System Health & Risk Management

# Chapter 5. Stress-Strain Relationships and Behavior

Mechanical Strengths and Behavior of Solids

# Contents

# 5.1 Introduction



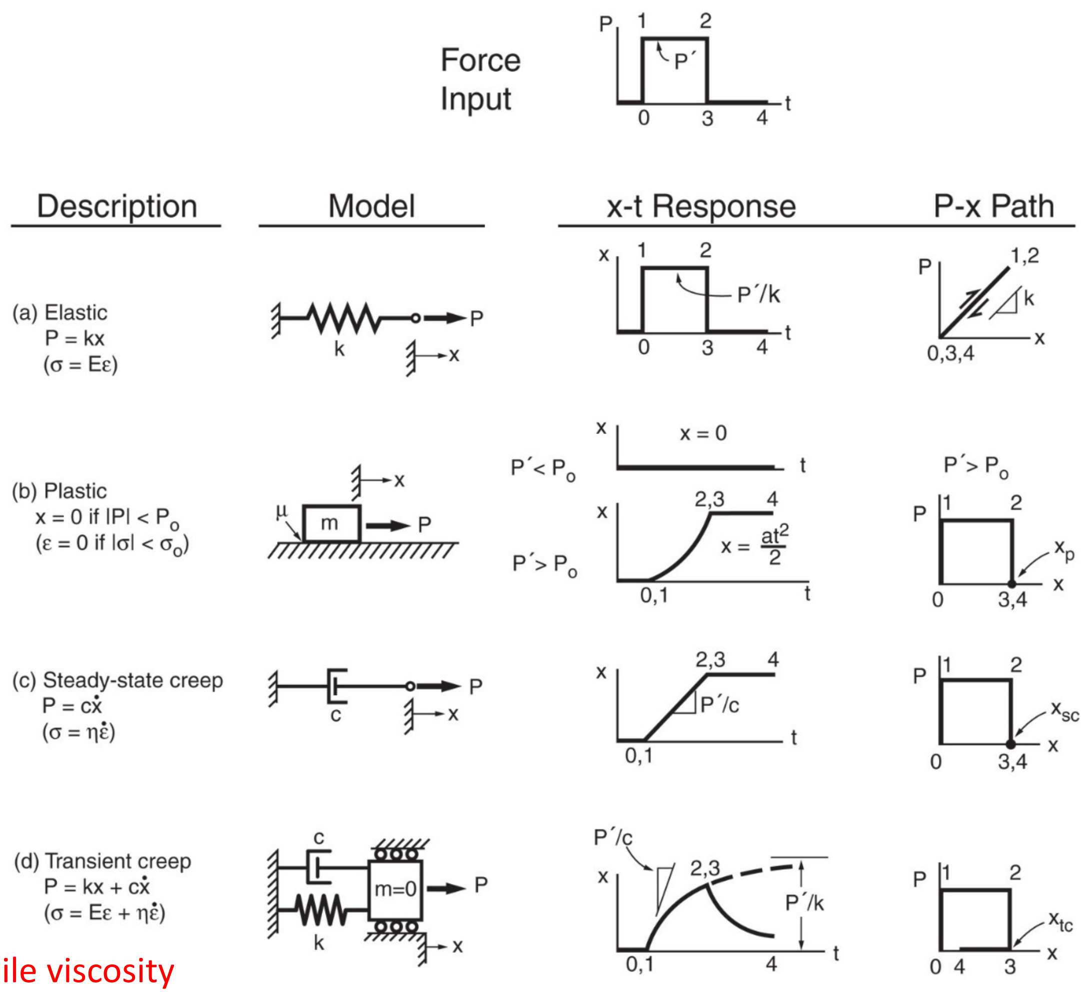
- **Three major types of deformation: Elastic, Plastic, and Creep deformation**

- **In engineering analysis, constitutive equations which describe stress-strain relationships are essential to calculate stresses and deflections in mechanical components.**

- **Consider one-dimensional stress-strain behavior and some corresponding simple physical models for each deformation.**

- **Three-dimensional elastic deformation: *Isotropy* and *Anisotropy***
  - Isotropy: The elastic properties are the same in all directions.
  - Anisotropy: The elastic properties vary with direction.

# 5.2 Models for Deformation Behavior




*η* : coefficient of tensile viscosity
(in analogous to *c*)

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**Figure 5.1** Mechanical models for four types of deformation. The displacement–time and force–displacement responses are also shown for step inputs of force $P$, which is analogous to stress $\sigma$. Displacement $x$ is analogous to strain $\varepsilon$.

