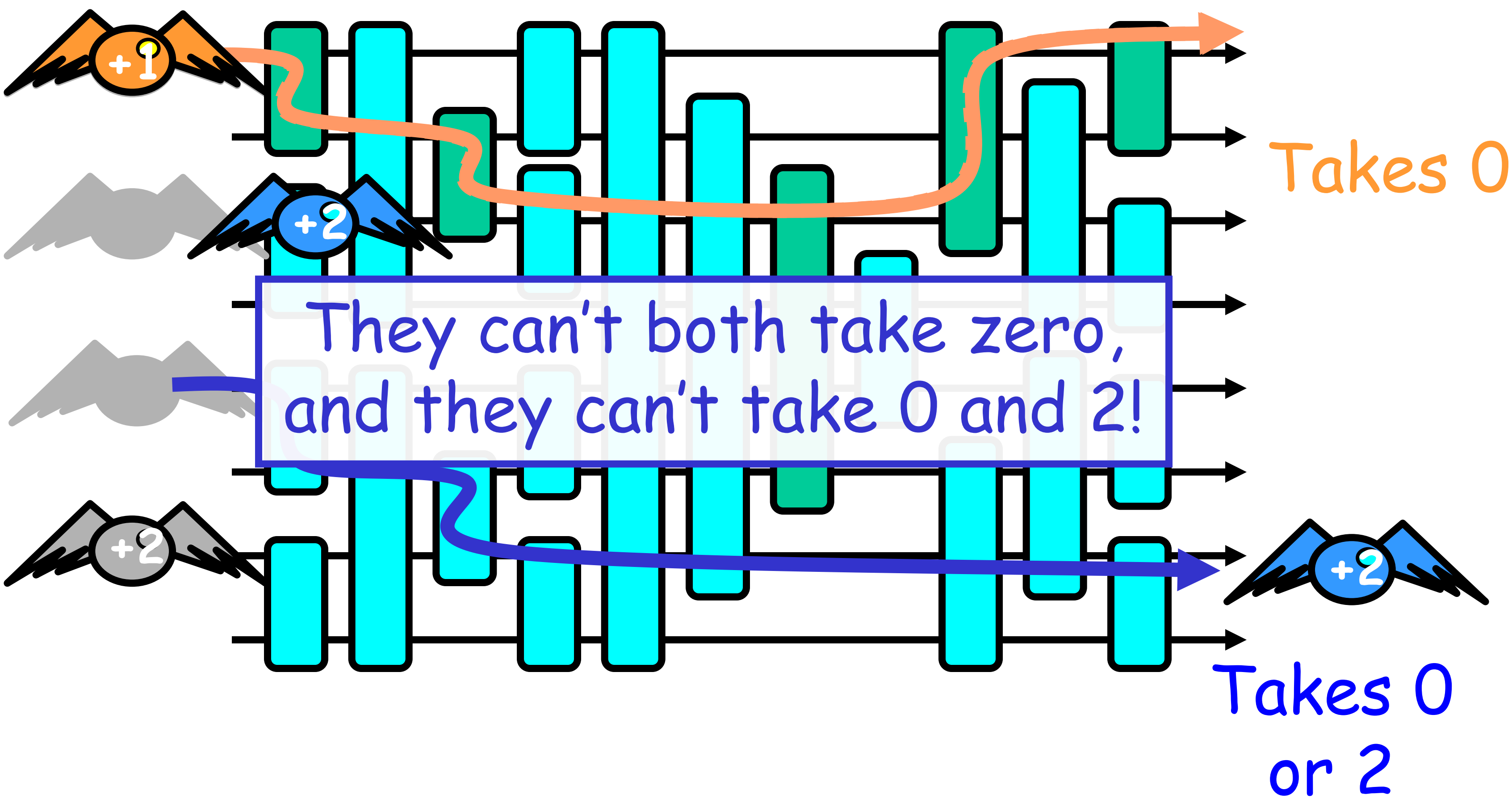
Halt blue token before first green balancer

Third Token



Takes 0 or 2

Third Token



+1
+2
+2
Takes 0
They can't both take zero,
and they can't take 0 and 2!
+2
Takes 0
or 2

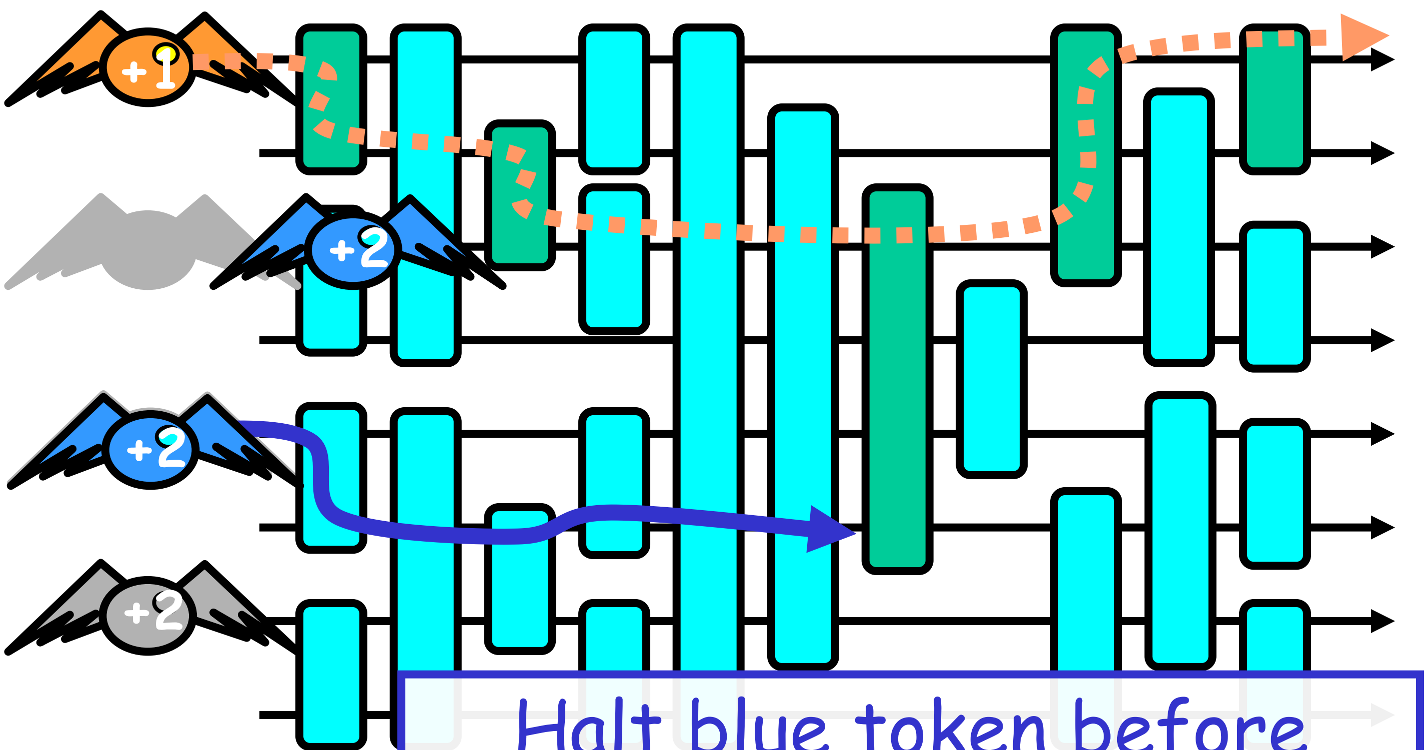
First,Second, & Third Tokens must be Ordered

