Chapter 14. Fatigue of Materials: Strain-Based Approach to Fatigue

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CHAPTER 14 Objectives

- Explore **strain versus fatigue life curves** and equations, including trends with material and adjustments for surface finish and size
- Extend strain-life curves to cases of **nonzero mean stress** and **multi-axial stress**
- Apply the **strain-based method** to make **life estimates** for engineering components, especially members with **geometric notches**, including cases of **irregular variation of load** with time
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2 Strain Versus Life Curves
3 Mean Stress Effects
4 Multi-axial Stress Effects
5 Life Estimates For Structural Components
6 Comparison of Methods*

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The fatigue life of a component is made up of **initiation** and **propagation** stages.

The size of crack is usually unknown and often depends on the size of the component being analyzed (e.g. A: 0.1mm-crack, B:0.15mm-crack).

**Stress & Strain-based Approach:** To determine crack initiation life.

**Fracture Mechanics:** To estimate the crack propagation life.

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**Figure 3.1** Initiation and propagation portions of fatigue life

**Figure 3.9** Extended service life of a cracked component

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14.1 Introduction – Why strain-based approach?

Brief Review of Stress-based Approach

- Key Idea is to develop stress-life relationship by employing elastic stress concentration factors and empirical modifications thereof.
- Stress-based approach emphasizes nominal stress, rather than local stresses and strains.
- Stress-based approach does not account for plastic strain.

Stress-based Approach vs Strain-based Approach

<table>
<thead>
<tr>
<th>High cycle fatigue</th>
<th>Low cycle fatigue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic range of the material (Does not account for plastic strain)</td>
<td>Account for plastic strain</td>
</tr>
<tr>
<td>More than 1000 cycles (~(N_t))</td>
<td>Less than 1000 cycles (~(N_t))</td>
</tr>
<tr>
<td>Stress-Life (Stress controlled)</td>
<td>Strain-Life (Strain controlled)</td>
</tr>
</tbody>
</table>

Source: [http://www.public.iastate.edu/~gkstarns/](http://www.public.iastate.edu/~gkstarns/)
14.1 Introduction

- Employment of cyclic stress-strain curve is a unique feature of the strain-based approach, as is the use of a strain versus life curve, instead of a nominal stress versus life (S-N) curve.

- Strain-based method gives improved estimates for intermediate and especially short fatigue lives.

- Strain-based approach employs the local mean stress at the notch, rather than the mean nominal stress.

- Certain concepts employed will be related to those introduced in Chapters 9 and 10. Also, we will draw upon the information in Chapters 12 and 13 on plastic deformation.

⇒ Be familiar with the contents in Chapters 9, 10, 12, and 13 before studying Chapter 14.
Review: Crack Initiation and Propagation

Introduction

Strain Versus Life Curves

Mean Stress Effects

Multi-axial Stress Effects

Life Estimates For Structural Components

Comparison of Methods*
14.2 Strain Versus Life Curves

- A strain versus life curve is a plot of **strain amplitude** versus **cycles to failure** (See Fig. 14.3)

**Figure 14.3** Elastic, plastic, and total strain versus life curves. (Adapted from [Landgraf 70]; copyright © ASTM; reprinted with permission.)
14.2.1 Strain-Life Tests and Equations

• Of particular relevance is the cyclic stress-strain curve (Also can be found in Eq. 12.54)

\[ \varepsilon_a = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{H'} \right)^{1/n'} \]  \hspace{1cm} (14.1)

• Note that the strain amplitude can be divided into elastic and plastic parts as

\[ \varepsilon_a = \varepsilon_{ea} + \varepsilon_{pa} \]  \hspace{1cm} (14.2)

\[ \varepsilon_{ea} = \frac{\sigma_a}{E} \]  

Measure of the half-width of the stress-strain hysteresis loop

• Strain-life curves are derived from fatigue tests under completely reversed cyclic loading between constant strain limits (See Fig. 14.2)

Measurement:
\[ \sigma_a, \varepsilon_a, \varepsilon_{pa} \]
14.2.1 Strain-Life Tests and Equations

- For each test, **three points** are plotted (Fig. 14.4)
- If data from several tests are plotted, the **elastic strains** often give a straight line of shallow slope on a log-log plot, and the **plastic strains** give a straight line of steeper slope.
- Equation can then be fitted to these lines
  \[ \varepsilon_{ea} = \frac{\sigma_a}{E} = \frac{\sigma'_f}{E} \left(2N_f\right)^b \]  
  \[ \varepsilon_{pa} = \varepsilon'_f \left(2N_f\right)^c \]
- Four constants \((\sigma'_f, b, \varepsilon'_f, c)\) are considered to be material properties
- Coffin-Manson Relationship
  \[ \varepsilon_a = \frac{\sigma'_f}{E} \left(2N_f\right)^b + \varepsilon'_f \left(2N_f\right)^c \]