# Plastic Deformation Models (1)

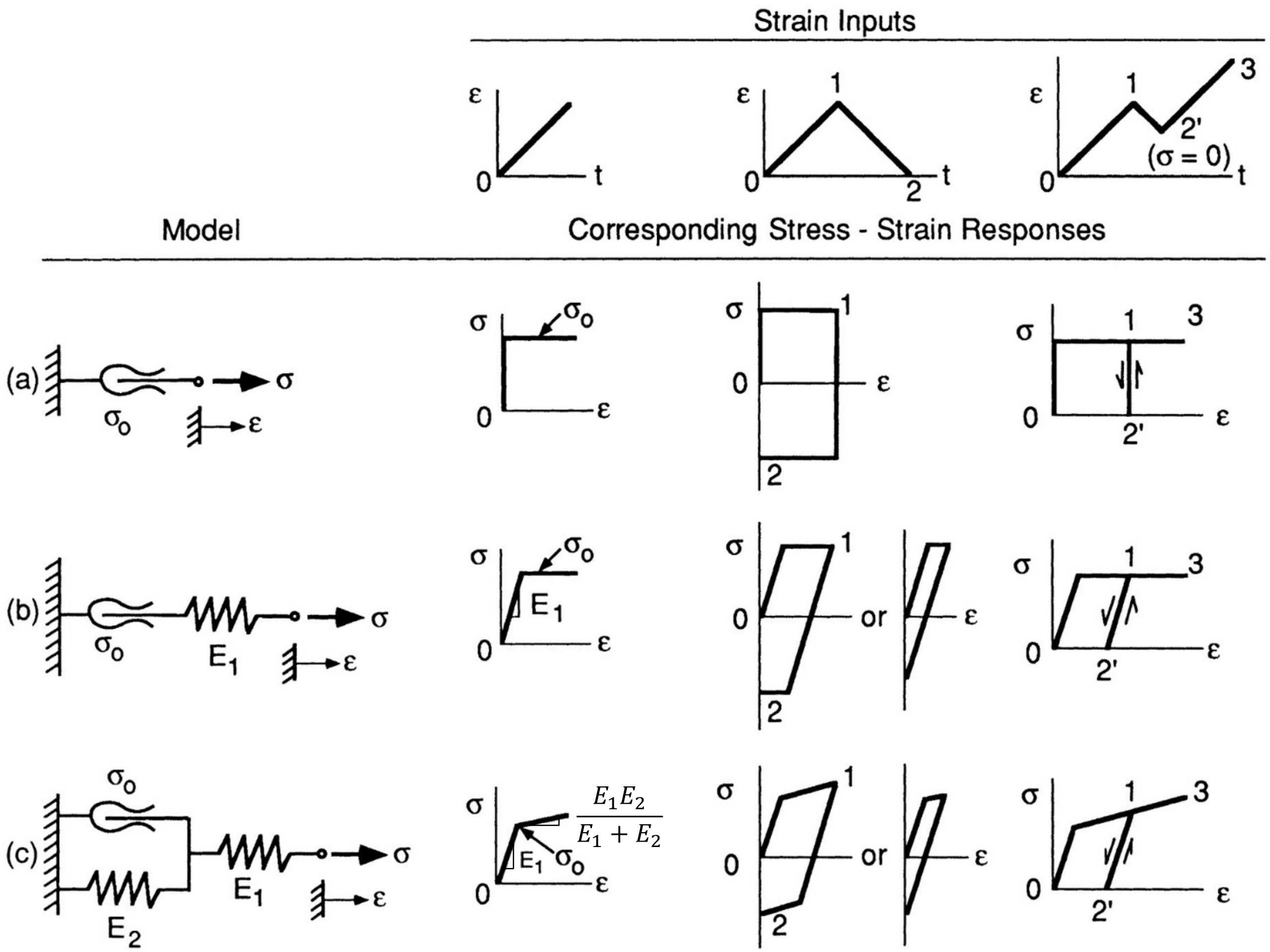


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**Figure 5.3** Rheological models for plastic deformation and their responses to three different strain inputs. Model (a) has behavior that is **rigid, perfectly plastic; (b) elastic, perfectly plastic; and (c) elastic, linear hardening**.

# Plastic Deformation Models (2)

- **The point during unloading where the stress passes through zero**
  - Elastic unloading of the same slope $E_1$ as the initial loading
  - The elastic strain $\varepsilon_e$ is recovered corresponds to the relaxation of spring $E_1$.
  - The permanent or plastic strain $\varepsilon_p$ corresponds to the motion of the slider up to the point of maximum strain.

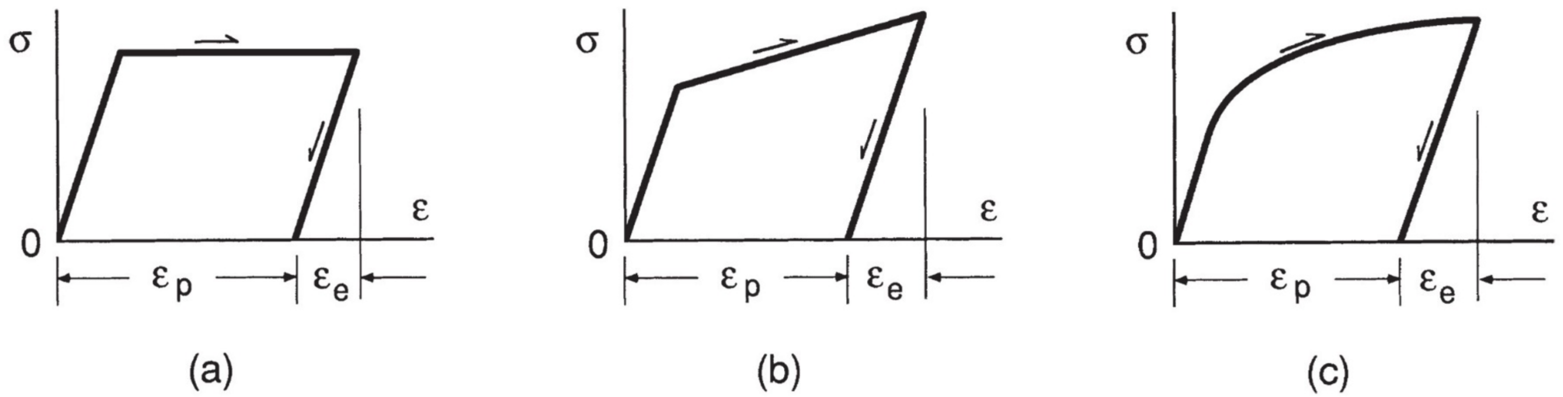


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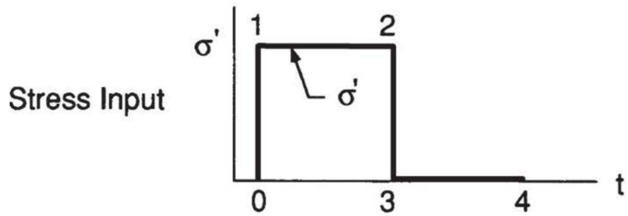
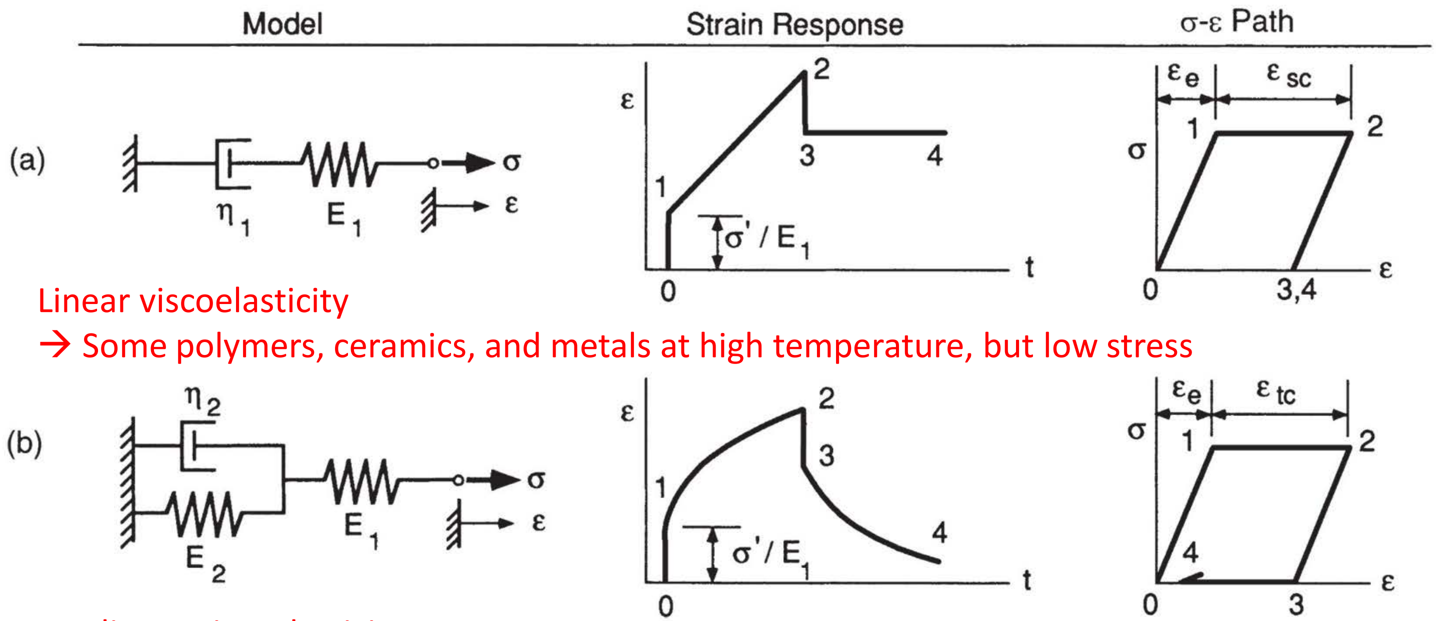
**Figure 5.4** Loading and unloading behavior of (a) an elastic, perfectly plastic model, (b) an elastic, linear-hardening model, and (c) a material with nonlinear hardening.