- Third (+2) token
  - Did not observe +1 token
  - May have observed earlier +2 token
  - Takes an even number

First,Second, & Third Tokens must be Ordered

- Because +1 token's path is disjoint
  - It chooses 0
  - Ordered first
  - Rest take odd numbers
- But last token takes an even number
- Something's wrong!

Third Token



Halt blue token before first green balancer

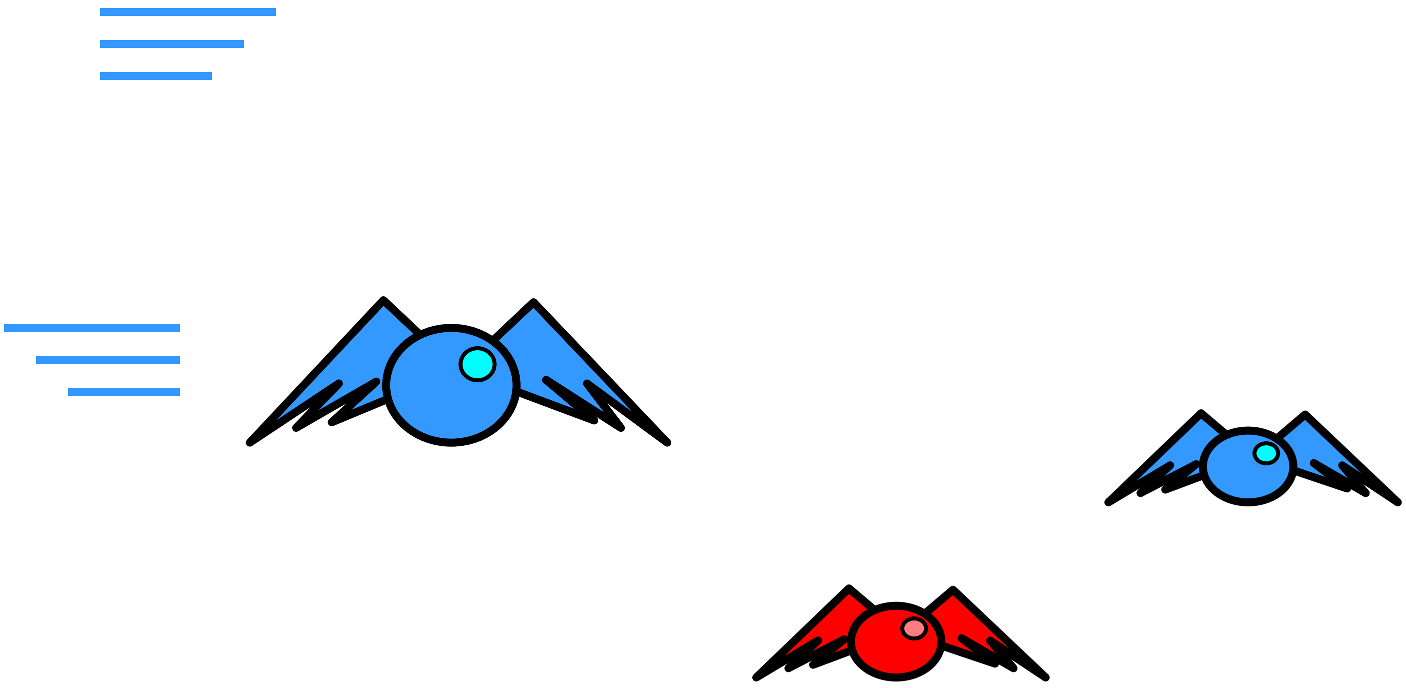
Continuing in this way

- We can "park" a token
  - In front of a balancer
  - That token #1 will visit
- There are n-1 other tokens
  - Two wires per balancer
  - Path includes n-1 balancers!

