Chapter 7. Yielding and Fracture under Combined Stresses

Mechanical Strengths and Behavior of Solids

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7.1 Introduction

- **A failure criterion is required to predict the safe limits for use of a material under combined stresses.**

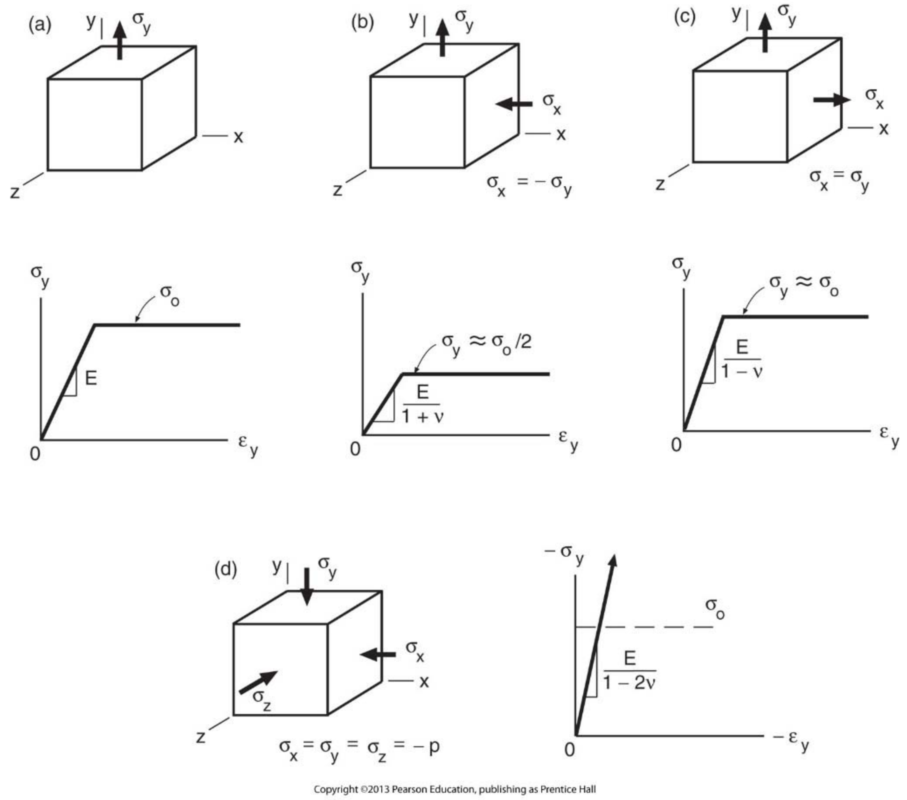


Figure 7.1 Yield strengths for a ductile metal under various states of stress: (a) uniaxial tension, (b) tension with transverse compression, (c) biaxial tension, and (d) hydrostatic compression.

7.2 General Form of Failure Criteria

- **A mathematical function of the principal normal stresses:**

$$f(\sigma_1, \sigma_2, \sigma_3) = \bar{\sigma}$$

Failure occurs when $\bar{\sigma} = \sigma_c$, where σ_c is the failure strength of the material.

Failure is not expected if $\bar{\sigma}$ is less than σ_c.

$$\bar{\sigma} = \sigma_c \qquad \text{(at failure)}$$

$$\bar{\sigma} < \sigma_c \qquad \text{(no failure)}$$

- **The safety factor against failure:**

$$X = \frac{\sigma_c}{\bar{\sigma}}$$

  - Safety factor describes the structural capacity of a system beyond expected loads or applied loads.
  - In other words, the applied stresses can be increased by a factor X before failure occurs.

- **A maximum normal stress fracture criterion:**

$$\sigma_u = \text{MAX}(|\sigma_1|, |\sigma_2|, |\sigma_3|) \qquad \text{(at fracture)}$$

For plane stress, as shown in Fig. 7.2(a), the box is the region that satisfies

$$\text{MAX}(|\sigma_1|, |\sigma_2|) \leq \sigma_u$$

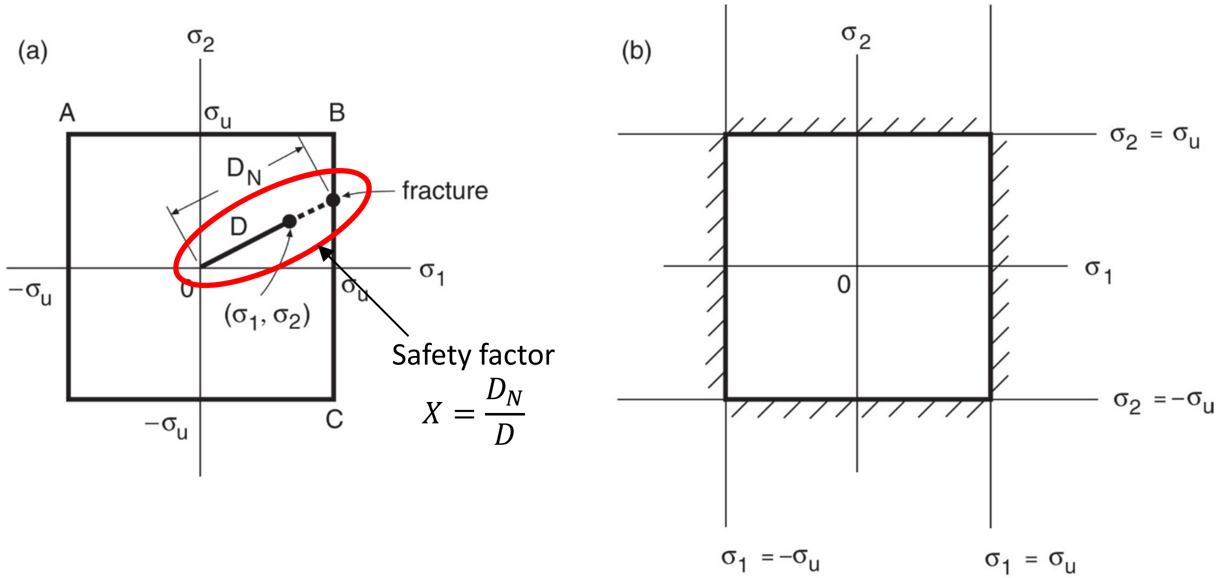


Figure 7.2 Failure locus for the maximum normal stress fracture criterion for plane stress.

7.3 Maximum Normal Stress Fracture Criterion (2)

For the general case, where all three principal normal stresses are nonzero, the failure surface is illustrated as Fig. 7.3.

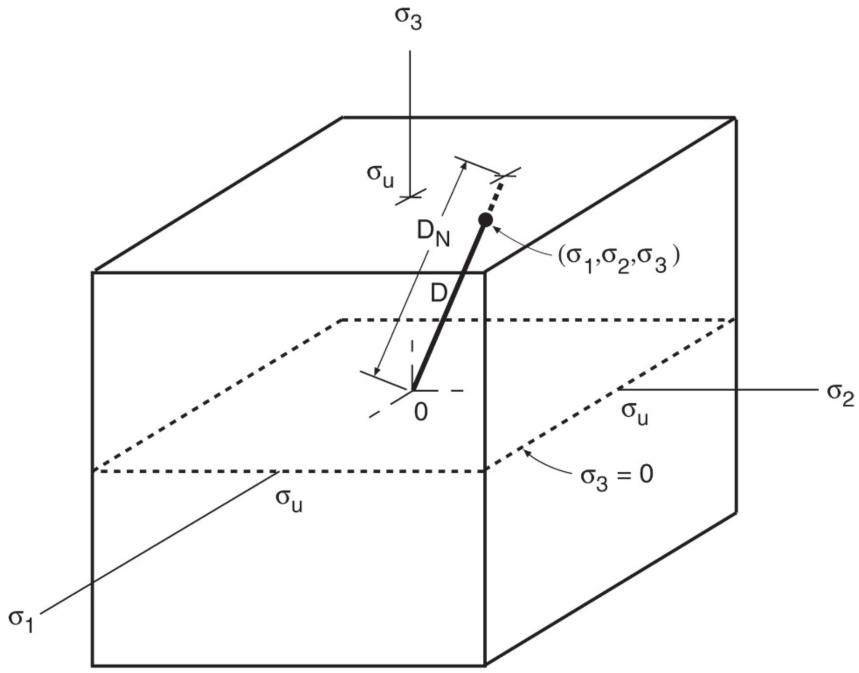


Figure 7.3 Three-dimensional failure surface for the maximum normal stress fracture criterion.