# Creep Deformation Models (1)

Spring: $\varepsilon_e = \dfrac{\sigma'}{E_1}$

Dashpot: $\dot{\varepsilon_c} = \dfrac{\sigma'}{\eta_1}$




Linear viscoelasticity

→ Some polymers, ceramics, and metals at high temperature, but low stress

Nonlinear viscoelasticity

→ Metals and ceramics at high stress

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**Figure 5.5** Rheological models having time-dependent behavior and their responses to a stress–time step. Both strain–time and stress–strain responses are shown. Model (a) exhibits steady-state creep with elastic strain added, and model (b) transient creep with elastic strain added.

# Creep Deformation Models (2)

- **For the steady-state creep model of Fig. 5.5(a),**

$$\varepsilon = \varepsilon_e + \varepsilon_c = \frac{\sigma'}{E_1} + \varepsilon_c$$

The rate of creep strain: $\dot{\varepsilon_c} = \frac{d\varepsilon_c}{dt} = \frac{\sigma'}{\eta_1}$

Solving $\varepsilon_c$ by integration, $\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma' t}{\eta_1}$

→ After removal of the stress, elastic strain disappears, but the creep strain accumulated during 1-2 remains as a permanent strain.

- **For the transient creep model of Fig. 5.5(b),**

The stress in the $(\eta_2, E_2)$ stage:

$$\sigma = E_2\varepsilon_2 + \eta_2\dot{\varepsilon_c}$$

This gives

$$\dot{\varepsilon_c} = \frac{d\varepsilon_c}{dt} = \frac{\sigma - E_2\varepsilon_c}{\eta_2}$$

Solving this differential equation,

$$\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma'}{E_2}\left(1 - e^{-\frac{E_2 t}{\eta_2}}\right)$$

→ Strain rate decreases with time and creep strain asymptotically approaches the limit $\sigma'/E_2$.

→ After stress removal, the transient creep strain decreases toward zero at infinite time due to the spring.

→ The equation for the recovery response can be obtained by solving ODE.

# Relaxation Behavior (1) (optional)

- **Creep:** Accumulation of strain with time, as under constant stress
- **Recovery:** Gradual disappearance of creep strain that occurs after removal of the stress
- **Relaxation:** Decrease in stress when a material is held at constant strain

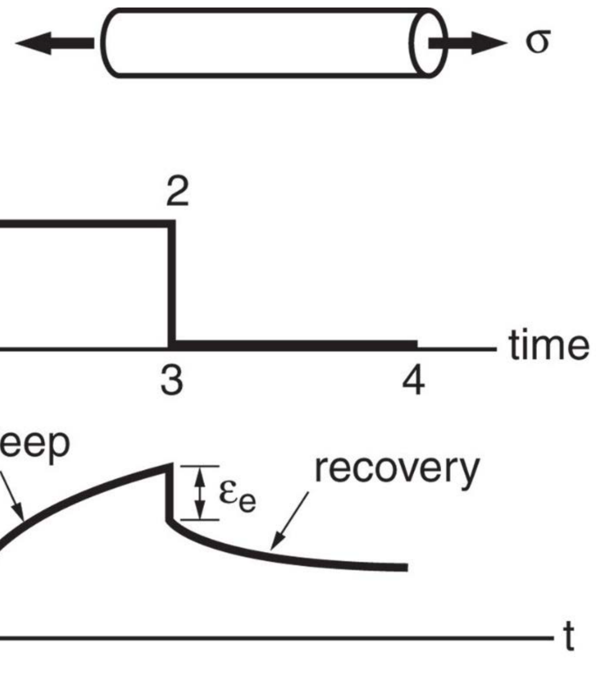


**Figure 5.7** Stress–time step applied to a material exhibiting strain response that includes elastic, plastic, and creep components.

# Relaxation Behavior (1) (optional)

- **Creep:** Accumulation of strain with time, as under constant stress
- **Recovery:** Gradual disappearance of creep strain that occurs after removal of the stress
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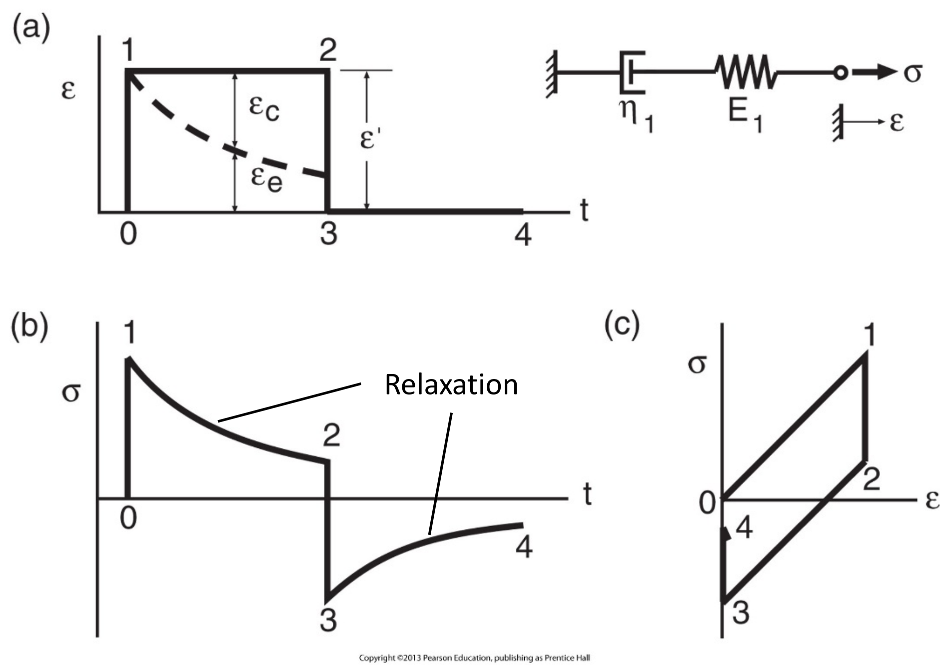


**Figure 5.6** Relaxation under constant strain for a model with steady-state creep and elastic behavior. The step in strain (a) causes stress–time behavior as in (b), and stress–strain behavior as in (c).