Theorem

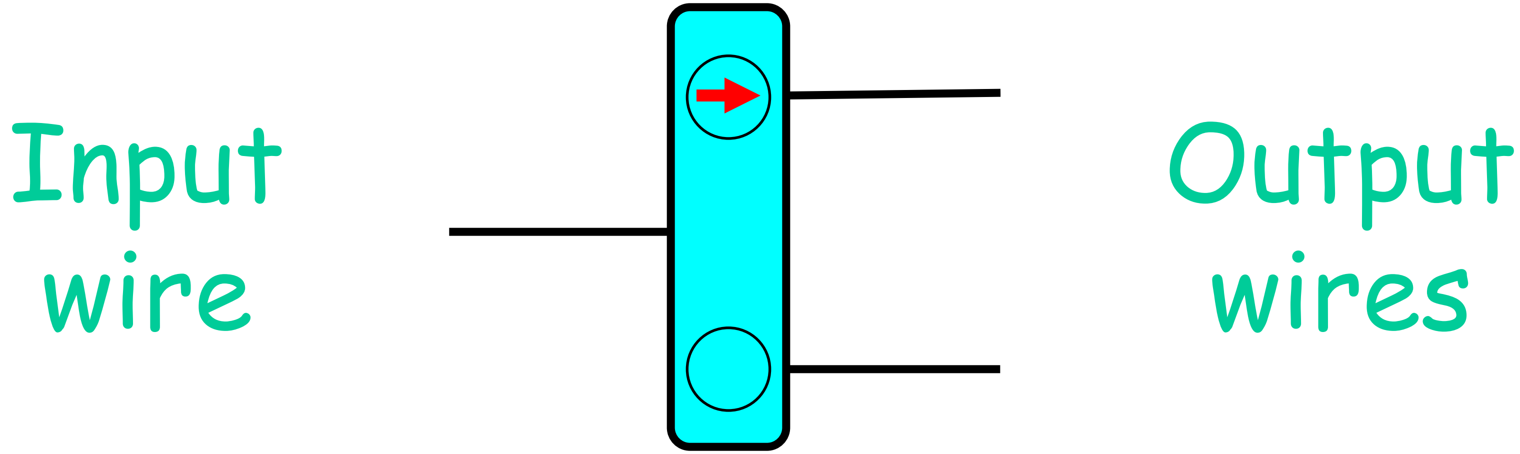
- In any adding network
  - In sequential executions
  - Tokens traverse at least n-1 balancers
- Same arguments apply to
  - Linearizable counting networks
  - Multiplying networks
  - And others

Clip Art

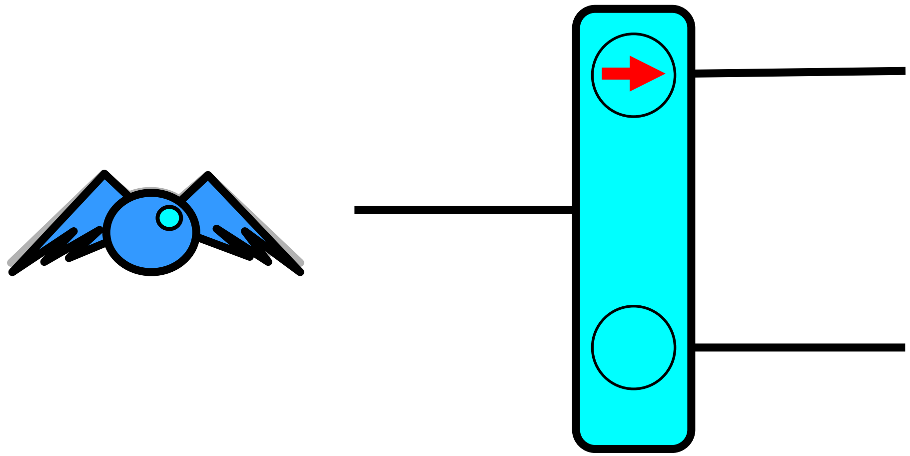


Diffracting Trees

A Simple Balancer

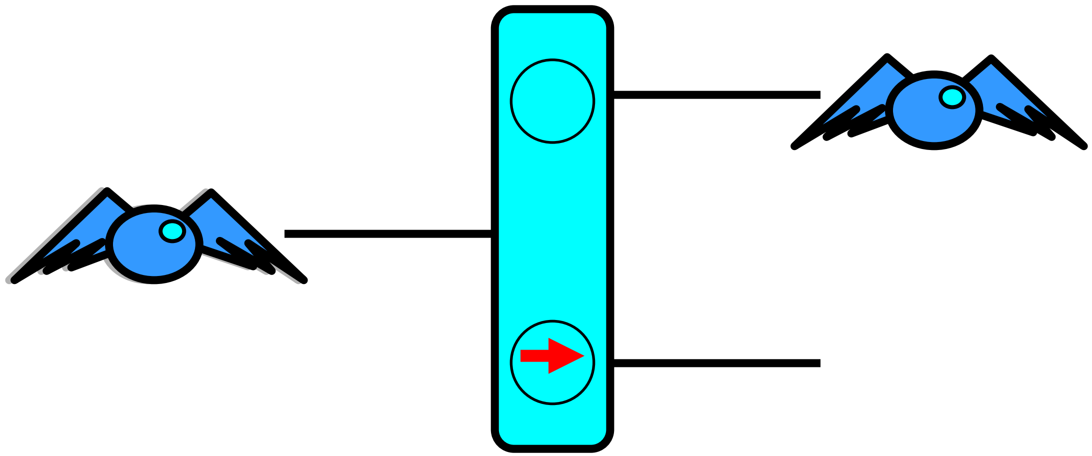


Tokens Traverse Balancers

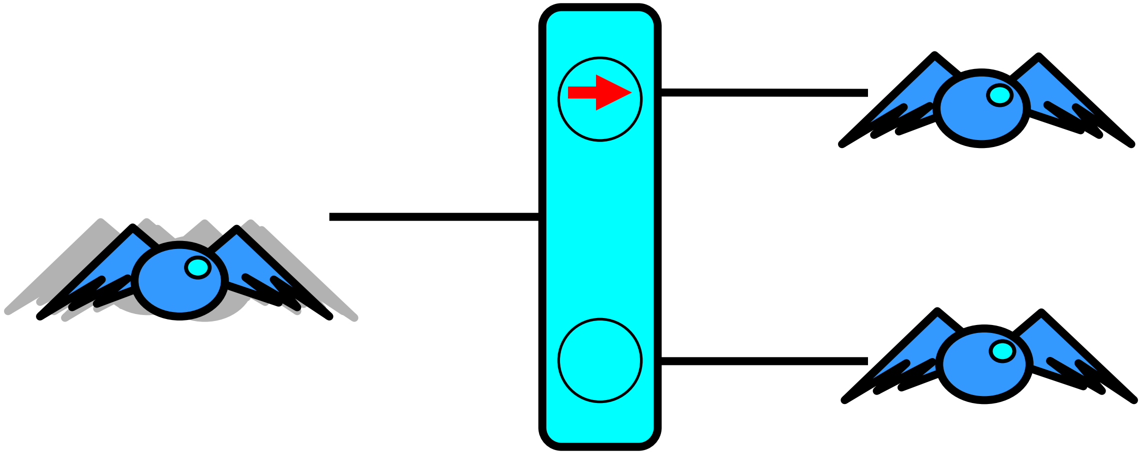


- Token i enters
- leaves on wire i mod (fan-out)