7.4 Maximum Shear Stress Yield Criterion (1)

- Yielding of ductile materials occurs when the maximum shear stress on any plane reaches a critical value τ_o ($\tau_o = \tau_{max}$).
- **Maximum shear stress yield criterion (Tresca criterion)**

$$\tau_o = \mathrm{MAX}(\tau_1, \tau_2, \tau_3) = \mathrm{MAX}(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2})$$

In uniaxial tension test,

$$\sigma_1 = \sigma_o, \qquad \sigma_2 = \sigma_3 = 0$$

Substitution of these values into the yield criterion gives

$$\tau_o = \frac{\sigma_o}{2}$$

Thus,

$$\frac{\sigma_o}{2} = \mathrm{MAX}(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2})$$

Let the effective stress be $\bar{\sigma}_S$, the safety factor is then

$$X = \frac{\sigma_o}{\bar{\sigma}_S}$$

7.4 Maximum Shear Stress Yield Criterion (2)

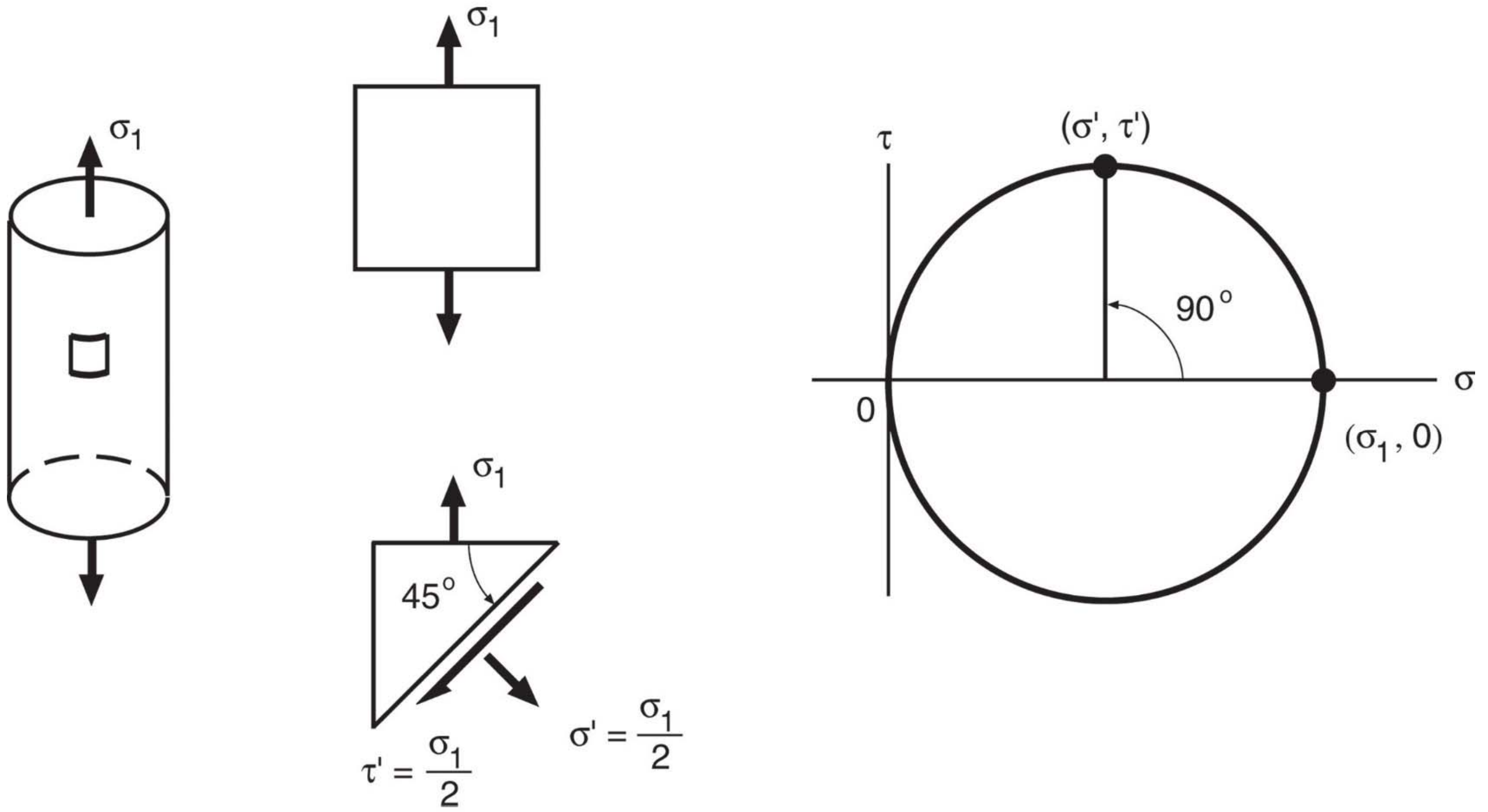


Figure 7.4 The plane of maximum shear in a uniaxial tension test.

7.4 Maximum Shear Stress Yield Criterion (3)

For plane stress, a maximum shear stress yield criterion:

$$\sigma_o = \mathrm{MAX}(|\sigma_1 - \sigma_2|, |\sigma_2|, |\sigma_1|)$$

Boundaries:

$$\sigma_1 - \sigma_2 = \pm\sigma_o, \qquad \sigma_2 = \pm\sigma_o, \qquad \sigma_1 = \pm\sigma_o$$

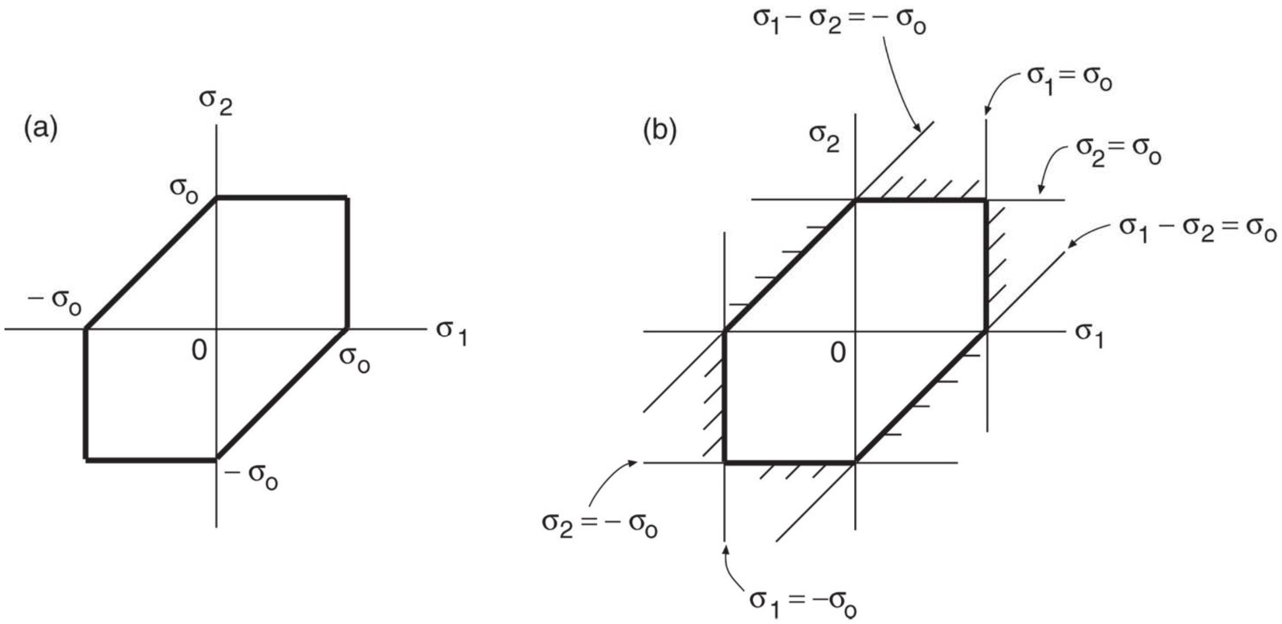


Figure 7.5 Failure locus for the maximum shear stress yield criterion for plane stress.