# Relaxation Behavior (2) (optional)

- About sudden strain $\varepsilon'$, it is absorbed by the spring entirely because the dashpot requires a finite time to respond.
- Due to the total strain being held constant, the strain in the spring decreases, as the strain in the dashpot increases.

$$\varepsilon' = \varepsilon_e + \varepsilon_c = const.$$

The stress necessary to maintain the constant strain:

$$\sigma = E_1 \varepsilon_e$$

The rate of creep strain:

$$\dot{\varepsilon_c} = \frac{d\varepsilon_c}{dt} = \frac{\sigma}{\eta_1}$$

Combining these equations and solving the differential equation:

$$\sigma = E_1 \varepsilon' e^{-\frac{E_1 t}{\eta_1}}$$

- If the strain is returned to zero, the stress is forced into compression. Additional relaxation the occurs, but in the opposite direction, as the relaxation always proceeds toward zero stress.

# 5.3 Elastic Deformation

**Table 5.2** Elastic Constants for Various Materials at Ambient Temperature

| Material | Elastic Modulus $E$, GPa ($10^3$ ksi) | | Poisson's Ratio $\nu$ |
|---|---|---|---|
| *(a) Metals* | | | |
| Aluminum | 70.3 | (10.2) | 0.345 |
| Brass, 70Cu-30Zn | 101 | (14.6) | 0.350 |
| Copper | 130 | (18.8) | 0.343 |
| Iron; mild steel | 212 | (30.7) | 0.293 |
| Lead | 16.1 | (2.34) | 0.44 |
| Magnesium | 44.7 | (6.48) | 0.291 |
| Stainless steel, 2Ni-18Cr | 215 | (31.2) | 0.283 |
| Titanium | 120 | (17.4) | 0.361 |
| Tungsten | 411 | (59.6) | 0.280 |
| *(b) Polymers* | | | |
| ABS, medium impact | 2.4 | (0.35) | 0.35 |
| Acrylic, PMMA | 2.7 | (0.40) | 0.35 |
| Epoxy | 3.5 | (0.51) | 0.33 |
| Nylon 66, dry | 2.7 | (0.39) | 0.41 |
| Nylon 66, 33% glass fibers | 9.5 | (1.38) | 0.39 |
| Polycarbonate | 2.4 | (0.345) | 0.38 |
| Polyethylene, HDPE | 1.08 | (0.157) | 0.42 |
| *(c) Ceramics and glasses* | | | |
| Alumina, $Al_2O_3$ | 400 | (58.0) | 0.22 |
| Diamond | 960 | (139) | 0.20 |
| Magnesia, MgO | 300 | (43.5) | 0.18 |
| Silicon carbide, SiC | 396 | (57.4) | 0.22 |
| Fused silica glass | 70 | (10.2) | 0.18 |
| Soda-lime glass | 69 | (10.0) | 0.20 |
| Type E glass | 72.4 | (10.5) | 0.22 |
| Dolomitic limestone | 69.0 | (10.0) | 0.281 |
| Westerly granite | 49.6 | (7.20) | 0.213 |

Sources: Data in [Boyer 85] p. 216, [Creyke 82] p. 222, [Kaplan 95] pp. B-146 to B-206, [Karfakis 90], [Kelly 86] pp. 376, 392, [Kelly 94] p. 285, [Morrell 85] Pt. 1, p. 96, [PDL 91] Vol. I-B, pp. 133–136, and [Schwartz 92] p. 2.75.

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# Elastic Constants (1)

- ***Homogeneous*: ** A material that has the same properties at all points within the solid
- ***Isotropic*: ** A material whose properties are the same in all directions

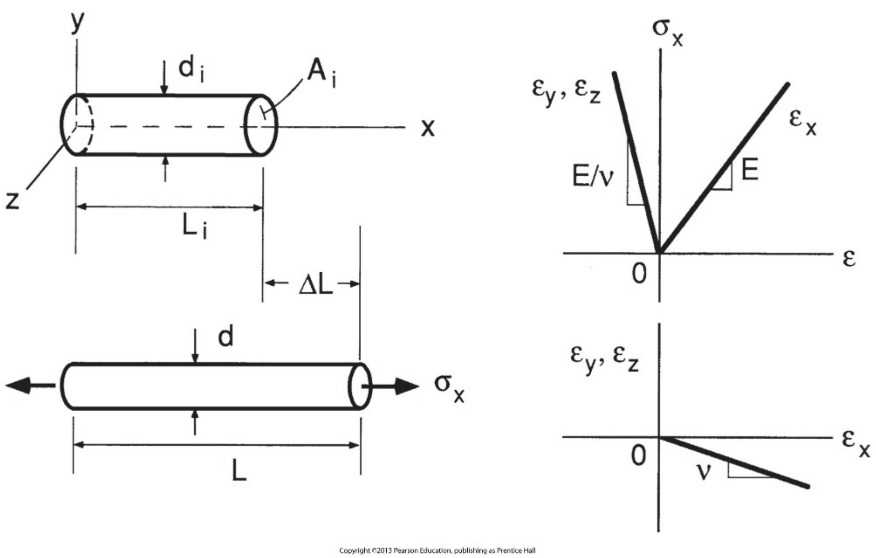


**Figure 5.8** Longitudinal extension and lateral contraction used to obtain constants for a linear-elastic material that is isotropic and homogeneous.

- Elastic modulus

$$E = \frac{\sigma_x}{\varepsilon_x}$$

- Poisson's ratio

$$\nu = -\frac{transverse\ strain}{longitudinal\ strain}$$

$$= -\frac{\varepsilon_y}{\varepsilon_x}$$

- From elastic modulus and Poisson's ratio

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x$$

# Elastic Constants (2)

- Poisson's ratio is often around 0.3, $0 < \nu < 0.5$.
- Negative values of $\nu$ imply lateral expansion during axial tension.
- $\nu = 0.5$ implies constant volume, and values lager than 0.5 imply a decrease in volume for tensile loading.
- ***Small-strain theory*:** When dimensional changes are small, original dimensions and cross-sectional areas are used to determine stresses and strains.
- If a given metal is alloyed with small percentages of other metals, the elastic constants $E$ and $\nu$ can be approximated as being the same as the corresponding pure metal values.
  - Aluminum and titanium alloys: less than 10%
  - Low-alloy steels: less than 5%

# Hooke's Law for Three Dimensions (1)

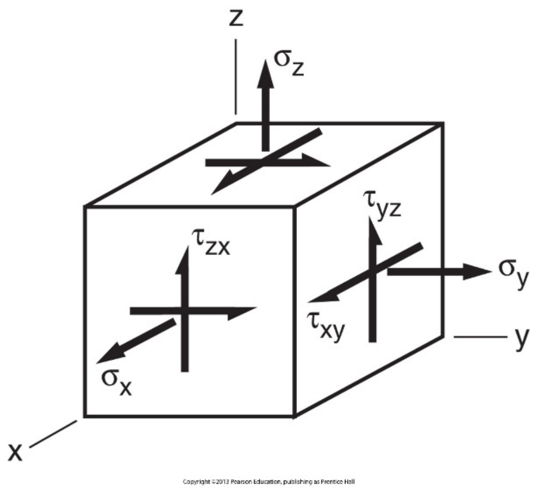


**Figure 5.9** The six components needed to completely describe the state of stress at a point.

- **Three normal stresses:**
  $\sigma_x, \sigma_y$, and $\sigma_z$
- **Three shear stresses:**
  $\tau_{xy}, \tau_{yz}$, and $\tau_{zx}$