**Figure 14.4** Strain versus life curves for RQC-100 steel. For each of several tests, elastic, plastic, and total strain data points are plotted versus life, and fitted lines are also shown. (From the author’s data on the ASTM Committee E9 material.)
14.2 Strain Versus Life Curves

- Coffin-Manson Relationship

\[ \varepsilon_a = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c \]  

(14.4)

Four constants \((\sigma'_f, b, \varepsilon'_f, c)\)

Figure 14.3 Elastic, plastic, and total strain versus life curves. (Adapted from [Landgraf 70]; copyright © ASTM; reprinted with permission.)
14.2.3 Trends for Engineering Metals

- **At long lives**: the curve approaches the *elastic strain* line
- **At short lives**: the curve approaches the *plastic strain* line
- **Near the crossing point**: the two types of strain are of similar magnitude
  \[ \rightarrow \text{Transition Fatigue Life, } N_t \]
- \( N_t \) is the most **logical point** for separating *low-cycle fatigue* and *high-cycle fatigue*
- Special analysis of *plasticity effects* by the *strain-based approach* may be needed if lives around or less than \( N_t \) are of interest

Let \( \varepsilon_{ea} = \varepsilon_{pa} \)

\[
N_t = \frac{1}{2} \left( \frac{\sigma_f'}{\varepsilon_f' E} \right)^{1/(c-b)}
\]  
\( \varepsilon_{ea} = \varepsilon_{pa} \)  
(14.6)

Strain-based Approach
14.2.3 Trends for Engineering Metals

• The strain-life equation requires four empirical constants \((\sigma'_f, b, \varepsilon'_f, c)\)

• **Several points must be considered** in attempting to obtain these constants from fatigue data*

1) **Generalization**
   Not all materials may be represented by the four-parameter strain-life equation. (Examples: some high strength aluminum alloys and titanium alloys)

2) **Data Size**
   The four fatigue constants may represent a curve fit to a limited number of data points. They may be changed if more data points are included in the curve fit.

3) **Range of data**
   The fatigue constants are determined from a set of data points over a given range. **Gross error** may occur when extrapolating fatigue life estimates outside this range.

4) **Physical phenomenon**
   The use of this equation is strictly a matter of mathematical convenience and is **not based on a physical phenomenon**

Nevertheless, the following approximate methods may be useful (cont.).

### 14.2.3 Trends for Engineering Metals

<table>
<thead>
<tr>
<th>Fatigue Strength Coefficient, $\sigma_f'$</th>
<th>Fatigue Ductility Coefficient, $\varepsilon_f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximation*</td>
<td></td>
</tr>
<tr>
<td>$\sigma_f' \approx \sigma_f$ (corrected for necking)</td>
<td>$\varepsilon_f' \approx \varepsilon_f$, where $\varepsilon_f = \ln\left(\frac{1}{1 - RA}\right)$</td>
</tr>
<tr>
<td>For steels ($&lt;500$ BHN): $\sigma_f' \approx S_u + 50ksi$</td>
<td>RA is the reduction in area</td>
</tr>
<tr>
<td>A ductile Metal</td>
<td>Low</td>
</tr>
<tr>
<td>A brittle metal</td>
<td>High</td>
</tr>
</tbody>
</table>

- All pass near the strain $\varepsilon_a = 0.01$ for a life of $N_f = 1000$ cycles


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**Figure 14.6** Trends in strain–life curves for strong, tough, and ductile metals. (Adapted from [Landgraf 70]; copyright © ASTM; reprinted with permission.)

\[ \varepsilon_a = \frac{\sigma_f'}{E} \left(2N_f\right)^b + \varepsilon_f' \left(2N_f\right)^c \]
14.2.3 Trends for Engineering Metals

- **Fatigue Strength Exponent, \( b \)**
  Average: -0.085
  
  For soft metals: \( b \approx -0.12 \)
  For highly hardened metals: \( b \approx -0.05 \)
  For steels with ultimate tensile strengths below about \( \sigma_u = 1400 \text{MPa} \)
  (Fatigue limit occurs near \( N_f = 10^6 \) cycles at a stress amplitude around \( \sigma_{erb} = \sigma_u / 2 \))

  \[
  \varepsilon_a = \frac{\sigma_f'}{E} \left(2N_f\right)^b + \varepsilon_f' \left(2N_f\right)^c
  \]

  \[
  b = - \frac{1}{\log(2N_f)} \log \left( \frac{2\sigma_f'}{\sigma_u} \right) = - \frac{1}{6.3} \log \left( \frac{2\tilde{\sigma}_f}{\sigma_u} \right) \quad (14.3) \quad (a)
  \]

  \[
  \varepsilon_{ea} = \frac{\sigma_a}{E} = \frac{\sigma_f'}{E} \left(2N_f\right)^b
  \]

  \[
  b = - \frac{1}{\log(2N_f)} \log \left( \frac{2\sigma_f'}{\sigma_u} \right) = - \frac{1}{6.3} \log \left( \frac{2\tilde{\sigma}_f}{\sigma_u} \right) \quad (14.10)
  \]