7.4 Maximum Shear Stress Yield Criterion (4)

For the general case, boundaries:

$$\sigma_1 - \sigma_2 = \pm\sigma_o, \qquad \sigma_2 - \sigma_3 = \pm\sigma_o, \qquad \sigma_1 - \sigma_3 = \pm\sigma_o$$

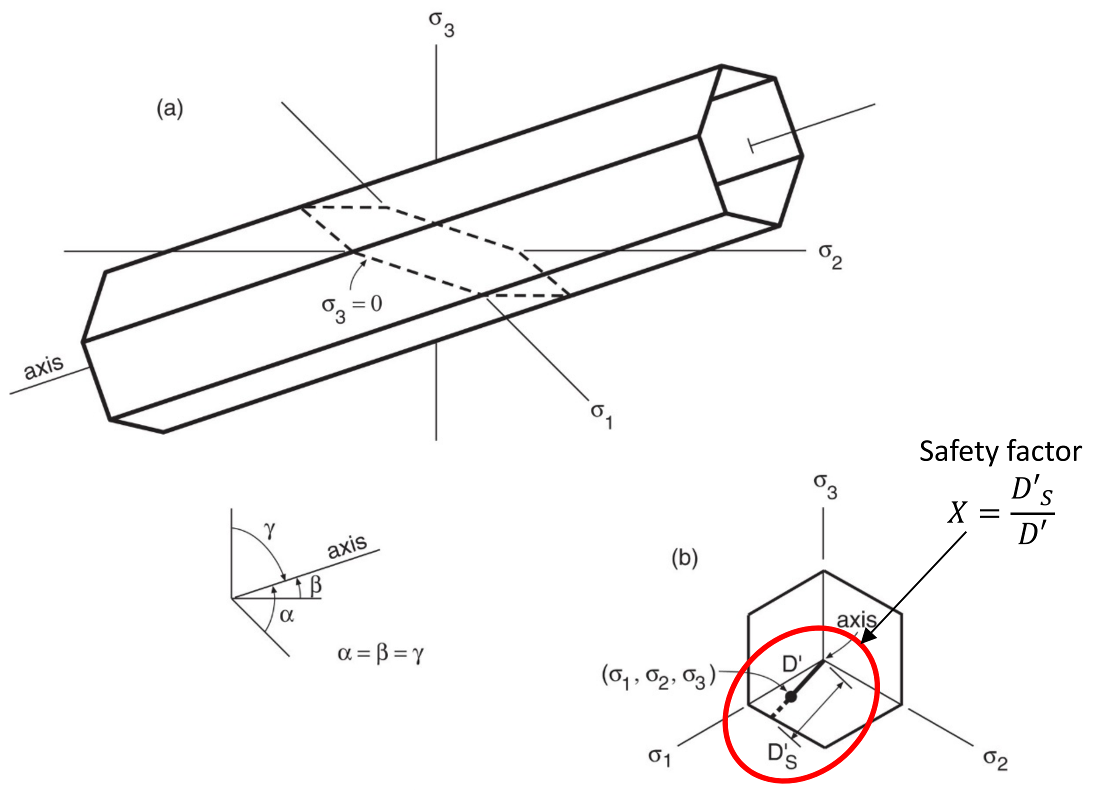


Figure 7.6 Three-dimensional failure surface for the maximum shear stress yield criterion.

7.5 Octahedral Shear Stress Yield Criterion (1)

- Yielding of ductile materials occurs when the maximum shear stress on octahedral plane reaches a critical value τ_{ho} ($\tau_h = \tau_{ho}$).
- ***Octahedral Plane*:** For the special case where $\alpha = \beta = \gamma$, the oblique plane which intersects the principal axes at equal distances from the origin

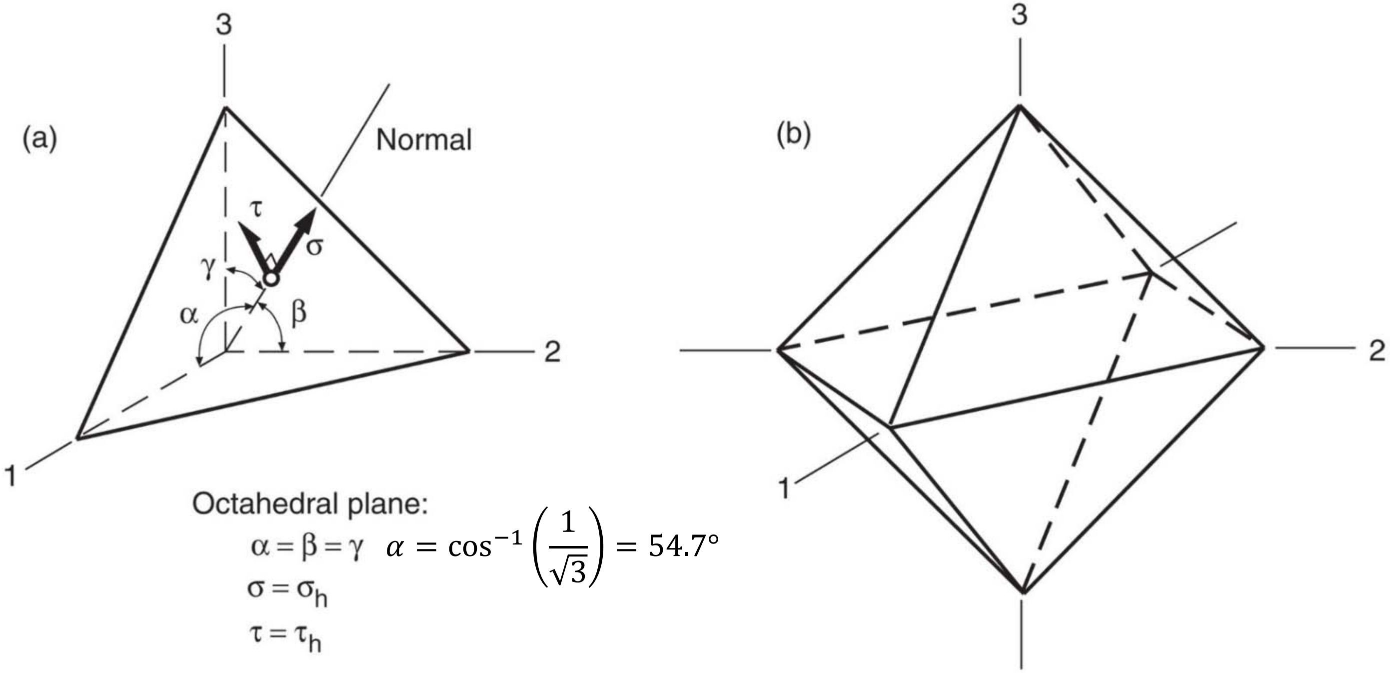


Figure 6.15 Octahedral plane shown relative to the principal normal stress axes (a), and the octahedron formed by the similar such planes in all octants (b).

7.5 Octahedral Shear Stress Yield Criterion (2)

- **Octahedral shear stress yield criterion (von Mises or distortion energy criterion):**

$$\tau_{ho} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

In uniaxial tension with $\sigma_1 = \sigma_o$, and $\sigma_2 = \sigma_3 = 0$,

$$\tau_{ho} = \frac{\sqrt{2}}{3}\sigma_o$$

The same result can be obtained from Mohr's circle that the normal stress is

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_1}{3}$$

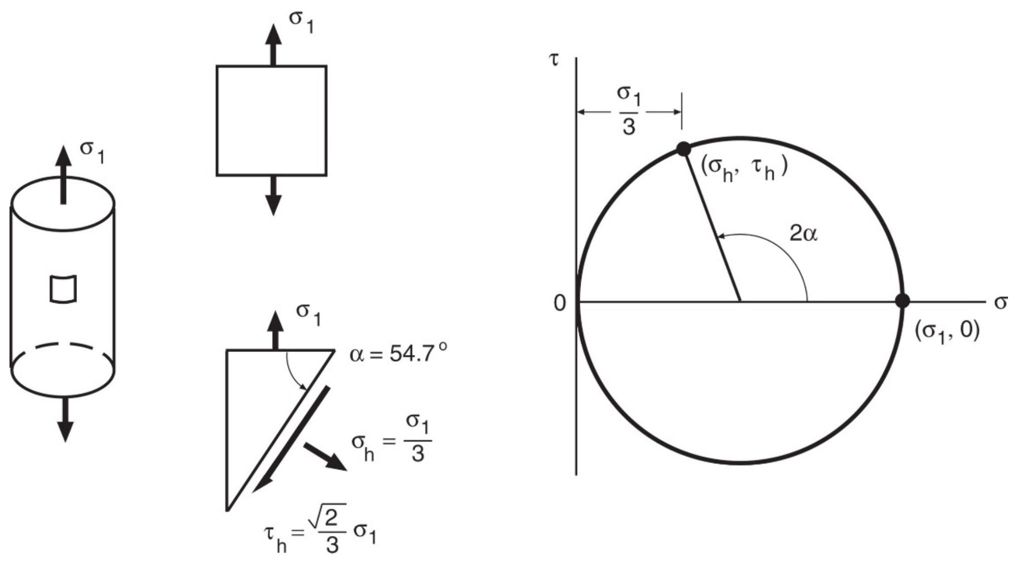


Figure 7.7 The plane of octahedral shear in a uniaxial tension test.

7.5 Octahedral Shear Stress Yield Criterion (3)

- **The yield criterion expressed in terms of the uniaxial yield strength:**

$$\sigma_o = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \qquad \text{(at yielding)}$$

For plane stress,

$$\sigma_o = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+\sigma_2^2+\sigma_1^2}$$

Manipulation gives

$$\sigma_o^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

Which is the equation of an ellipse, as shown in Fig. 7.8.