The strains caused by stresses in each direction:

| | Resulting Strain Each Direction | | |
|---|---|---|---|
| Stress | X | Y | Z |
| $\sigma_x$ | $\frac{\sigma_x}{E}$ | $-\frac{\nu\sigma_x}{E}$ | $-\frac{\nu\sigma_x}{E}$ |
| $\sigma_y$ | $-\frac{\nu\sigma_y}{E}$ | $\frac{\sigma_y}{E}$ | $-\frac{\nu\sigma_y}{E}$ |
| $\sigma_z$ | $-\frac{\nu\sigma_z}{E}$ | $-\frac{\nu\sigma_z}{E}$ | $\frac{\sigma_z}{E}$ |

Total strain in each direction:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

# Hooke's Law for Three Dimensions (2)

- **Shear strains** ($G$: *shear modulus*):

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}, \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

- The shear strain on a given plane is unaffected by the shear stresses on other planes.

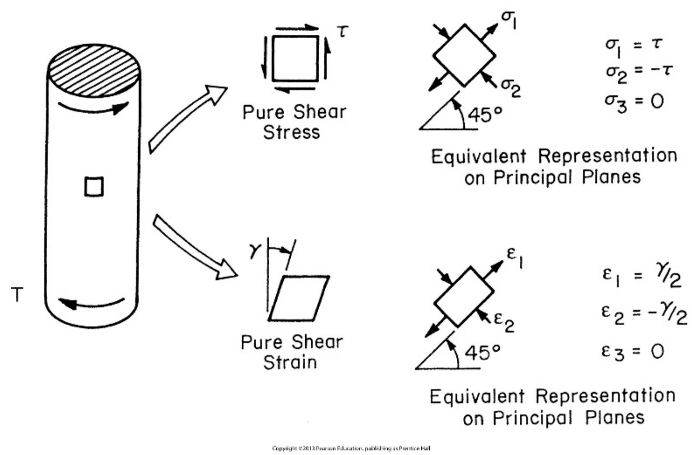


**Figure 4.41** A round bar in torsion and the resulting state of pure shear stress and strain. The equivalent normal stresses and strains for a 45° rotation of the coordinate axes are also shown.

Consider a state of pure shear stress, as in a round bar under torsion in Fig. 4.41.

$$\sigma_x = \tau, \quad \sigma_y = -\tau, \quad \sigma_z = 0, \quad \varepsilon_x = \frac{\gamma}{2}$$

This yields

$$\gamma = \frac{2(1+\nu)}{E}\tau$$

From $G = \tau/\gamma$,

$$G = \frac{E}{2(1+\nu)}$$

# Volumetric Strain

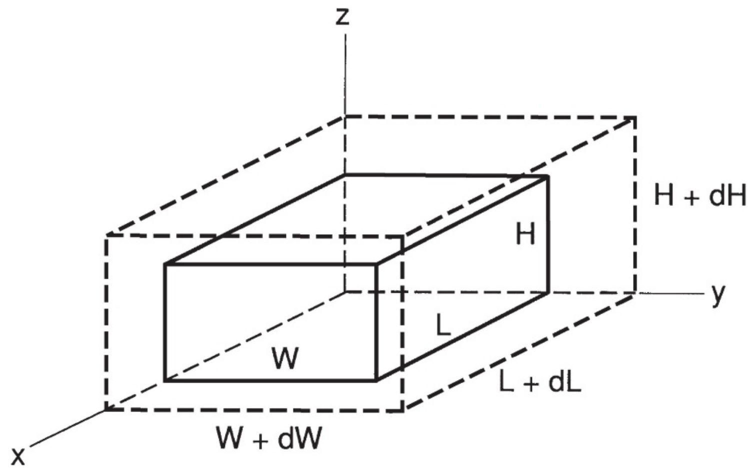


**Figure 5.10** Volume change due to normal strains.

The normal strains:

$$\varepsilon_x = \frac{dL}{L}, \qquad \varepsilon_y = \frac{dW}{W}, \qquad \varepsilon_z = \frac{dH}{H}$$

From $V = LWH$,

$$dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial W} dW + \frac{\partial V}{\partial H} dH$$

Evaluating the partial derivatives and dividing both sides by $V = LWH$,

$$\frac{dV}{V} = \frac{dL}{L} + \frac{dW}{W} + \frac{dH}{H}$$

*Volumetric strain* or *dilatation*, $\varepsilon_v$: Ratio of the change in volume

$$\varepsilon_v = \frac{dV}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

For an isotropic material (by Hooke's law),

$$\varepsilon_v = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

# Hydrostatic Stress

- ***Hydrostatic stress***: The average normal stress

$$\sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

Substituting this into volumetric strain for an isotropic material,

$$\varepsilon_v = \frac{3(1-2\nu)}{E}\sigma_h$$

- ***Bulk modulus***: The constant of proportionality between volumetric strain and hydrostatic stress

$$B = \frac{\sigma_h}{\varepsilon_v} = \frac{E}{3(1-2\nu)}$$

- $\varepsilon_v$ and $\sigma_h$ are invariant quantities which always have the same values, regardless of the choice of coordinate system.
- In other words, the sum of the normal strains and the sum of the normal stresses will have the same value for any coordinate system.

# Thermal Strains

- ***Thermal Strain*: Elastic strain caused by the temperature changes**

$$\varepsilon = \alpha(T - T_0) = \alpha(\Delta T)$$

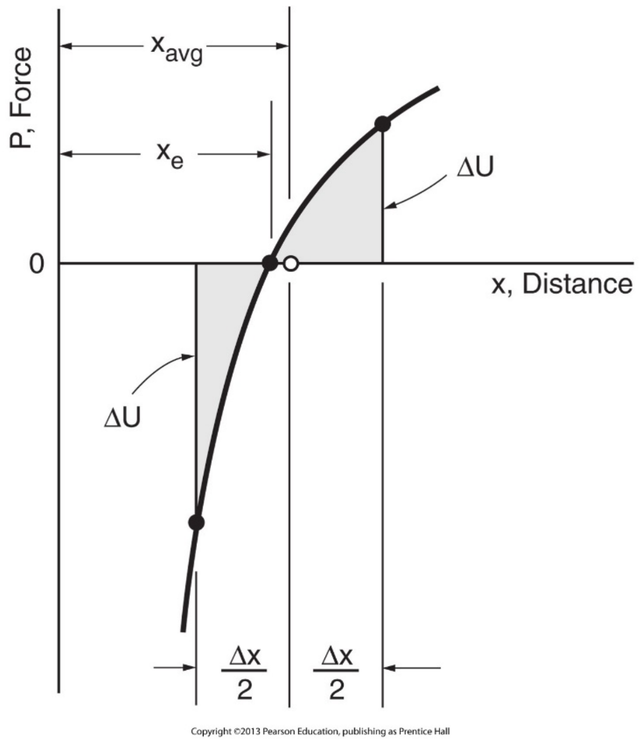


**Figure 5.11** Force vs. distance between atoms. A thermal oscillation of equal potential energies about the equilibrium position $x_e$ gives an average distance $x_{avg}$ greater than $x_e$.