  In other cases, where the fatigue limit (or long-life fatigue strength) at \( N_e \) cycles is given by \( \sigma_{erb} = m_e \sigma_u \), the estimate becomes

  \[
  b = - \frac{1}{\log(2N_e)} \log \left( \frac{\tilde{\sigma}_f}{m_e \sigma_u} \right) \quad (14.11)
  \]
14.2.3 Trends for Engineering Metals

- **Fatigue Ductility Exponent**, $c^*$

$c$ is not well defined as the other parameters.
A rule of thumb approach must be followed rather than an empirical equation
- Coffin found $c$ to be about -0.5
- Manson found $c$ to be about -0.6
- Morrow found that $c$ varied between -0.5 and -0.7

Fairly ductile metals (where $\varepsilon_f \approx 1$) have average values of $c=-0.6$
For strong metals (where $\varepsilon_f \approx 0.5$) have average values of $c=-0.5$

\[
\varepsilon_a = \frac{\sigma_f'}{E} (2N_f)^b + \varepsilon_f' (2N_f)^c
\]
14.2.3 Trends for Engineering Metals

- Hardness varies inversely with ductility so that $N_t$ decreases as hardness is increased.

Figure 14.8  Transition fatigue life versus hardness for a wide range of steels. (Adapted from [Landgraf 70]; copyright © ASTM; reprinted with permission.)
14.2.4 Factors Affecting Strain-Life Curves

- Is a hostile chemical environment or elevated **temperature** is present, smaller numbers of cycles to failure are expected
- **Temperature**
  - At temperature exceeding about half of the absolute melting temperature of a given material, nonlinear deformation due to time-dependent creep-relaxation behavior generally become significant
- **Residual Stress**
  - Residual stress are quickly removed by cycle-dependent relaxation if cyclic plastic strains are present
  - These have only **limited effect at lives around and below** $N_t$.

Some additional comments on this topic are given in Chapter 15.
14.2.4 Factors Affecting Strain-Life Curves: Surface Finish

- **High-cycle fatigue** (most of the life is spent initiating a crack)
  - Surface finish is important
- **Fatigue with significant plastic strain** (most of the life is spent growing crack size)
  - Surface finish cannot have an effect
- A reasonable method to include the effect of surface finish is **to change only the elastic slope** $b$.
- When the fatigue limit at $N_e$ cycles is given by $\sigma_a = m_s \sigma_e$, where $m_s$ is a surface effect factor, as in Chapter 10

$$b_s = -\frac{1}{\log(2N_e)} \log\left(\frac{\bar{\sigma}_f}{m_s \sigma_e}\right) = b + \frac{\log(m_s)}{\log(2N_e)}$$  \hspace{1cm} (14.12)

$$\sigma_e = \sigma_f'(2N_e)^b$$
14.2.4 Factors Affecting Strain-Life Curves: Size Effect

• Size effect (as discussed in Chapter 10) are also a concern in applying a strain-based approach to large-size members, but experimental data are limited.

• A study suggested lowering the entire strain-life curve by this factors so that the intercept constants \( \sigma_f' \) and \( \varepsilon_f' \) are replaced by reduced values and \( \sigma_{f_d} \) and \( \varepsilon_{f_d} \)

\[
\sigma_{f_d} = m_d \sigma_f', \quad \varepsilon_{f_d} = m_d \varepsilon_f'
\]

(14.14)

where \( m_d \) for the shafts up to 200 mm in diameter (low-carbon and low-alloy steels) was found to vary with shaft diameter as

\[
m_d = \left( \frac{d}{25.4\, mm} \right)^{-0.093}
\]

(14.13)
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14.3.1 Mean Stress Effects*

- Cyclic fatigue properties of a material are obtained from completely reversed, constant amplitude strain-controlled tests. However, components seldom experience this type of loading.
- Mean stress effect can either increase the fatigue life with a nominally compressive load or decrease it with a nominally tensile value (Fig. 2.21*)
- At high strain amplitude (0.5% to 1% or above), where plastic strains are significant, mean stress relaxation occurs and the mean stress tends toward zero
- Modifications to the strain-life equation have been made to account for mean stress effects by 1) Morrow, 2) Smith, Watson, and Topper (SWT), and 3) Walker.