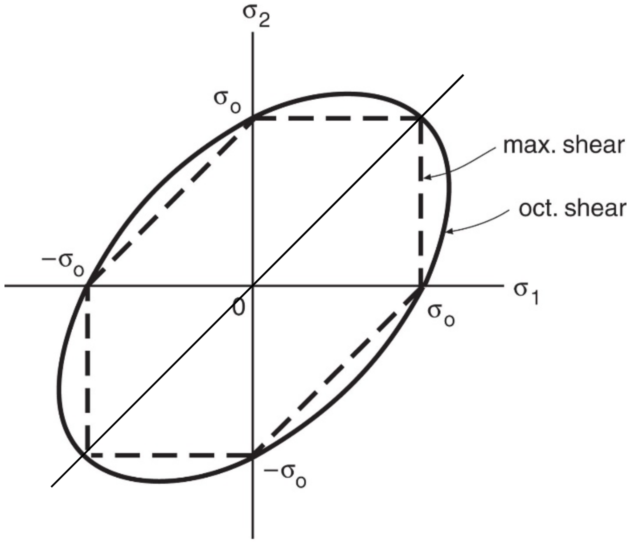


Figure 7.8 Failure locus for the octahedral shear stress yield criterion for plane stress, and comparison with the maximum shear criterion.

7.5 Octahedral Shear Stress Yield Criterion (4)

For the general case, boundaries:

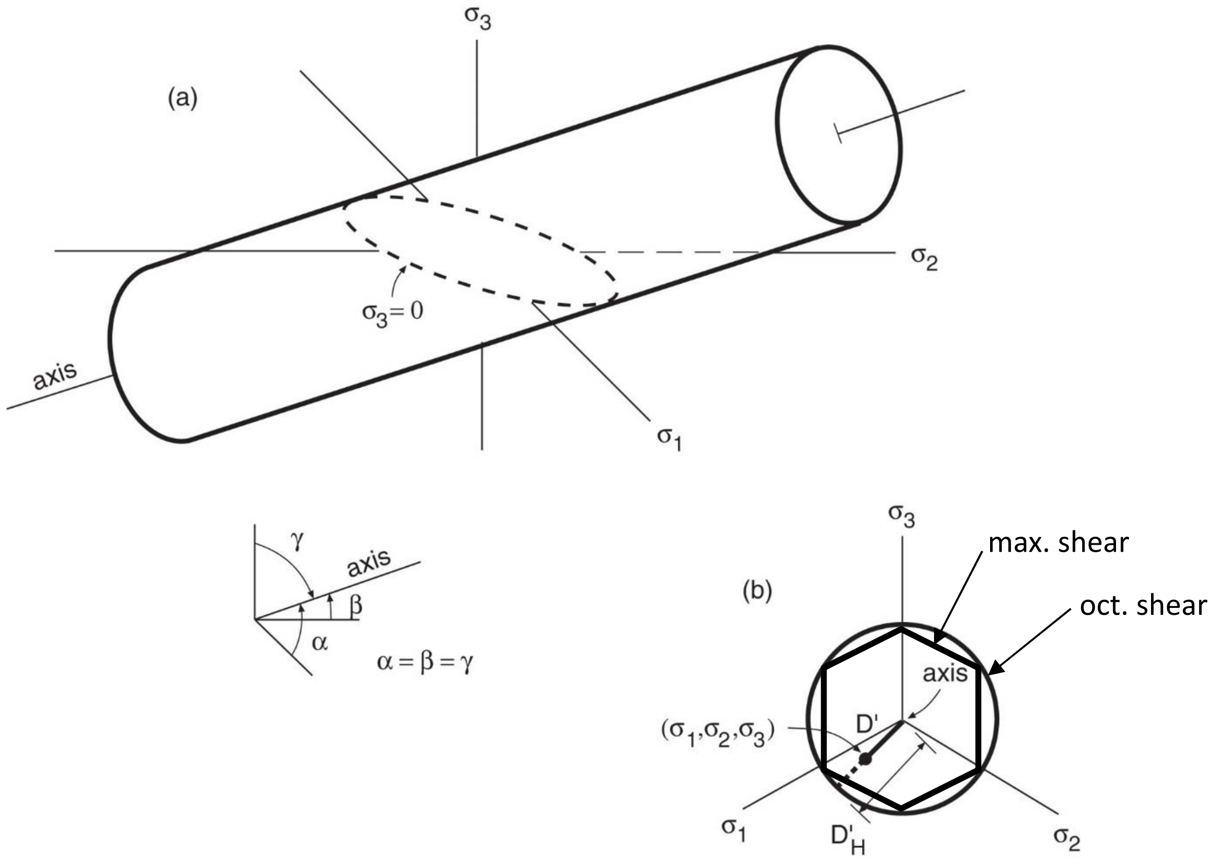


Figure 7.9 Three-dimensional failure surface for the octahedral shear stress yield criterion.

7.6 Comparison of Failure Criteria

- **Ductile materials**
  - Tresca and von Mises criteria are widely used to predict yielding of ductile materials.
  - Both of these indicate that hydrostatic stress does not affect yielding.
  - Because of the small difference between the two, a choice between the two is not a matter of major importance.
- **Brittle materials**
  - They usually follow the normal stress criterion.

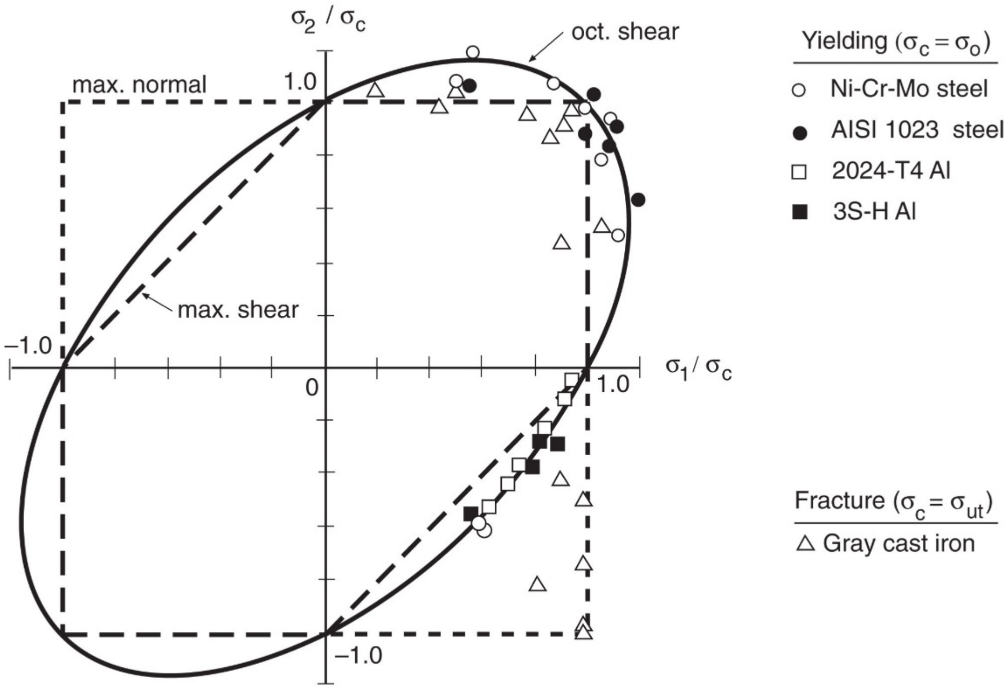


Figure 7.11 Plane stress failure loci for three criteria. These are compared with biaxial yield data for ductile steels and aluminum alloys, and also with biaxial fracture data for gray cast iron. (The steel data are from [Lessells 40] and [Davis 45], the aluminum data from [Naghdi 58] and [Marin 40], and the cast iron data from [Coffin 50] and [Grassi 49].)

- **The anisotropic yield criterion described in Hill:**

$$H(\sigma_X-\sigma_Y)^2+F(\sigma_Y-\sigma_Z)^2+G(\sigma_Z-\sigma_X)^2+2N\tau_{XY}^2+2L\tau_{YZ}^2+2M\tau_{ZX}^2=1$$

$$H+G=\frac{1}{\sigma_{oX}^2},\qquad H+F=\frac{1}{\sigma_{oY}^2},\qquad F+G=\frac{1}{\sigma_{oZ}^2}$$

$$2N=\frac{1}{\tau_{oXY}^2},\qquad 2L=\frac{1}{\tau_{oYZ}^2},\qquad 2M=\frac{1}{\tau_{oZX}^2}$$

In case of polymers, yield strengths in compression is often higher than those in tension.