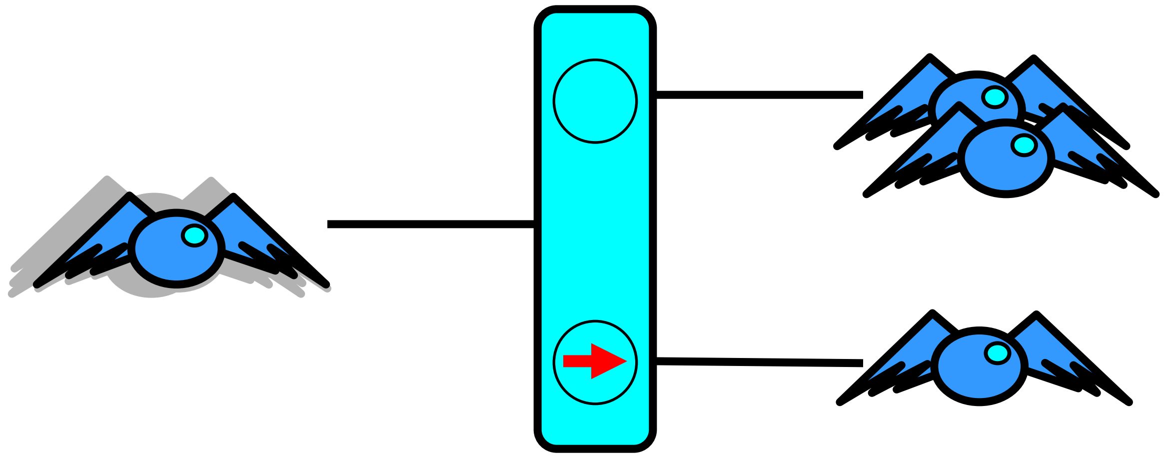
Tokens Traverse Balancers



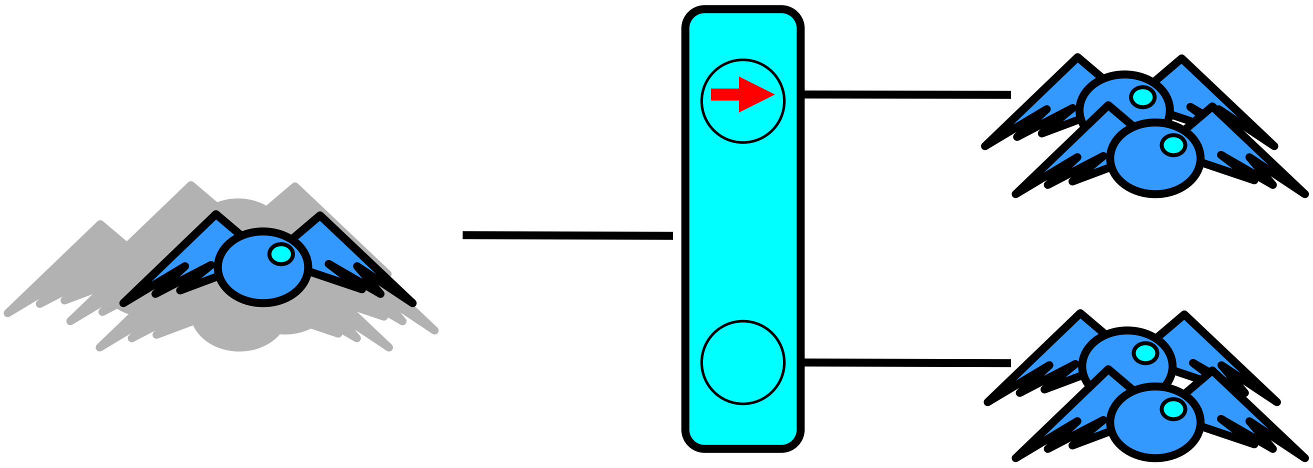