14.3.3 Mean Stress Equation of Morrow*

- Morrow and Halford (1981) found that ratio of elastic to plastic strain in the equation proposed by Morrow (1968) is dependent on mean stress, which is not true.
- They modified both the elastic and plastic terms of the strain-life equation to maintain the independence of the elastic-plastic strain ratio from mean stress as:

\[
\varepsilon_a = \frac{\sigma_f - \sigma_m}{E} \left(2N_f\right)^b + \varepsilon'_f \left(\frac{\sigma_f - \sigma_m}{\sigma'_f}\right)^{c/b} \left(2N_f\right)^c
\]  

(14.24)

* Refer to the Appendix for more details

14.3.3 Mean Stress Equation of Morrow (modified)*

- Morrow (1968) suggested to modify the elastic term in the strain-life equation

\[
\varepsilon_{ea} = \frac{\sigma_a}{E} = \frac{\sigma_f'}{E} \left(2N_f\right)^b
\]

(14.3) \quad (a)

\[
\varepsilon_a = \frac{\sigma_f'}{E} \left(2N_f\right)^b + \varepsilon_f' \left(2N_f\right)^c
\]

(14.4)

\[
\varepsilon_{ea} = \frac{\sigma_a}{E} = \frac{\sigma_f' - \sigma_m}{E} \left(2N_f\right)^b
\]

\[
\varepsilon_a = \frac{\sigma_f' - \sigma_m}{E} \left(2N_f\right)^b + \varepsilon_f' \left(2N_f\right)^c
\]

(14.27)

**Figure 2.23** Morrow’s mean stress correction to the strain-life curve for a tensile mean

**Figure 14.12** Family of strain–life curves given by the modified Morrow approach

14.3.5 Smith, Watson, and Topper (SWT) Parameters

- This approach assumes that the life for any situation of mean stress depends on the product

\[
\sigma_{\text{max}}\varepsilon_a = \frac{(\sigma'_f)^2}{E} (2N_f)^{2b} + \sigma'_f \varepsilon'_f (2N_f)^{b+c}
\]  

(14.29)

where \(\sigma_{\text{max}} = \sigma_m + \sigma_a\)

14.3.5 Walker Mean Stress Equation

\[
\varepsilon_a = \frac{\sigma'_f}{E} \left(\frac{1 - R}{2}\right)^{(1-\gamma)/b} (2N_f)^b + \varepsilon'_f \left(\frac{1 - R}{2}\right)^{c(1-\gamma)/b} (2N_f)^c
\]  

(14.33)
14.3.5 Smith, Watson, and Topper (SWT) Parameters

Figure 14.13 Plot of the Smith, Watson, and Topper parameter versus life for the data of Fig. 14.10.

AISI 4340 Steel
\( \sigma_u = 1172 \text{ MPa} \)
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4 Multi-axial Stress Effects
  • Fatigue under multi-axial loading where plastic deformations occur is currently an area of active research.
  • Reasonable estimates are possible for relatively simple situations
5 Life Estimates For Structural Components
6 Comparison of Methods*
14.4.1 Effective Strain Approach

- Consider situations where all cyclic loadings have the same frequency and are either in-phase or 180° out-of-phase.
- Recall three-dimensional stress-strain relationships (Ch. 12)

\[
\bar{\sigma} = \frac{1}{\sqrt{2}} \times \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}
\]  
(12.21)

\[
\bar{\varepsilon}_p = \frac{\sqrt{2}}{3} \times \sqrt{(\varepsilon_{p1} - \varepsilon_{p2})^2 + (\varepsilon_{p2} - \varepsilon_{p3})^2 + (\varepsilon_{p3} - \varepsilon_{p1})^2}
\]  
(12.22)

\[
\bar{\varepsilon} = \frac{\bar{\sigma}}{E} + \bar{\varepsilon}_p
\]  
(12.23)

- Then, we can define an effective strain amplitude

\[
\bar{\varepsilon}_a = \frac{\bar{\sigma}_a}{E} + \bar{\varepsilon}_{pa}
\]  
(14.34)

- \(\bar{\sigma}_a\) and \(\bar{\varepsilon}_{pa}\) are obtained by substituting amplitudes of the principal stresses and plastic strains.
- A negative sign is employed for amplitude quantities that are 180° out-of-phase.
14.4.1 Effective Strain Approach

- For uniaxial loading ($\sigma_2 = \sigma_3 = 0$), its value reduces to the uniaxial strain amplitude ($\bar{\varepsilon}_a = \varepsilon_{1a}$)

$$\bar{\varepsilon}_a = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c$$ (14.35)

$$\bar{\sigma}_a = \sigma'_f (2N_f)^b, \quad \bar{\varepsilon}_{pa} = \varepsilon'_f (2N_f)^b$$ (14.36)