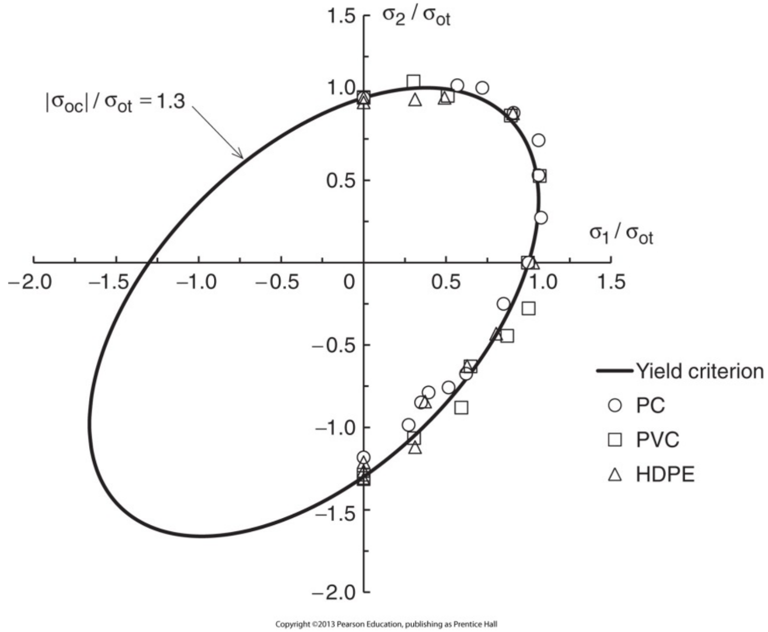


Figure 7.12 Biaxial yield data for various polymers compared with a modified octahedral shear stress theory. (Data from [Raghava 72].)

7.6 Fracture in Brittle Materials

- In brittle materials, deviation from maximum normal stress criterion occur if the normal stress having the largest absolute value is compressive.
- If the dominant stresses are compressive, the planar flaws (cracks, etc.) tend to have their opposite sides pressed together so that they have less effect on the behavior. This explains the higher strengths in compression for brittle materials.
- Also, failure occurs on planes inclined to the planes of principal normal stress and more nearly aligned with planes of maximum shear. (See Figs. 4.23 and 4.24)
- One possibility to handle this fracture behavior in tension and compression is simply to modify the maximum normal stress criterion as shown in Fig. 7.13.

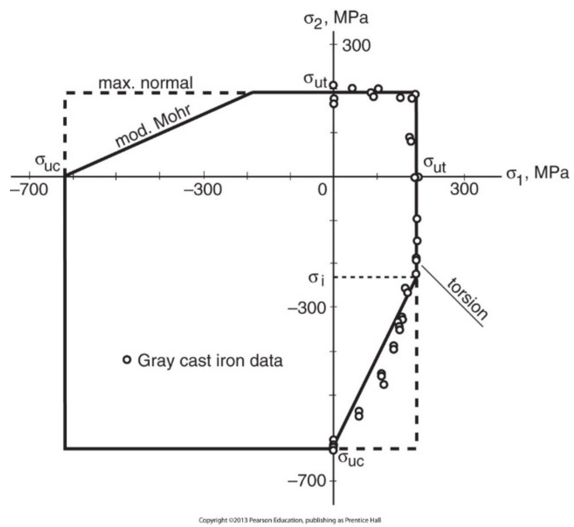


Figure 7.13 Biaxial fracture data of gray cast iron compared with two fracture criteria. (Data from [Grassi 49].)

7.7 Coulomb-Mohr Fracture Criterion (1)

- Hypothesis: Fracture occurs when a critical combination of shear and normal stress acts on the plane.
- Mathematical function (at fracture):

$$|\tau| + \mu\sigma = \tau_i$$

where τ and σ are stresses acting on the fracture plane and μ and τ_i are material constants.

- The failure condition is satisfied if the largest of the three circles is tangent to the Eq. 7.42 line. Otherwise a safety factor greater than unity exists.

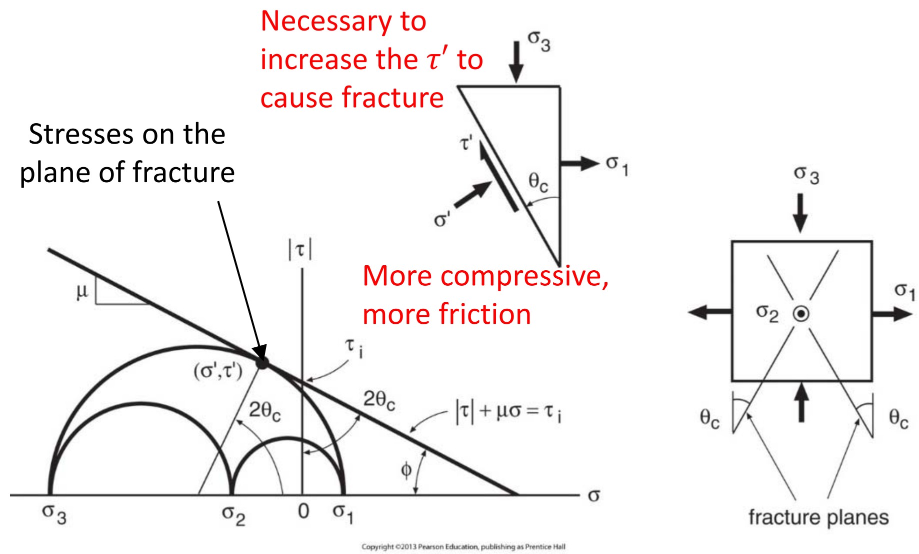


Figure 7.14 Coulomb–Mohr fracture criterion as related to Mohr's circle, and predicted fracture planes.

7.7 Coulomb-Mohr Fracture Criterion (2)

Assume $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (with signs considered).

From the largest Mohr's circle at the point (σ', τ'),

$$\sigma' = \frac{\sigma_1 + \sigma_3}{2} + \left|\frac{\sigma_1 - \sigma_3}{2}\right| \cos 2\theta_c, \qquad |\tau'| = \left|\frac{\sigma_1 - \sigma_3}{2}\right| \sin 2\theta_c$$

$$\cos 2\theta_c = \sin\phi, \qquad \sin 2\theta_c = \cos\phi, \qquad \mu = \tan\phi = \frac{\sin\phi}{\cos\phi}, \qquad \sin^2\phi + cos^2\phi = 1$$

After manipulation,

$$|\sigma_1 - \sigma_3| + (\sigma_1 + \sigma_3)\sin\phi = 2\tau_i \cos\phi$$

$$|\sigma_1 - \sigma_3| + m(\sigma_1 + \sigma_3) = 2\tau_i\sqrt{1 - m^2} \qquad (m = \sin\phi)$$