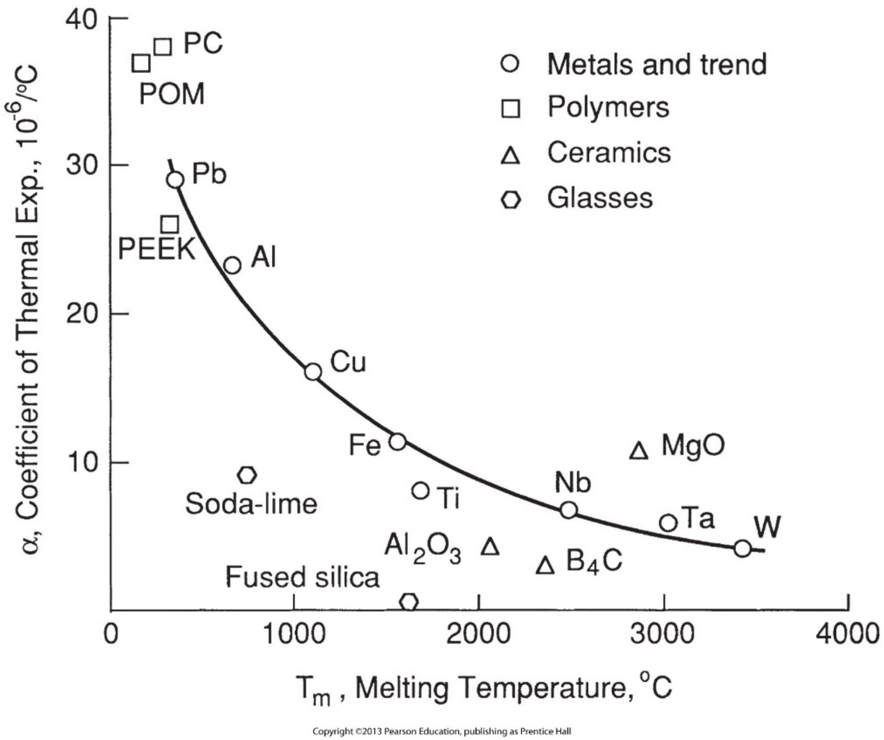


**Figure 5.12** Coefficients of thermal expansion at room temperature versus melting temperature for various materials. (Data from [Boyer 85] p. 1.44, [Creyke 82] p. 50, and [ASM 88] p. 69.)

# 5.4 Anisotropic Materials

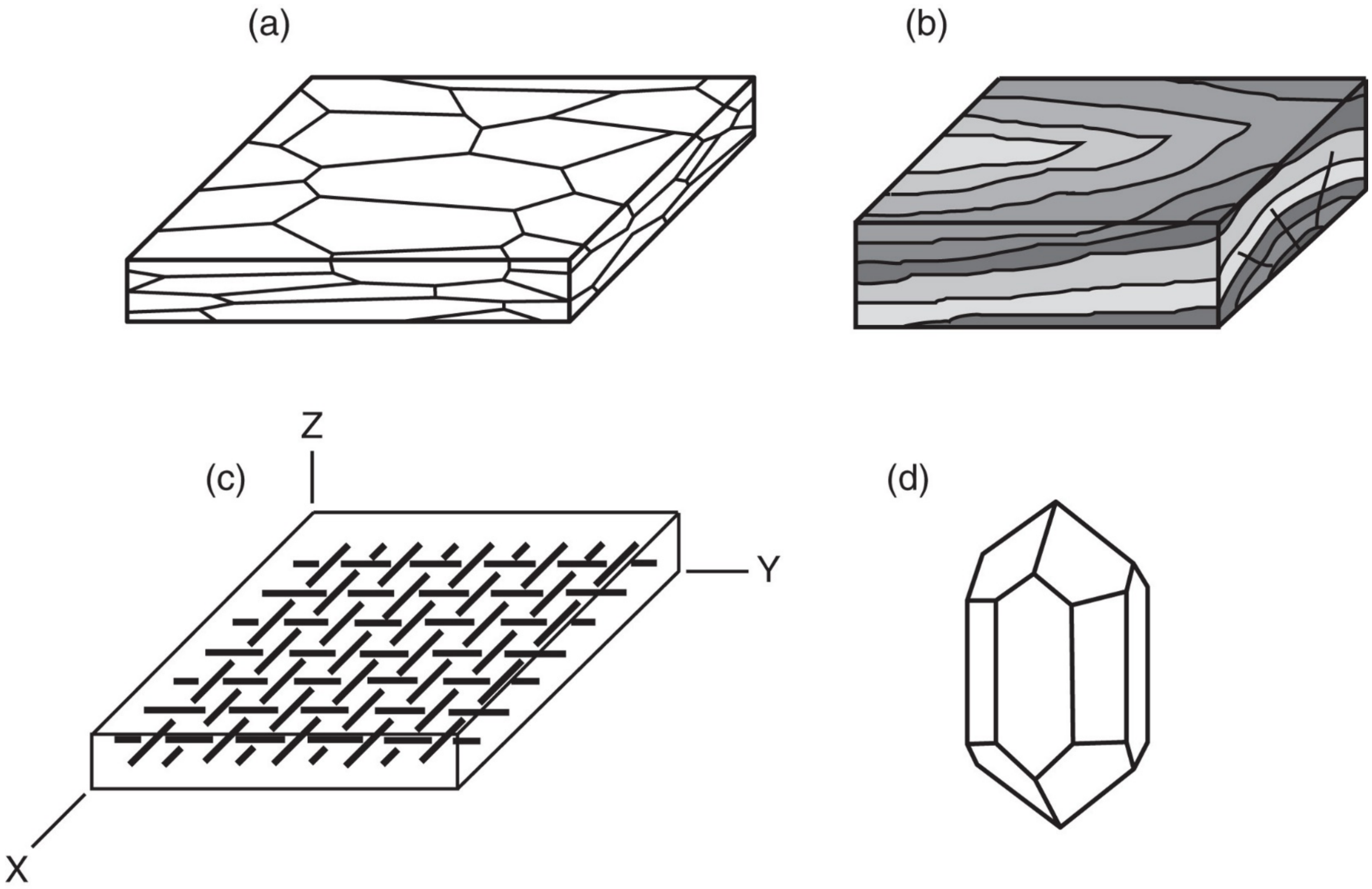


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**Figure 5.14** Anisotropic materials: (a) metal plate with oriented grain structure due to rolling, (b) wood, (c) glass-fiber cloth in an epoxy matrix, and (d) a single crystal.