- Consider the special case of plane stress (Refer to Ch. 12.3.4)

$$\sigma_2a = \lambda \sigma_{1a}, \quad \sigma_3a = 0, \quad \varepsilon_{1a} = \varepsilon_{e1a} + \varepsilon_{p1a}$$ (14.37)

- Combining Eqs. (12.19), (12.24) and (12.32) with Eqs. (14.35), (14.37)

$$\varepsilon_{1a} = \frac{\sigma'_f}{E} (1 - \nu \lambda)(2N_f)^b + \varepsilon'_f (1 - 0.5\lambda)(2N_f)^c \sqrt{1 - \lambda + \lambda^2}$$ (14.38)

- For the special state of plane stress that is pure shear ($\sigma_{2a} = \lambda \sigma_{1a}, \lambda = -1$)

$$\gamma_{xya} = \frac{\sigma'_f}{\sqrt{3}G} (2N_f)^b + \sqrt{3}\varepsilon'_f (2N_f)^c$$ (14.39)

Shear strain amplitude Shear modulus

Refer to Ch. 13.4 for more information
14.4.2 Discussion of the Effective Strain Approach

• Consider the hydrostatic stress not accounted for by the octahedral shear strain

\[
\sigma_{ha} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}
\]  

(14.40)

• Relative value of \( \sigma_{ha} \) may be expressed as a triaxiality factor for plane stress \((\sigma_3 = 0)\)

\[
T = \frac{1 + \lambda}{\sqrt{1 - \lambda + \lambda^2}}, \quad \sigma_2 = \lambda \sigma_1,
\]  

(14.41)

(a) Pure planar shear \((\lambda = -1) \Rightarrow T = 0\)
(b) Uniaxial stress \((\lambda = 0) \Rightarrow T = 1\)
(c) Equal biaxial stress \((\lambda = 1) \Rightarrow T = 2\)

• Marloff (1985) proposed to include this effect to strain-life equation as:

\[
\bar{\varepsilon}_a = \frac{\sigma'_f}{E} (2N_f)^b + 2^{1-T} \varepsilon'_f (2N_f)^c
\]  

(14.42)
14.4.3 Critical Plane Approaches

• Critical plane approach is needed where the loading is non-proportional to a significant degree

• Stresses and strains normal to the crack plane may have a major effect on the behavior, accelerating the growth if they tend to open the crack.

  1) Stresses and strains are determined for various orientations (planes) in the material.
  2) Stresses and strains acting on the most severely loaded plane are used for analysis.
14.4.3 Critical Plane Approaches

- Fatemi and Socie

\[ \gamma_{ac} \left( 1 + \frac{\alpha \sigma_{\text{max}}}{\sigma'_o} \right) \bar{\varepsilon}_a = \frac{\tau'_f}{G} \left( 2N_f \right)^b + \gamma'_f \left( 2N_f \right)^c \]  

(14.43)

where \( \gamma_{ac} \) is the largest amplitude of shear strain for any plane, 
\( \sigma_{\text{max}} \) is the peak tensile stress normal to the plane of \( \gamma_{ac} \), 
\( \alpha \) is an empirical constant ranging from 0.6 to 1.0, and 
\( \sigma'_o \) is the yield strength for the cyclic stress-strain curve. 
\( \tau'_f, b, \gamma'_f, c \) give the strain-life curve from completely reversed tests in pure shear.  
(specifically torsion tests on thin-walled tubes)

Figure 14.14 Crack under pure shear (a), where irregularities retard growth, compared with a situation (b) where a normal stress acts to open the crack, enhancing its growth. (Adapted from [Socie 87]; used with permission of ASME.)
14.4.3 Critical Plane Approaches

- **Fatemi and Socie**

\[
\gamma_{ac} \left( 1 + \frac{\alpha \sigma_{\text{max}}}{\sigma_o'} \right) \varepsilon_a = \frac{\tau_f'}{G} \left( 2 N_f \right)^b + \gamma_f' \left( 2 N_f \right)^c \tag{14.43}
\]

- **Smith, Watson, and Topper (SWT) Parameters**

\[
\sigma_{\text{max}} \varepsilon_a = \frac{\left( \sigma_f' \right)^2}{E} \left( 2 N_f \right)^{2b} + \sigma_f' \gamma_f' \left( 2 N_f \right)^{b+c} \tag{14.35}
\]

The shortest life estimated from either Eqs. (14.43) or (14.35) is the final life estimate.