$$|\sigma_1 - \sigma_3| + m(\sigma_1 + \sigma_3) = |\sigma'_{uc}|(1 - m)$$

In simple compression, $\sigma_3 = \sigma'_{uc}, \sigma_1 = \sigma_2 = 0.$

Therefore, the expected strength σ'_{uc} (negative) is

$$\sigma'_{uc} = -2\tau_i\sqrt{\frac{1+m}{1-m}}$$

In simple tension, $\sigma_1 = \sigma'_{ut}, \sigma_2 = \sigma_3 = 0.$

Therefore, the expected strength σ'_{ut} is

$$\sigma'_{ut} = 2\tau_i\sqrt{\frac{1-m}{1+m}}$$

7.7 Coulomb-Mohr Fracture Criterion (3)

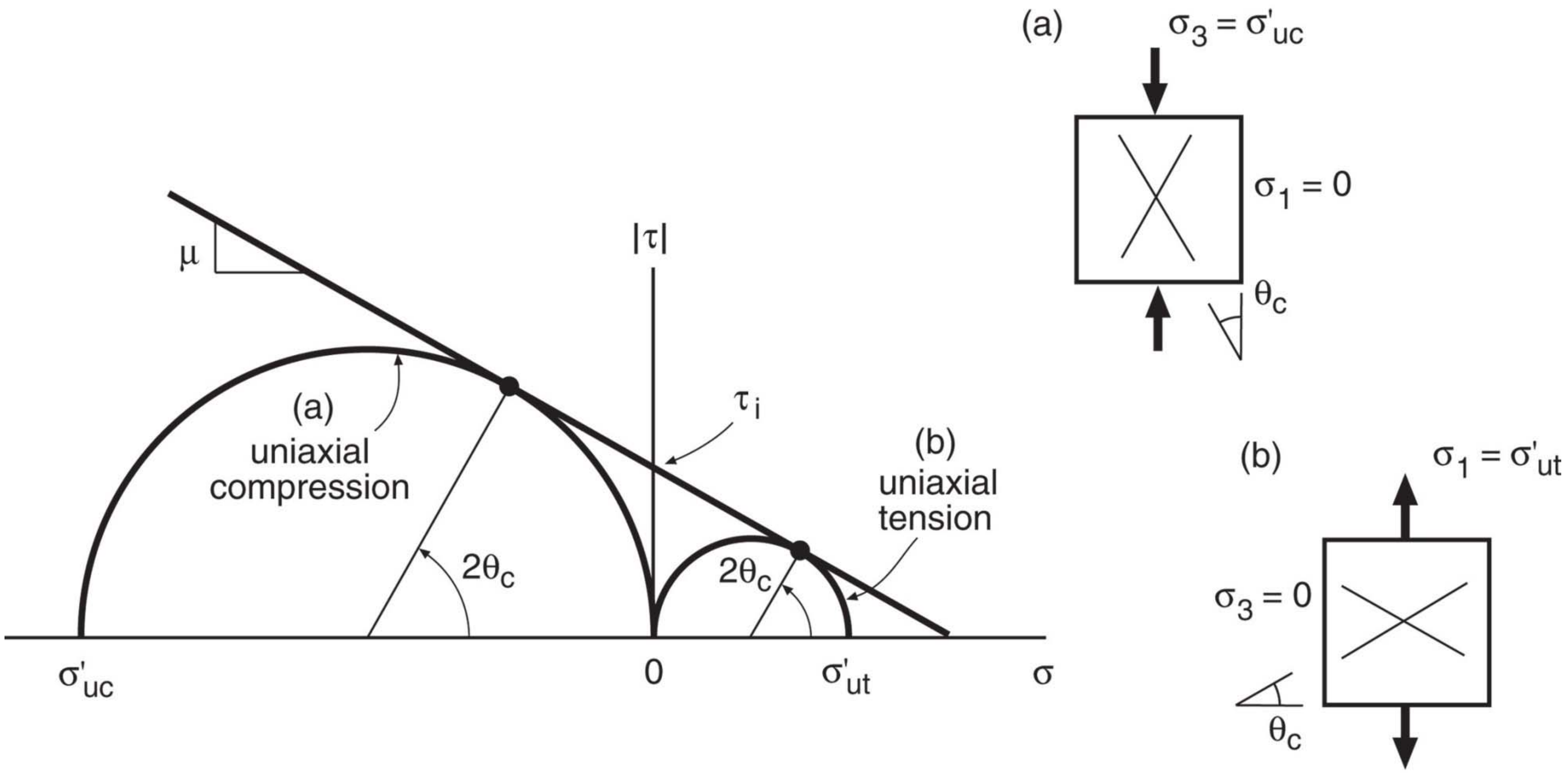


Figure 7.15 Fracture planes predicted by the Coulomb–Mohr criterion for uniaxial tests in tension and compression.

7.7 Coulomb-Mohr Fracture Criterion (4)

In simple torsion, $\sigma_1 = -\sigma_3 = \tau'_u, \sigma_2 = 0.$

Therefore, the fracture strength τ'_u is

$$\tau'_u = \tau\sqrt{1 - m^2}$$

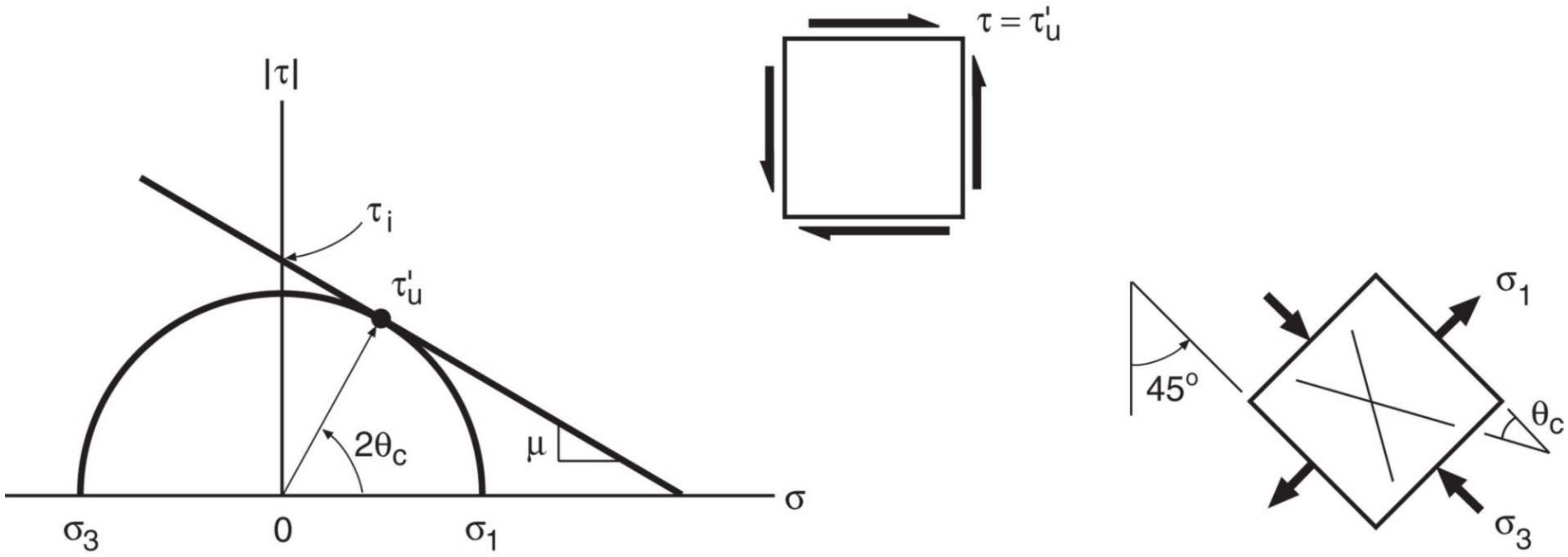


Figure 7.16 Pure torsion and the fracture planes predicted by the Coulomb–Mohr criterion.

7.7 Coulomb-Mohr Fracture Criterion (5)

If the assumption $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is dropped,

$$|\sigma_1 - \sigma_2| + m(\sigma_1 + \sigma_2) = |\sigma'_{uc}|(1-m)$$
$$|\sigma_2 - \sigma_3| + m(\sigma_2 + \sigma_3) = |\sigma'_{uc}|(1-m)$$
$$|\sigma_3 - \sigma_1| + m(\sigma_3 + \sigma_1) = |\sigma'_{uc}|(1-m)$$

For plane stress with $\sigma_3 = 0$,

$$|\sigma_1 - \sigma_2| + m(\sigma_1 + \sigma_2) = |\sigma'_{uc}|(1-m)$$
$$|\sigma_2| + m(\sigma_2) = |\sigma'_{uc}|(1-m)$$
$$|\sigma_1| + m(\sigma_1) = |\sigma'_{uc}|(1-m)$$

And relationship between the fracture strengths in tension and compression is

$$\sigma'_{ut} = |\sigma'_{uc}|\frac{1-m}{1+m}$$

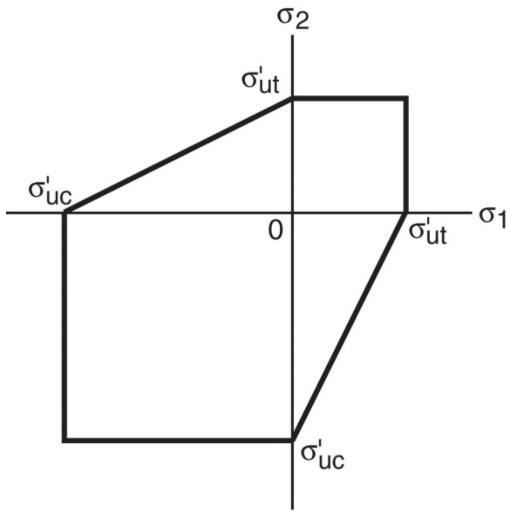


Figure 7.17 Failure locus for the Coulomb–Mohr fracture criterion for plane stress.