Tokens Traverse Balancers



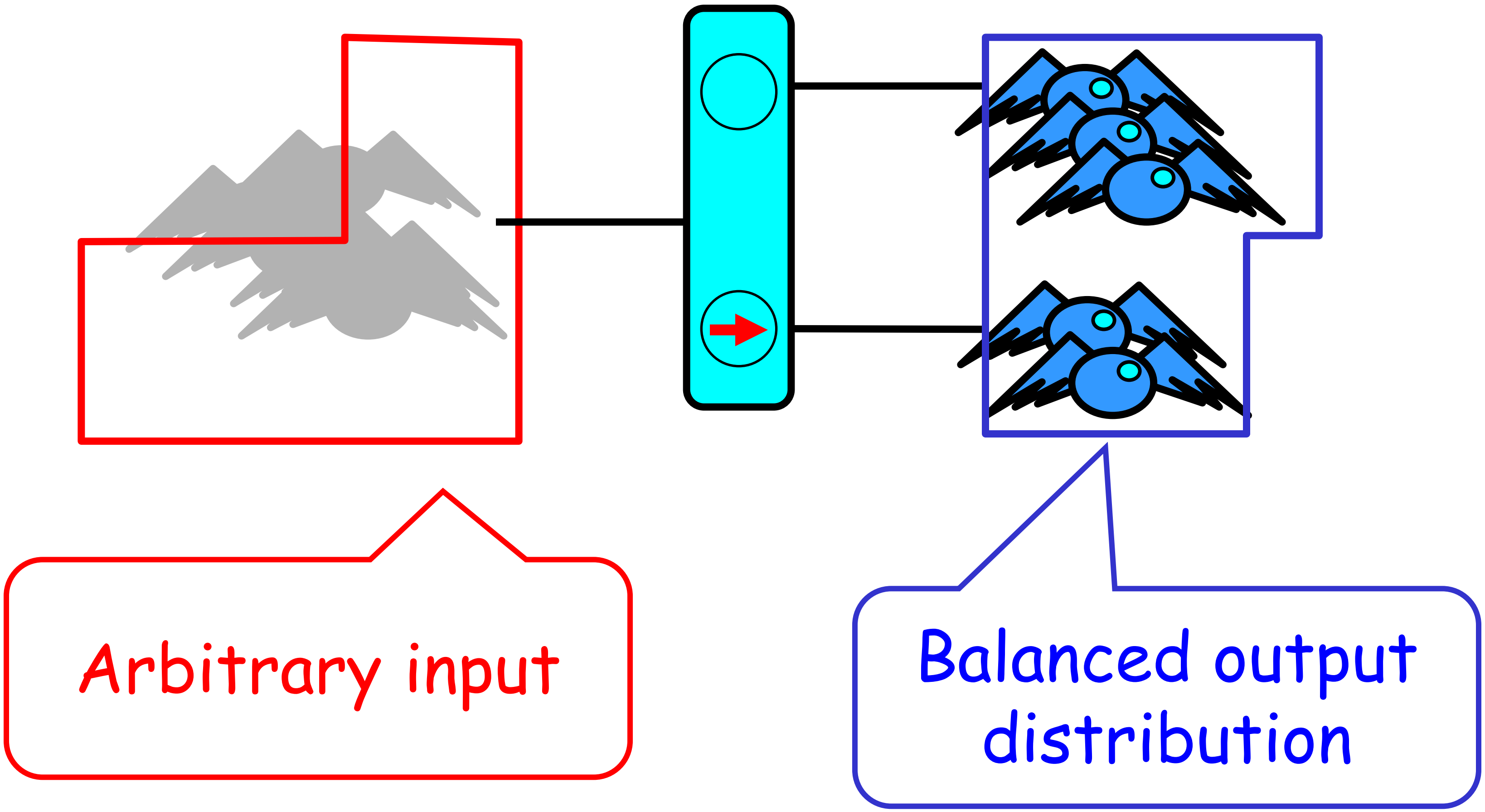
Tokens Traverse Balancers



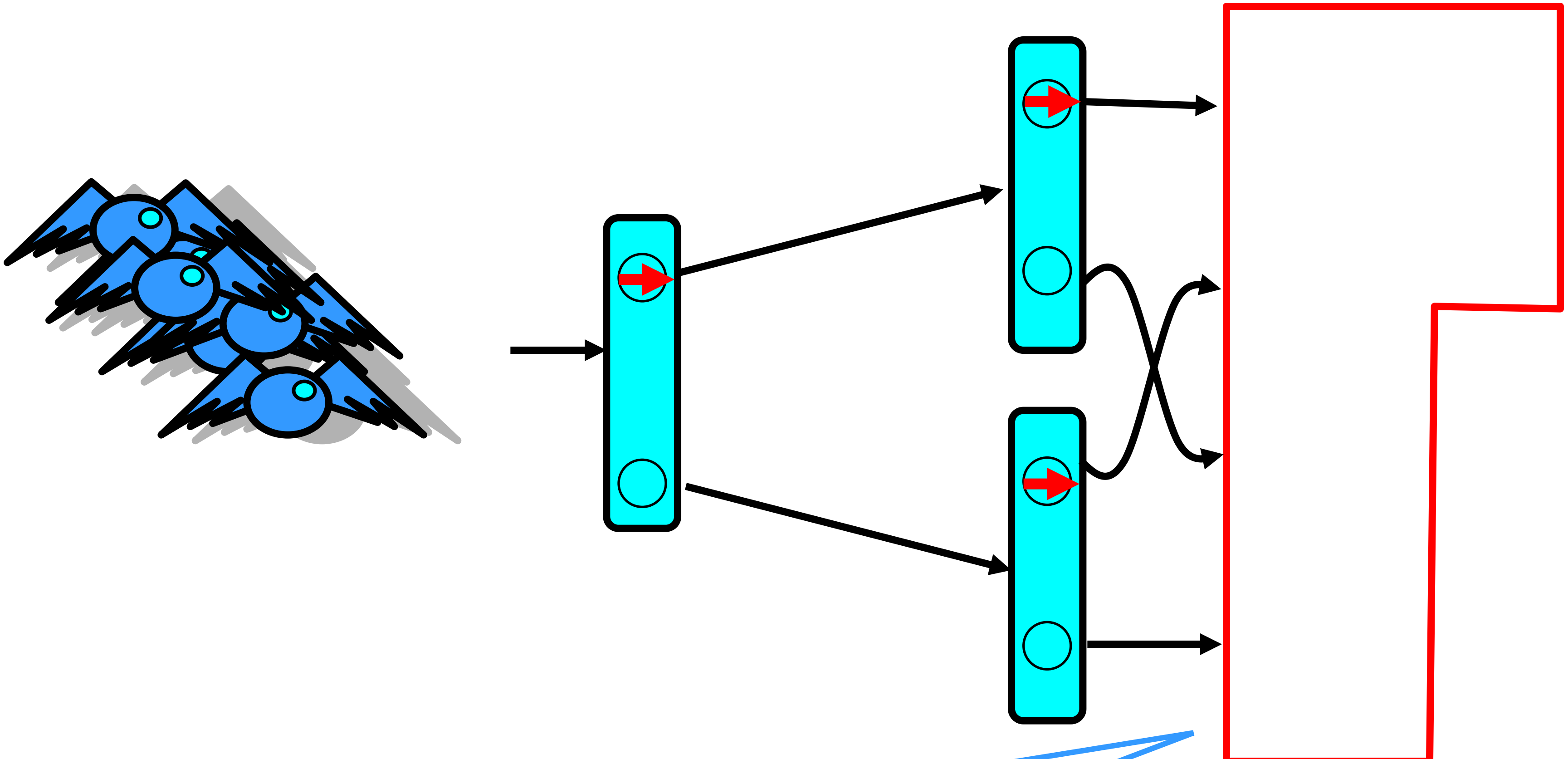