- **A single multiaxial fatigue criterion that considers both the shear and normal stress cracking mode is that of Chu (1995)**

\[
2 \tau_{\text{max}} \gamma_a + \sigma_{\text{max}} \varepsilon_a = f \left( N_f \right) \tag{14.44}
\]

where \( f \left( N_f \right) \) can be obtained from uniaxial test data.
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14.5.1 Constant Amplitude Loading

- Assuming idealized behavior for the material

\[ \varepsilon = f(\sigma), \quad \varepsilon_a = f(\sigma_a) \]

(Monotonic) (Cyclic)

\[ \frac{\Delta \varepsilon}{2} = f \left( \frac{\Delta \sigma}{2} \right) \]  (14.45)

\[ \frac{\Delta \varepsilon}{2} = f \left( \frac{\Delta \sigma}{2} \right) \]  (14.46)

Figure 12.14 Stress–strain unloading and reloading behavior consistent with a spring and slider rheological model. The example curves plotted correspond to a Ramberg–Osgood stress–strain curve with constants as in Fig. 12.9.
14.5.1 Constant Amplitude Loading

- Consider Neuber’s rule to analyze a notched member
  ➔ Ch. 13 will be reviewed in the following two slides
13.5.3 Estimates of Notch Stress and Strain for Local Yielding

- Theoretical stress concentration factor, \( K_t \), is used to relate the nominal stress or strain to the local values.*
- Upon yielding, the local values are no longer linearly related to the nominal values by \( K_t \)
- Instead, the values are related in terms of stress and strain concentration factors as:

\[
K_\sigma = \frac{\sigma_{\text{local}}}{\sigma_{\text{nom}}}, \quad K_\varepsilon = \frac{\varepsilon_{\text{local}}}{\varepsilon_{\text{nom}}}
\]  

(13.56)

- After yielding, the actual local stress is less than that predicted using \( K_t \)
- After yielding, the actual local strain is greater than that predicted using \( K_t \)
- This is due to residual stresses at the notch root

13.5.3 Estimates of Notch Stress and Strain for Local Yielding

- Neuber’s rule states simply that the geometric mean of the stress and strain concentration factors remains equal to \( k_t \) during plastic deformation.
- If fully plastic yielding does not occur, \( e = S/E \) applies.

\[
k_t = \sqrt{k_\sigma k_\varepsilon} \quad (13.57)
\]

\[
\sigma \varepsilon = \frac{(k_t S)^2}{E} \quad (13.58)
\]

**Figure 13.16** For a given notched member and stress–strain curve (a), Neuber’s rule may be used to estimate local notch stresses and strains, \( \sigma \) and \( \varepsilon \), corresponding to a particular value of nominal stress \( S \). Stress and strain concentration factors vary as in (b).
14.5.1 Constant Amplitude Loading

- Consider Neuber’s rule to analyze a notched member

**Step 1**

1. Assemble Input
   - Loading: \( S_{\text{peak}}, S_0 \)
   - Geometry: \( k_i \)
   - Material constants:
     - \( E, h', h'' \) for \( \epsilon_s = f(h) \)
     - \( \sigma, b, c \) for \( \epsilon_s = f(h) \)

**Step 2**

2. Analyze \( S_{\text{peak}}, S_0 \)
   - a) \( \sigma_{\text{max}} = \frac{S_{\text{max}}}{k} \)
   - b) \( \epsilon_s = \frac{S_0}{E} \)

**Step 3**

3. Plot \( \sigma - \epsilon \) Response
   - a) \( \sigma_{\text{max}} = \epsilon_{\text{max}} + 2 \sigma_0 \)
   - b) \( \sigma_{\text{min}} = \epsilon_{\text{min}} - 2 \sigma_0 \)
   - c) \( \epsilon_{\text{min}} = \epsilon_{\text{max}} - \epsilon_0 \)

**Step 4**

4. Determine Life, \( N_L \)
   - a) Morrow: \( \epsilon_s = h(N)^2 \)
     or \( \epsilon_s = h(N_L, N_L) \)
   - b) or use SWT: \( \sigma_{\text{max}} - \sigma_{\text{min}} = f(N_L) \)

**Figure 14.15** Steps required in strain-based life prediction for a notched member under constant amplitude loading.
**14.5.1 Constant Amplitude Loading**

- **Step 1**

1. **Assemble Input**
   - (a) Loading: $S_{\text{max}}$, $S_a$
   - (b) Geometry: $k_t$
   - (c) Material constants:
     - $E$, $H'$, $n'$ for $\varepsilon_a = f(\sigma_a)$
     - $\sigma'$, $b$, $\varepsilon'$, $c$ for $\varepsilon_a = h(N_f)$
14.5.1 Constant Amplitude Loading

- **Step 2 & Step 3**

\[
\sigma_\varepsilon = \frac{(k_S)^2}{E}
\]

\[
\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{H'}\right)^{1/n'}
\]