7.7 Coulomb-Mohr Fracture Criterion (6)

For the general case, fracture criterion represents six planes as shown in Fig. 7.18. The surface forms a vertex along the line $\sigma_1 = \sigma_2 = \sigma_3$ at the point

$$\sigma_1 = \sigma_2 = \sigma_3 = |\sigma'_{uc}|\frac{1-m}{2m}$$

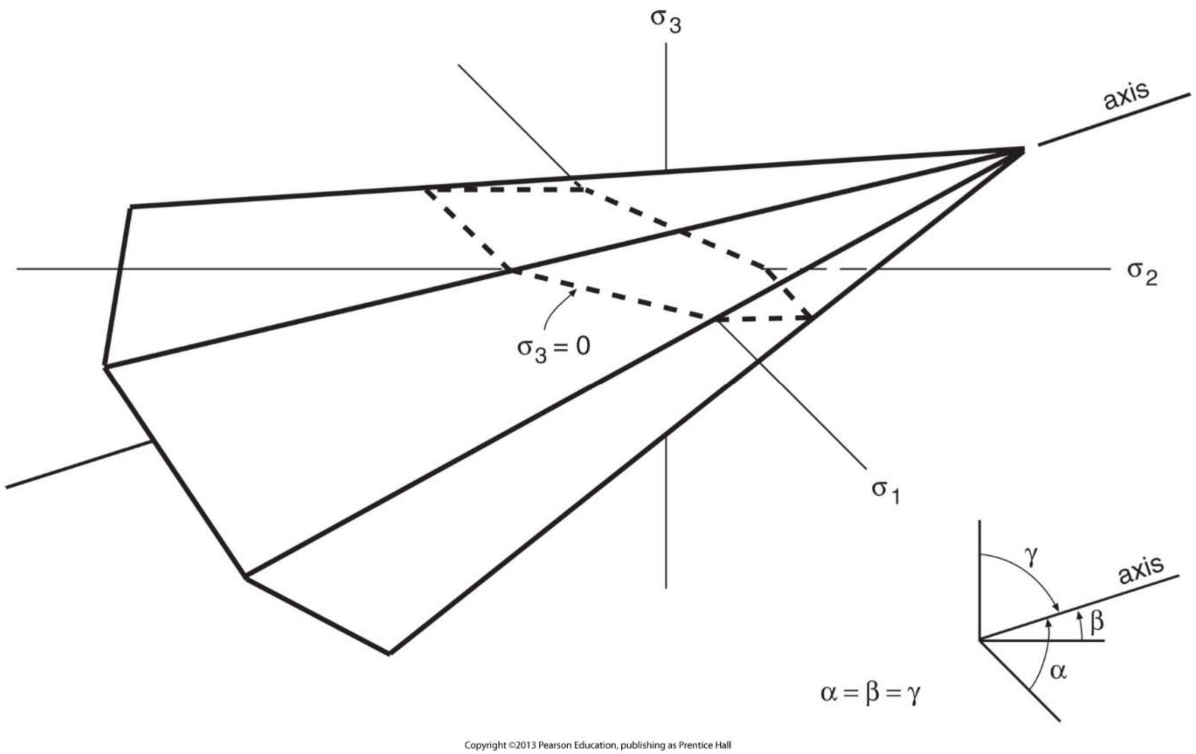


Figure 7.18 Three-dimensional failure surface for the Coulomb–Mohr fracture criterion.

7.7 Coulomb-Mohr Fracture Criterion (7)

- **Effective stress for the C-M criterion**

$$C_{12} = \frac{1}{1-m}[|\sigma_1 - \sigma_2| + m(\sigma_1 + \sigma_2)]$$
$$C_{23} = \frac{1}{1-m}[|\sigma_2 - \sigma_3| + m(\sigma_2 + \sigma_3)]$$
$$C_{31} = \frac{1}{1-m}[|\sigma_3 - \sigma_1| + m(\sigma_3 + \sigma_1)]$$

Hence, effective stress is

$$\bar{\sigma}_{CM} = \mathrm{MAX}(C_{12}, C_{23}, C_{31})$$

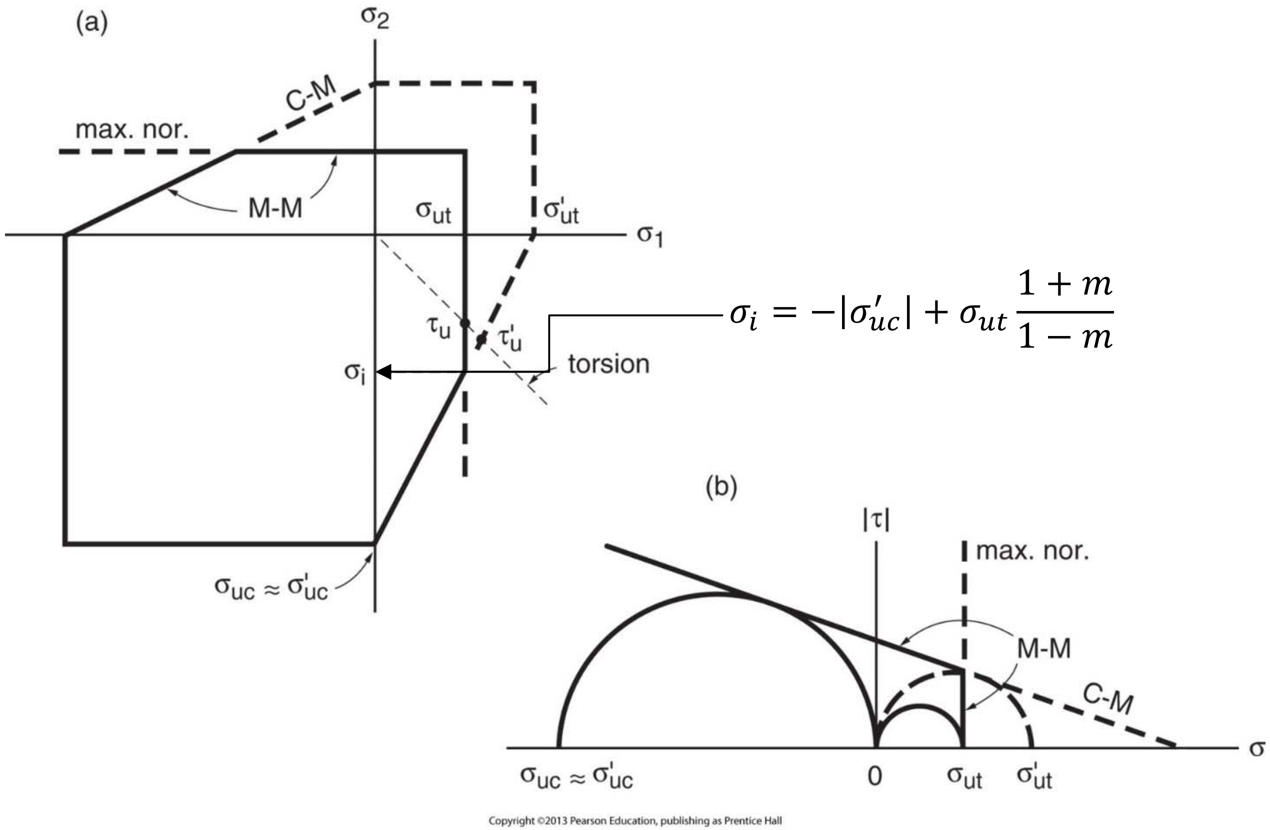
And safety factor against fracture is

$$X_{CM} = \frac{|\sigma'_{uc}|}{\bar{\sigma}_{CM}}$$

- **Discussion**
  - The fracture planes predicted for both tension and torsion are incorrect. Brittle materials generally fail on planes normal to the maximum tension stress in both cases, not on the planes predicted by the C-M theory, as in Figs. 7.15 and 7.16.
  - The fracture strengths in tension, compression, and shear are not typically related to one another as predicted by a single value of *m* used with the previous equations.

7.8 Modified Mohr Fracture Criterion (1) (optional)

- **Modified Mohr (M-M) fracture criterion:** Combination of the C-M criterion and the maximum normal stress fracture criterion



$$\sigma_i = -|\sigma'_{uc}| + \sigma_{ut}\frac{1+m}{1-m}$$

Figure 7.20 The modified Mohr (M-M) fracture criterion, formed by the maximum normal stress criterion truncating the Coulomb–Mohr (C–M) criterion.