Tokens Traverse Balancers



Tokens Traverse Balancers

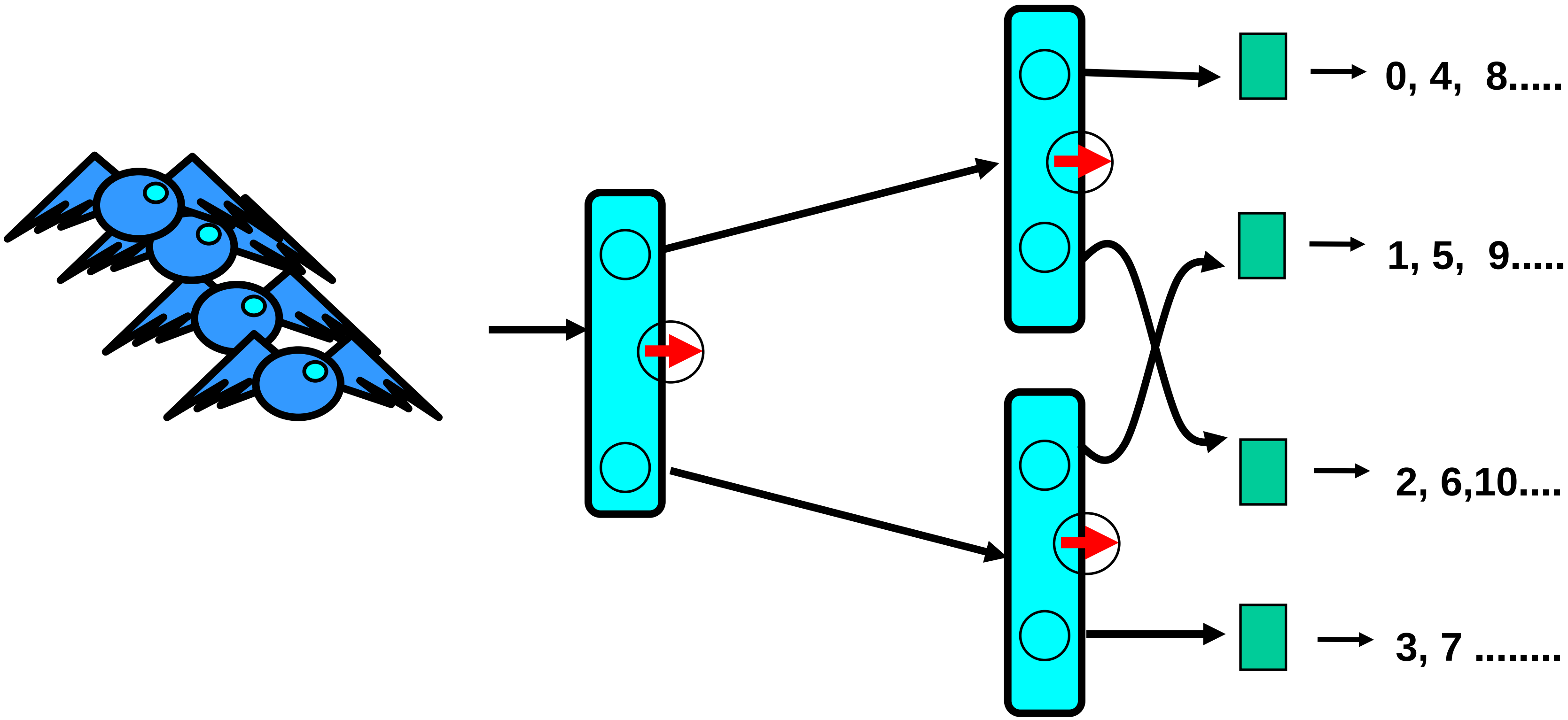


Counting Tree