1. **Analyze** \( S_{\text{max}} / S_a \)
   - (a) \( \sigma_{\text{max}} = \frac{(k_S S_{\text{max}})^2}{E} \)
   - (b) \( \sigma_a = \frac{(k_S S_a)^2}{E} \)

2. **Plot** \( \sigma - \varepsilon \) Response
   - (a) \( \varepsilon_{\text{min}} = \varepsilon_{\text{max}} - 2\varepsilon_a \)
   - (b) \( \sigma_{\text{min}} = \sigma_{\text{max}} - 2\sigma_a \)
   - (c) \( \sigma_m = \sigma_{\text{max}} - \sigma_a \)
14.5.1 Constant Amplitude Loading

- Step 4

(4) Determine Life, \( N_f \)

(a) Morrow: \( \varepsilon_a = h(N^*) \)

or \( \varepsilon_a = h'(\sigma_m, N_f) \)

(b) or use SWT: \( \sigma_{\text{max}} \varepsilon_a = h''(N_f) \)
14.5.2 Irregular Load Versus Time Histories

- The life to failure $N_f$ corresponding to each hysteresis loop can be determined from its combination of strain amplitude and mean stress.
- If the SWT parameter is used, $\sigma_{max}$ is simply the highest stress ($\sigma_G$) for F-G-F'
- Having obtained the $N_f$ value for each loop, we can apply the Palmgren Miner rule (Ch. 9), where each closed stress-strain hysteresis loop is considered to represent a cycle.

**Figure 14.16** Analysis of a notched member subjected to an irregular load versus time history. Notched member (a), made of 2024-T351 aluminum, is subjected to load history (b). The resulting local stress–strain response at the notch is shown in (c).
Contents

0  Review: Crack Initiation and Propagation
1  Introduction
2  Strain Versus Life Curves
3  Mean Stress Effects
4  Multi-axial Stress Effects
5  Life Estimates For Structural Components
6  Comparison of Methods*
6. Comparison of Methods*

General Points for Comparison

• Are the methods to be used in the design cycle or to analyze an existing component?
• What is the accuracy of the methods compared to input variables such as load history and material properties?
• What are the relative economics?
• What is the level of acceptance?
• Uses in design versus research.

6. Comparison of Methods*

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<tr>
<td><strong>Residual Stress</strong></td>
<td>Not considered</td>
<td>Considered</td>
</tr>
</tbody>
</table>

Where should we use this approach?

- For constant amplitude loading and long fatigue lives.
- When elastic strains are dominant (e.g. Transmission shaft, valve springs, and gears).
- For variable amplitude loading and short fatigue lives.
- When plastic strains are significant.
- High temperature applications with fatigue-creep (e.g. Gas turbine engine).

Appendix
14.3.2 Including Mean Stress Effects in Strain-Life Equation

• Let us define *equivalent completely reversed stress amplitude* \( \sigma_{ar} \) (as Eq. 9.22), which is repeated here:

\[
\sigma_{ar} = \sigma_f' (2N_f)^b
\]

(14.15)

• An additional equation is needed to calculate \( \sigma_{ar} \) for the mean stress situation

\[
\sigma_{ar} = f(\sigma_a, \sigma_m) = \sigma_a \frac{f(\sigma_a, \sigma_m)}{\sigma_a} = \sigma_f' (2N_f)^b
\]

(14.17)

• Then, we can define *zero-mean-stress-equivalent life*, \( N^* \)

\[
\sigma_a = \sigma_f' \left[ 2N_f \left( \frac{\sigma_a}{f(\sigma_a, \sigma_m)} \right)^{1/b} \right]^b = \sigma_f' (2N^*)^b
\]

(14.18)

where \( N^* = N_f \left( \frac{\sigma_a}{f(\sigma_a, \sigma_m)} \right)^{1/b} \)
14.3.3 Mean Stress Equation of Morrow*

- Equivalent completely reversed stress amplitude (Also can be found at Eq. (9.21))

\[ \sigma_{ar} = \frac{\sigma_a}{1 - \frac{\sigma_m}{\sigma_f}} \]  \hspace{1cm} (14.21)

\[ N_{mi}^* = N_f \left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{1/b} \] \hspace{1cm} (14.22)

\[ N_f = N_{mi}^* \left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{-1/b} \] \hspace{1cm} (14.23)

\[ \varepsilon_a = \frac{\sigma_f'}{E} (2N^*)^b + \varepsilon_f' (2N^*)^c \]

\[ \varepsilon_a = \frac{\sigma_f'}{E} \left(1 - \frac{\sigma_m}{\sigma_f'}\right) (2N_f)^b + \varepsilon_f' \left(1 - \frac{\sigma_m}{\sigma_f'}\right)^{c/b} (2N_f)^c \] \hspace{1cm} (14.24)
14.3.3 Mean Stress Equation of Morrow*

\[ N_{mi}^* = N_f \left(1 - \frac{\sigma_m}{\sigma_f}\right)^{1/b} \]