7.8 Modified Mohr Fracture Criterion (2) (optional)

- In three dimensions, the M-M failure locus is similar to Fig. 7.21.
- The three positive faces of the normal stress cube correspond to

$$\sigma_1 = \sigma_{ut}, \qquad \sigma_2 = \sigma_{ut}, \qquad \sigma_3 = \sigma_{ut}$$

- Three requirements for M-M criterion
  - The slope of the C-M failure envelope (specified by μ, ϕ, θ_c, or m)
  - The intercept τ_i of the C-M failure envelope
  - The ultimate tensile strength σ_{ut}
- For the maximum normal stress components,

$$\bar{\sigma}_{NP} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad X_{NP} = \frac{\sigma_{ut}}{\bar{\sigma}_{NP}}$$

- Safety factor for the M-M criterion:

$$X_{MM} = \mathrm{MIN}(\mathrm{X_{CM}}, \mathrm{X_{NP}}) \text{ or}$$

$$\frac{1}{X_{MM}} = \mathrm{MAX}\left(\frac{\bar{\sigma}_{CM}}{|\sigma'_{uc}|}, \frac{\bar{\sigma}_{NP}}{\sigma_{ut}}\right)$$

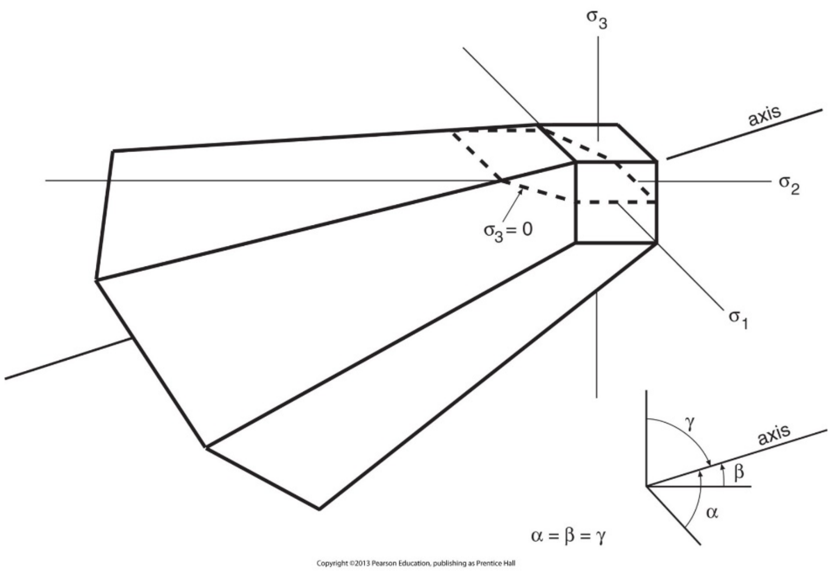


Figure 7.21 Three-dimensional failure surface for the modified Mohr fracture criterion. The Coulomb–Mohr surface is truncated by three faces of the maximum normal stress cube.

7.9 Brittle Versus Ductile Behavior (1)

- Brittle materials may exhibit considerable ductility when tested under loading such that hydrostatic component σ_h of the applied stress is highly compressive.
- Also, materials normally considered ductile fail with increased ductility if the hydrostatic stress is compressive, or reduced ductility if it is tensile.

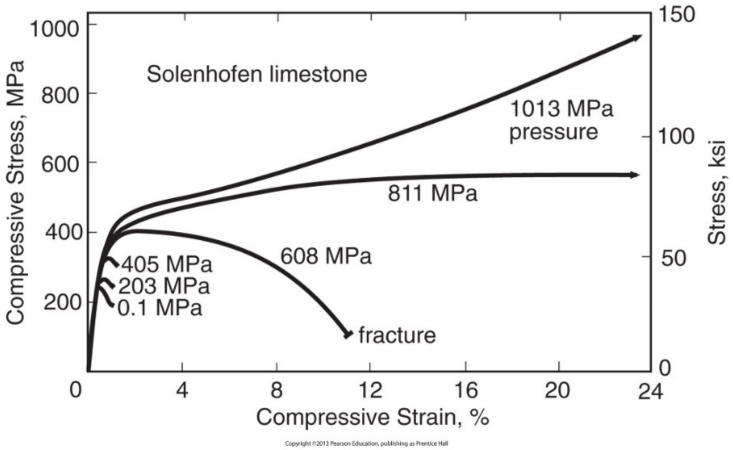


Figure 7.22 Stress–strain data for limestone cylinders tested under axial compression with various hydrostatic pressures ranging from one to 10,000 atmospheres. The applied compressive stress plotted is the stress in the *pressurized laboratory*, that is, the compression in excess of pressure. (Adapted from [Griggs 36]; used with permission; © 1936 The University of Chicago Press.)

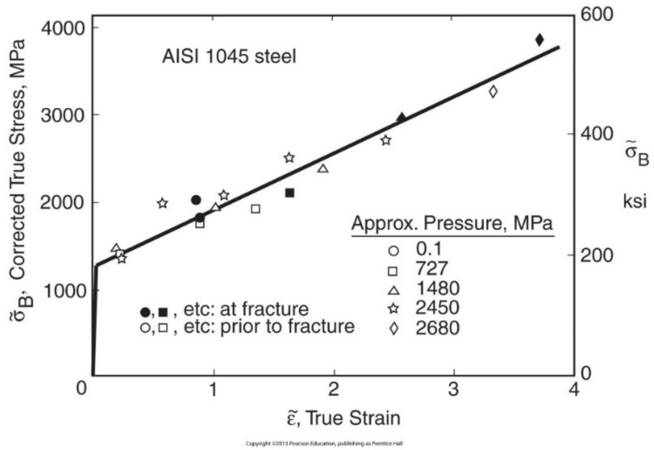


Figure 7.23 Effect of pressures ranging from one to 26,500 atmospheres on the tensile behavior of a steel, specifically AISI 1045 with *HRC* = 40. Stress in the *pressurized laboratory* is plotted. (Data from [Bridgman 52] pp. 47–61.)

7.9 Brittle Versus Ductile Behavior (2)

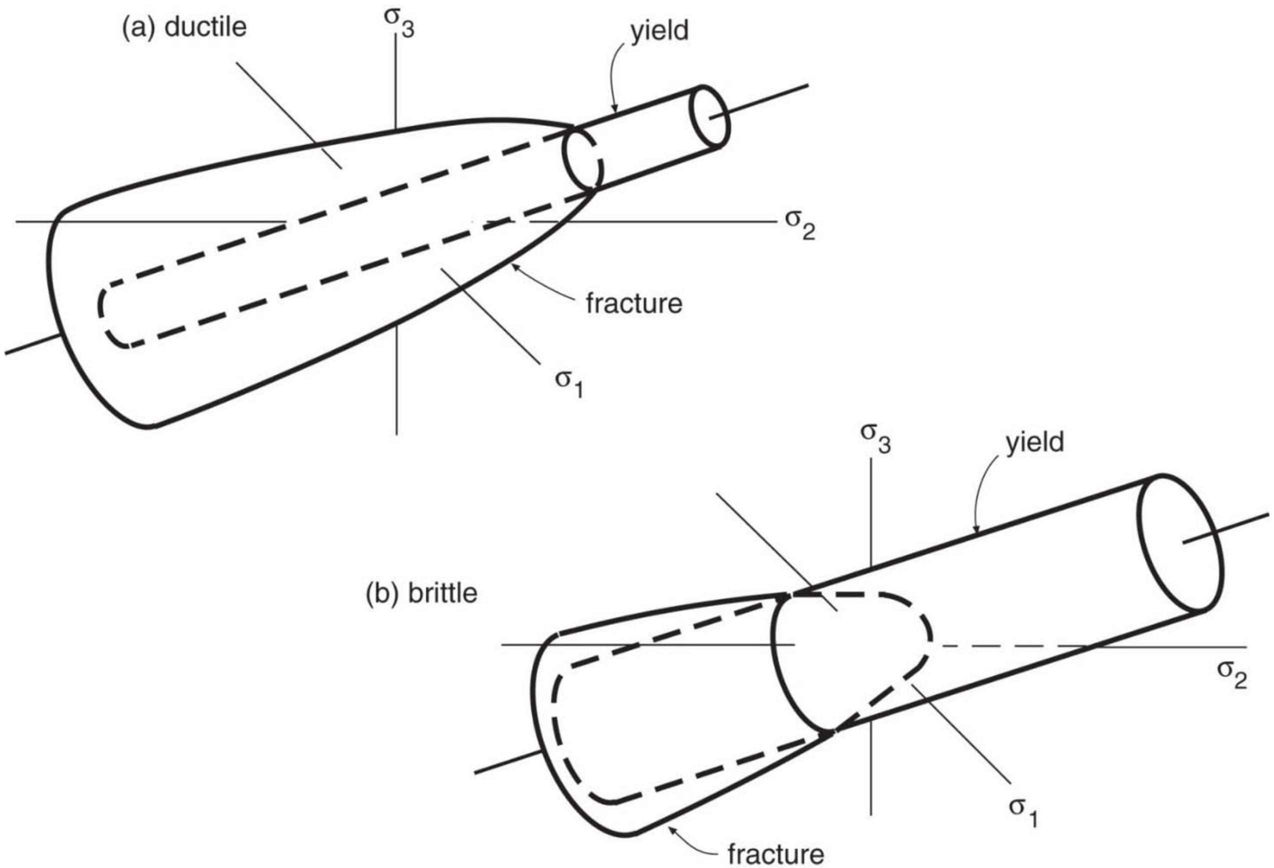


Figure 7.24 Relationships of the limiting surfaces for yielding and fracture for materials that usually behave in a ductile manner, and also for materials that usually behave in a brittle manner.