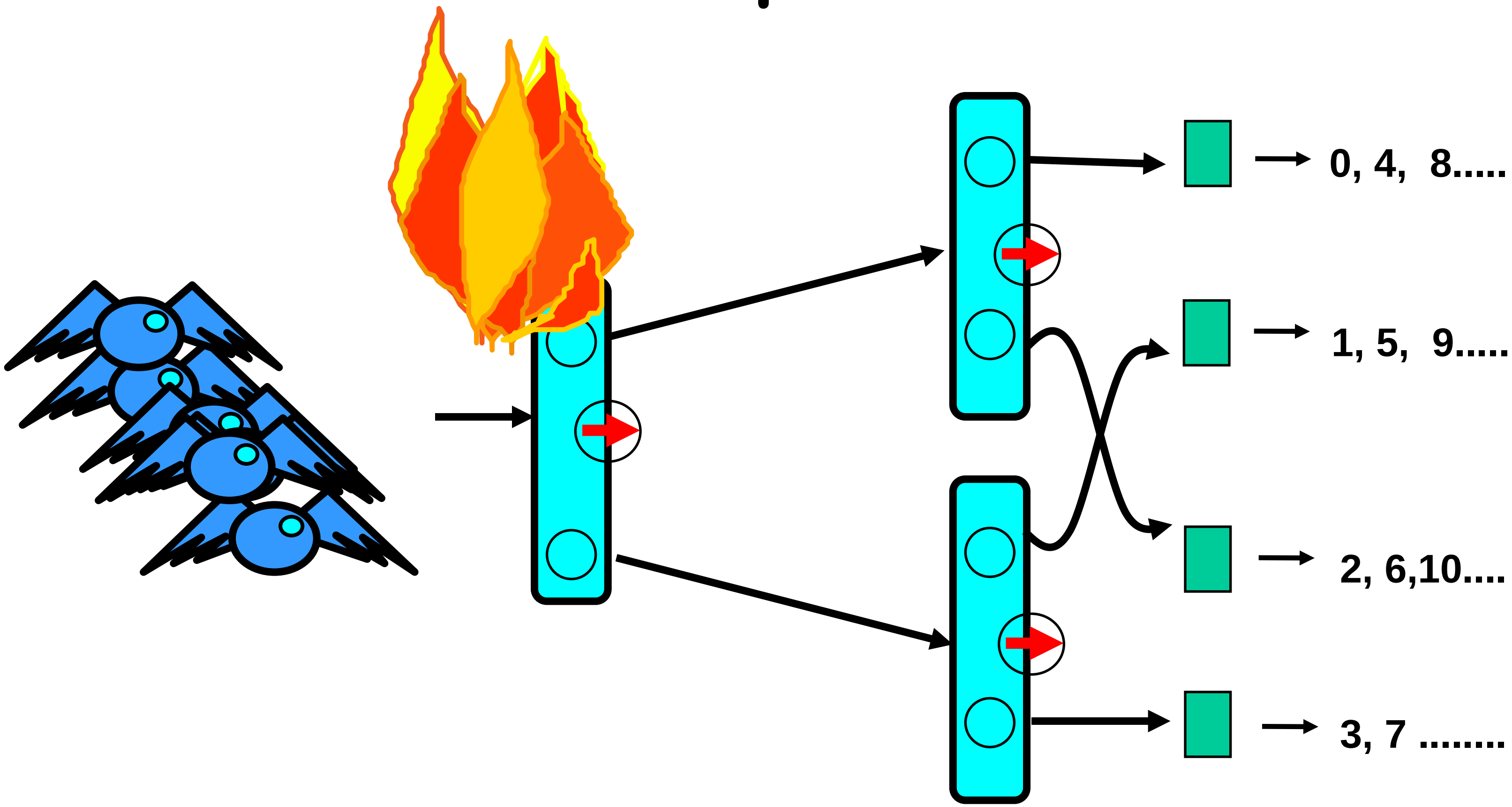


Step property

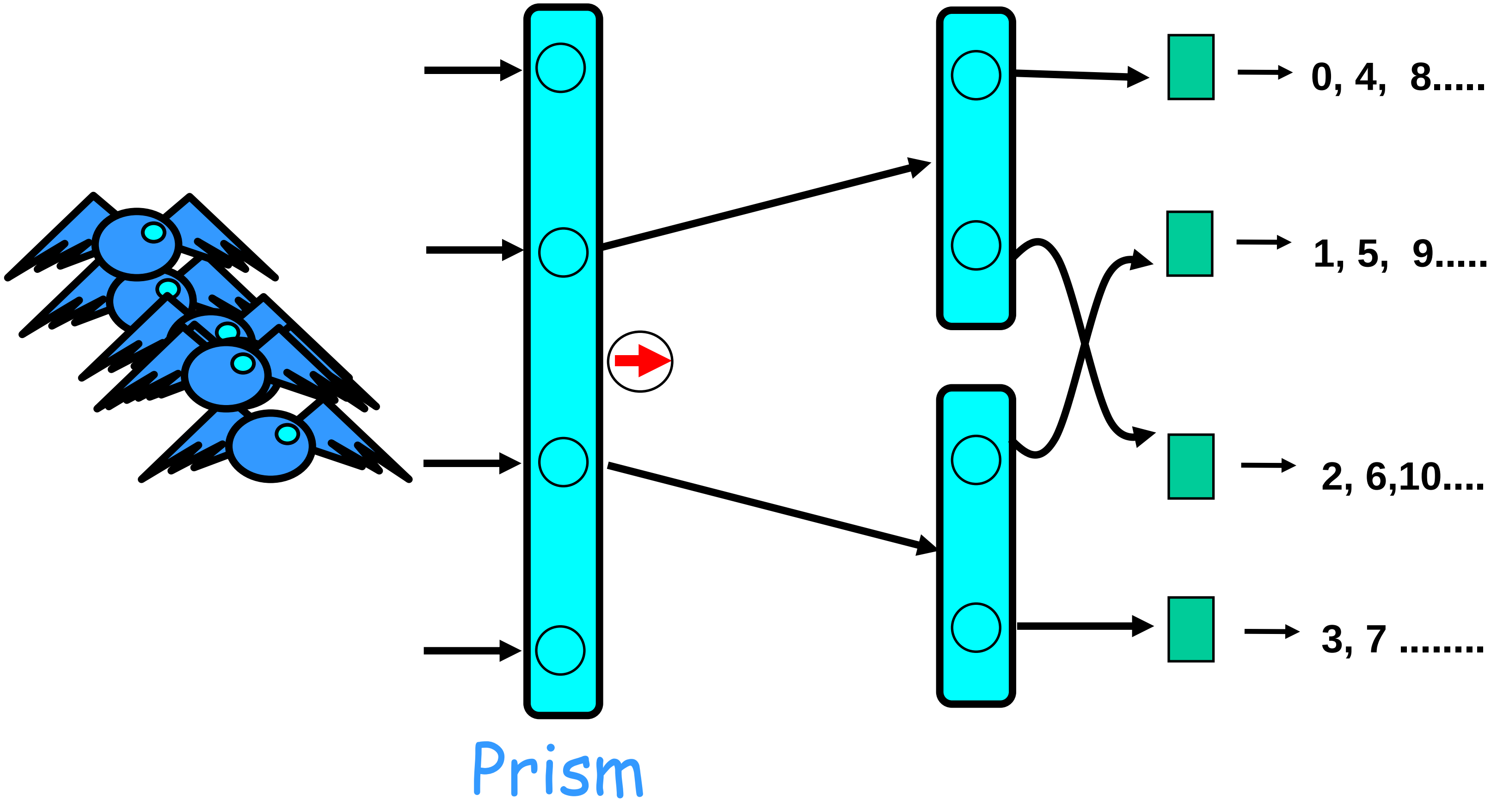
Counting Tree Counts!