# Anisotropic Hooke's Law

- The general anisotropic form of Hooke's law:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & \cdots & S_{16} \\ \vdots & \ddots & \vdots \\ S_{16} & \cdots & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- $S_{ij}$ change if the orientation of the x-y-z coordinate system is changed.
- Each unique $S_{ij}$ has a different nonzero value.
- The matrix is symmetrical with 21 independent variables.
- In the isotropic case,

$$[S_{ij}] = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & 0 & 0 \\ & 0 & & 0 & 1/G & 0 \\ & & & 0 & 0 & 1/G \end{bmatrix}$$

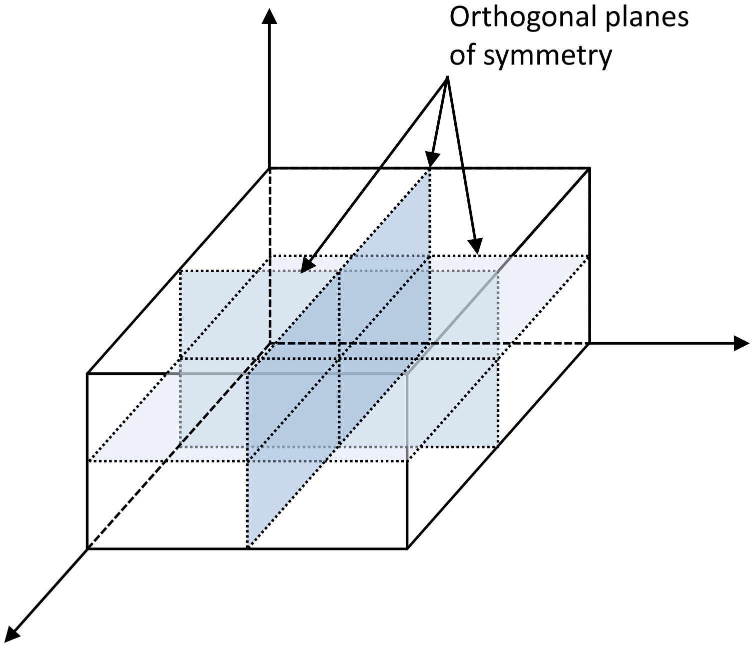
# Orthotropic Materials; Other Special Cases

- ***Orthotropic material***: The material that possesses symmetry about three orthogonal planes
  - The coefficients for Hooke's law for an orthotropic material:

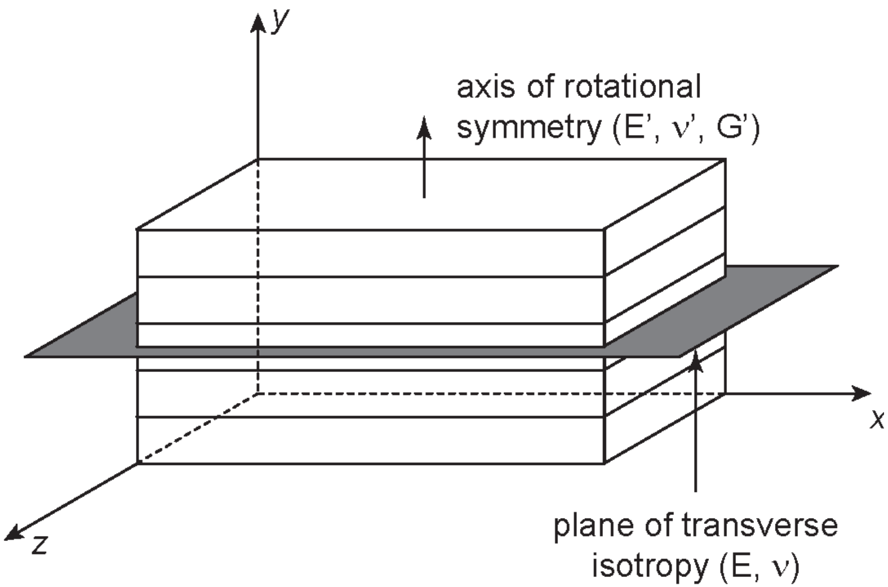
$$[S_{ij}] = \begin{bmatrix} 1/E_X & -\nu_{YX}/E_Y & -\nu_{ZX}/E_Z & & & \\ -\nu_{XY}/E_X & 1/E_Y & -\nu_{ZY}/E_Z & & 0 & \\ -\nu_{XZ}/E_X & -\nu_{YZ}/E_Y & 1/E_Z & & & \\ & & & 1/G_{YZ} & 0 & 0 \\ & 0 & & 0 & 1/G_{ZX} & 0 \\ & & & 0 & 0 & 1/G_{XY} \end{bmatrix}$$

- ***Cubic material***: The material that has the same properties in the X-, Y-, and Z-directions
  - Three independent constants: $E_X, G_{XY}, \nu_{XY}$
  - There is still one more independent constant than for the isotropic case, and the elastic constants still apply only for the special X-Y-Z coordination system.
- ***Transversely isotropic material***: The properties are the same all directions in a plane, such as the X-Y plane
  - Five independent constants: $E_X, \nu_{XY}, E_Z, \nu_{XZ}, G_{XZ}$

# Orthotropic Materials




*Orthotropic material*




*Transversely isotropic material*

# Fibrous Composites

- For thin plates or sheets, we assume that $\sigma_Z = \tau_{YZ} = \tau_{ZX} = 0$.
- Therefore, Hooke's law can be used in following form:

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{bmatrix} 1/E_X & -\nu_{YX}/E_Y & 0 \\ -\nu_{XY}/E_X & 1/E_Y & 0 \\ 0 & 0 & 1/G_{XY} \end{bmatrix} \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix}$$

**Table 5.3** Elastic Constants and Density for Fiber-Reinforced Epoxy with 60% Unidirectional Fibers by Volume

| (a) Reinforcement | | | (b) Composite, $V_r = 0.60$ | | | | |
|---|---|---|---|---|---|---|---|
| Type | $E_r$ GPa ($10^3$ ksi) | $\nu_r$ | $E_X$ GPa ($10^3$ ksi) | $E_Y$ GPa ($10^3$ ksi) | $G_{XY}$ GPa ($10^3$ ksi) | $\nu_{XY}$ | $\rho$ g/cm$^3$ |
| E-glass | 72.3 (10.5) | 0.22 | 45 (6.5) | 12 (1.7) | 4.4 (0.64) | 0.25 | 1.94 |
| Kevlar 49 | 124 (18.0) | 0.35 | 76 (11.0) | 5.5 (0.8) | 2.1 (0.3) | 0.34 | 1.30 |
| Graphite (T-300) | 218 (31.6) | 0.20 | 132 (19.2) | 10.3 (1.5) | 6.5 (0.95) | 0.25 | 1.47 |
| Graphite (GY-70) | 531 (77.0) | 0.20 | 320 (46.4) | 5.5 (0.8) | 4.1 (0.6) | 0.25 | 1.61 |

Note: For approximate matrix properties, use $E_m = 3.5$ GPa (510 ksi) and $\nu_m = 0.33$.
Sources: Data in [ASM 87] pp. 175–178, and [Kelly 94] p. 285.