\( \sigma_m, \text{ MPa} \)
- O 0
- □ 207
- △ 414
- ◊ 621
- ○ −207

AISI 4340 Steel
\( \sigma_u = 1172 \text{ MPa} \)

Figure 14.11 Mean stress data of Fig. 14.10 plotted versus \( N^* \) according to the Morrow equation.
14.3.5 Smith, Watson, and Topper (SWT) Parameters

- This approach assumes that the life for any situation of mean stress depends on the product

\[ \sigma_{\text{max}} \varepsilon_a = h''(N_f) \] (14.28)

where \( \sigma_{\text{max}} = \sigma_m + \sigma_a \) and \( h''(N_f) \) indicates a function of fatigue life \( N_f \)

- Life is expected to be the same as for completely reversed loading where this product has the same value.

- Let \( \sigma_{ar} \) and \( \varepsilon_{ar} \) be the completely reversed stress and strain amplitude that result in the same life \( N_f \) as the \( (\sigma_{\text{max}}, \varepsilon_a) \) combination.

- When \( \sigma_m = 0 \) and \( \sigma_{\text{max}} = \sigma_{ar} \), we find \( \sigma_{\text{max}} \varepsilon_a = \sigma_{ar} \varepsilon_{ar} \)

- Then, using Eqs. 14.5 and 14.4, we can define

\[
\sigma_{\text{max}} \varepsilon_a = \sigma_f' (2N_f)^b \left[ \frac{\sigma_f'}{E} (2N_f)^b + \varepsilon_f'(2N_f)^c \right] \] (14.29)

\[
\sigma_{\text{max}} \varepsilon_a = \frac{\left(\sigma_f'\right)^2}{E}(2N_f)^{2b} + \sigma_f'\varepsilon_f'(2N_f)^{b+c} \] (14.30)
14.3.5 Smith, Watson, and Topper (SWT) Parameters

Figure 14.13 Plot of the Smith, Watson, and Topper parameter versus life for the data of Fig. 14.10.
14.3.5 Walker Mean Stress Equation

- Recall Eq. (9.19)

\[
\sigma_{ar} = \sigma_{\text{max}}^{1-\gamma} \sigma_a^\gamma, \quad \sigma_{ar} = \sigma_{\text{max}} \left(\frac{1 - R}{2}\right)^\gamma
\]  (14.31)

where \( R = \sigma_{\text{min}} / \sigma_{\text{max}} \)

\[
N_f = N_{mi}^* \left(1 - \frac{\sigma_m}{\sigma_f^*}\right)^{-1/b}
\]  (14.23)

\[
N_w^* = N_f \left(\frac{\sigma_a}{\sigma_{\text{max}}}\right)^{(1-\gamma)/b}, \quad N_w^* = N_f \left(\frac{1 - R}{2}\right)^{(1-\gamma)/b}
\]  (14.32)

\[
\varepsilon_a = \frac{\sigma_f^*}{E} \left(\frac{1 - R}{2}\right)^{(1-\gamma)/b} \left(2N_f\right)^b + \varepsilon_f^* \left(\frac{1 - R}{2}\right)^{c(1-\gamma)/b} \left(2N_f\right)^c
\]  (14.33)
Backup
4.3 Strain-life Approach*

4.3.1 Notch Root Stresses and Strains

• Theoretical stress concentration factor, $K_t$, is used to relate the nominal stress or strain to the local values.
• Upon yielding, the local values are no longer linearly related to the nominal values by $K_t$.
• Instead, the values are related in terms of stress and strain concentration factors as:

\[ K_\sigma = \frac{\text{Local stress}}{\text{Nominal stress}} \quad (4.11)* \]
\[ K_\varepsilon = \frac{\text{Local strain}}{\text{Nominal strain}} \quad (4.12)* \]

• After yielding, the actual local stress is less than that predicted using $K_t$.
• After yielding, the actual local strain is greater than that predicted using $K_t$.
• This is due to residual stresses at the notch root.

14.1 Constant Amplitude Loading

- Let strain be expressed as a function of $S$ that denotes load, moment, nominal stress, etc.

\[ \varepsilon = g(S) \tag{14.47} \]

\[ \varepsilon_{\text{max}} = g(S_{\text{max}}) = f(\sigma_{\text{max}}) \quad \text{for tensile stress} \tag{14.48} \]

\[ \varepsilon_a = g(S_a) = f(\sigma_a) \quad \frac{\Delta \varepsilon}{2} = g\left(\frac{\Delta S}{2}\right) = f\left(\frac{\Delta \sigma}{2}\right) \]