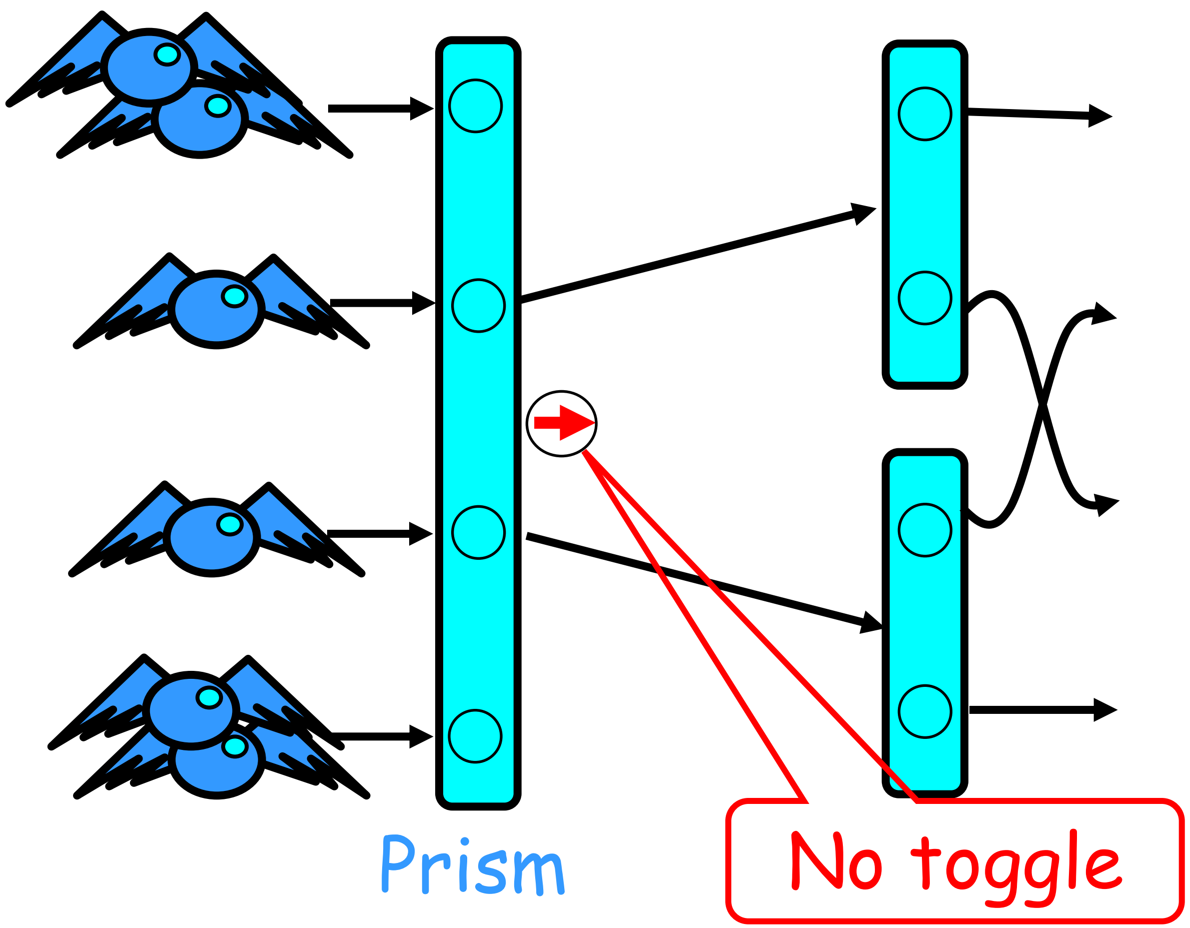
Back to square 1 !



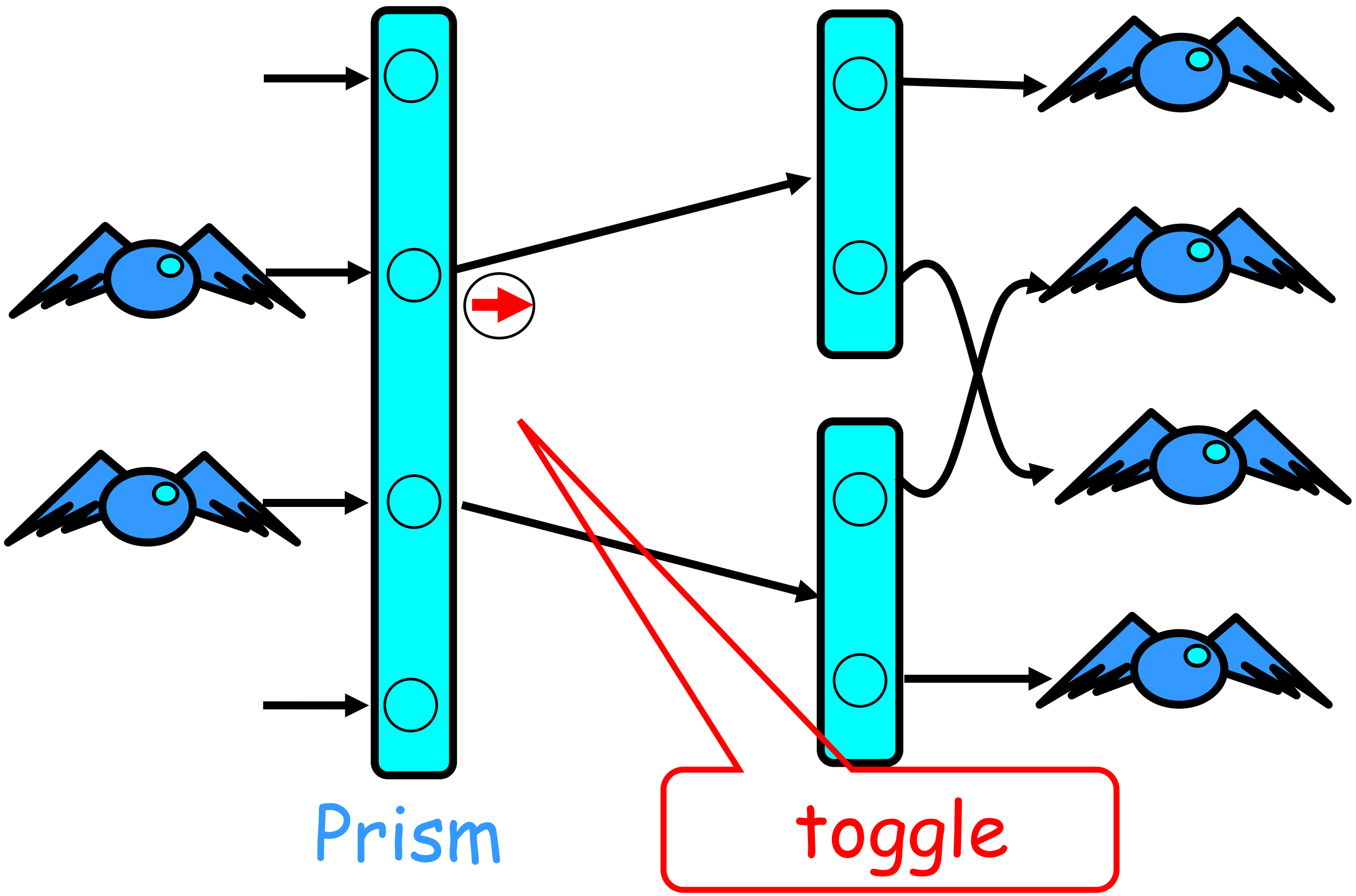
Introducing Diffracting Tree



Diffracting Tree



Diffracting Tree



Diffracting Tree