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# Elastic Modulus Parallel to Fibers (1)

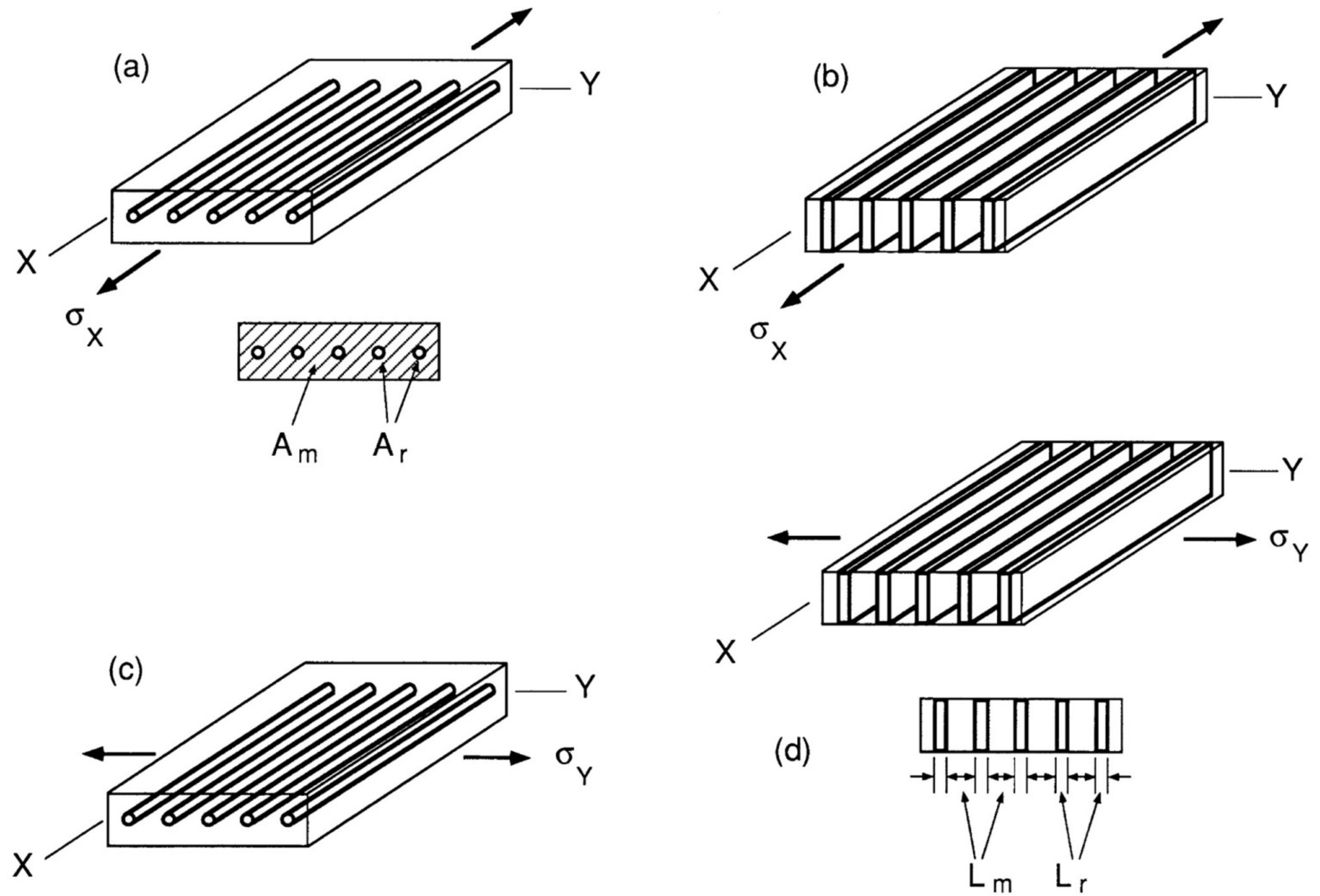


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**Figure 5.15** Composite materials with various combinations of stress direction and unidirectional reinforcement. In (a) the stress is parallel to fibers, and in (b) to sheets of reinforcement, whereas in (c) and (d) the stresses are normal to similar reinforcement.

# Elastic Modulus Parallel to Fibers (2)

- Fibers: isotropic material with elastic constant $E_r, \nu_r$, and $G_r$
- Matrix: isotropic material with elastic constant $E_m, \nu_m$, and $G_m$
- Consider a uniaxial stress $\sigma_X$ parallel to fibers.
- Assumption: The fibers are perfectly bonded to the matrix.

$$A = A_r + A_m$$
$$\sigma_X A = \sigma_r A_r + \sigma_m A_m$$
$$\sigma_X = E_X \varepsilon_X, \qquad \sigma_r = E_r \varepsilon_r, \qquad \sigma_m = E_m \varepsilon_m$$

From the assumption ($\varepsilon_X = \varepsilon_r = \varepsilon_m$), yielding the modulus of the composite material:

$$E_X = \frac{E_r A_r + E_m A_m}{A}$$

Volume fractions:

$$V_r = \frac{A_r}{A}, \qquad V_m = 1 - V_r = \frac{A_m}{A}$$

Thus,

$$E_X = V_r E_r + V_m E_m$$

# Elastic Modulus Transverse to Fibers

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- Consider uniaxial stress $\sigma_Y$ in orthogonal in-plane direction.

$$\sigma_Y = \sigma_r = \sigma_m$$

$$\varepsilon_Y = \frac{\Delta L}{L}, \qquad \varepsilon_r = \frac{\Delta L_r}{L_r}, \qquad \varepsilon_m = \frac{\Delta L_m}{L_m}$$

where

$$\Delta L = \Delta L_r + \Delta L_m$$

Therefore,

$$\varepsilon_Y = \frac{\varepsilon_r L_r + \varepsilon_m L_m}{L}$$

$$\frac{1}{E_Y} = \frac{1}{E_r}\frac{L_r}{L} + \frac{1}{E_m}\frac{L_m}{L}$$

Volume fractions:

$$V_r = \frac{L_r}{L}, \qquad V_m = 1 - V_r = \frac{L_m}{L}$$

Thus,

$$\frac{1}{E_Y} = \frac{V_r}{E_r} + \frac{V_m}{E_m}, \qquad E_Y = \frac{E_r E_m}{V_r E_m + V_m E_r}$$

# Other Elastic Constants, and Discussion

- **Estimate of the major Poisson's ration, $\nu_{XY}$**

$$\nu_{XY} = V_r \nu_r + V_m \nu_m$$

- **Estimate of the shear modulus**

$$G_{XY} = \frac{G_r G_m}{V_r G_m + V_m G_r}$$

- Actual values of $E_X$ are usually reasonably close to the estimate.
- Since $E_Y$ is a lower bound for the case of fibers, actual values are somewhat higher.
- ***Quasi-isotropic*:** A material whose elastic constants are approximately the same for any direction in the X-Y plane, but different in the Z-direction
  - Isotropic material for in-plane loading
  - Transversely isotropic material for general three-dimensional analysis