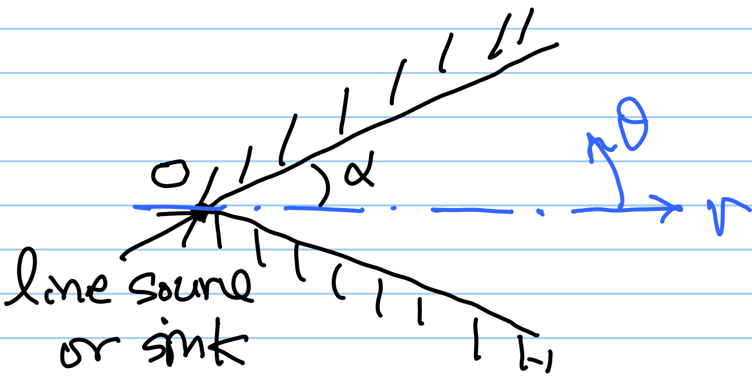
8-2) Flows in Wedge-shaped region (Jefferey-Hamel flows)



- radial flow is induced by source/sink @ $r=0$.
- $u_r = u_r(r, \theta)$, $u_r = \max$ @ $\theta = 0$.
- $u_\theta = u_z = 0$.

- continuity : $\frac{1}{r}\frac{\partial}{\partial r}(r\cdot u_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)} + \cancel{\frac{\partial}{\partial z}(u_z)} = 0.$

- mtm : $u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}\right)$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial u_r}{\partial \theta}$$

from Continuity: $\frac{\partial}{\partial r}(r u_r) = 0 \rightarrow u_r = \frac{1}{r} \cdot F(\theta)$.

i) Substitute $u_r = \frac{1}{r} F(\theta)$ into mtm eqns.

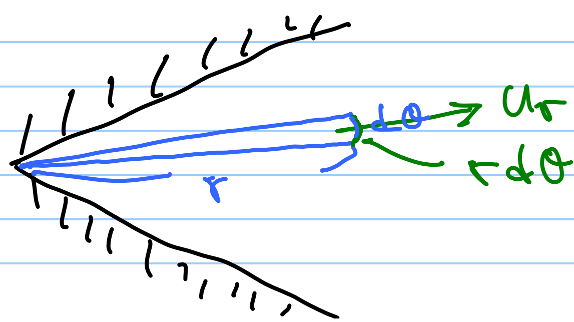
ii) $\frac{\partial}{\partial r}(\theta\text{-mtm eq})$, $\frac{\partial}{\partial \theta}(r\text{-mtm eq.})$

iii) $\frac{1}{\rho}\frac{\partial^2 p}{\partial r \partial \theta}$ as a common term.

$$\Rightarrow \underline{2FF' + \nu F''' + 4\nu F' = 0}.$$

BC) $F = 0$ @ $\theta = \pm\alpha$.

· Volume flow rate, $Q = \int_{-\alpha}^{\alpha} u_r \cdot r d\theta = \int_{-\alpha}^{\alpha} F \cdot d\theta$



· $Re \equiv \frac{UL}{\nu} \sim \frac{Q}{\nu}$ → Define $Re \equiv \frac{\alpha \cdot F_{max}}{\nu}$

$= \frac{\alpha \cdot u_{max} \cdot r}{\nu}$

$Re < 0$: inflow (sink)

$Re > 0$: outflow (source)

Let $\theta/\alpha \equiv \eta$, $F/F_{max} = f = u_r/u_{r,max}$

then, the gov. eqn becomes

$$f''' + 2\alpha Re f f' + 4\alpha^2 f' = 0.$$

$f = 0$ @ $\eta = \pm 1$.

$f = 1, f' = 0$ @ $\eta = 0$.

↓ integrate

$$\left[f'' + \alpha \cdot Re\, f^2 + 4\alpha^2 f = \underline{b} \right] \times f' \text{ and integrate}$$

$$\rightarrow \frac{1}{2}(f')^2 + \frac{1}{3}\alpha \cdot Re\, f^3 + 2\alpha^2 f^2 = \underline{b} f + \underline{d}.$$

apply BC @ $\eta = 0$, $(f')^2 = (1-f)\left\{\frac{2}{3}\alpha \cdot Re\,(f^2+f) + 4\alpha^2 f + \underset{\substack{\shortparallel \\ 2d}}{c}\right\}$

" " $\eta = \pm 1$, $(f')^2 = c$ @ $\eta = 1$.

limiting case

① case i ($\alpha \ll 1$, $\alpha \cdot Re \ll 1$)

$$f''' = 0. \rightarrow f = 1 - \eta^2 = \frac{u_r}{u_{r,max}} = 1 - \left(\frac{\theta}{\alpha}\right)^2.$$

→ similar to Poiseuille flow distribution.
(valid for inflow or outflow)

② case : $\alpha \ll 1$, but $\alpha \cdot Re \gg 1$.

(← most practical applications)

$$(f')^2 = \underline{\underline{\tfrac{2}{3}\alpha \cdot Re}} \cdot f(1-f)^2 + c(1-f).$$

← flow depends on the single parameter,

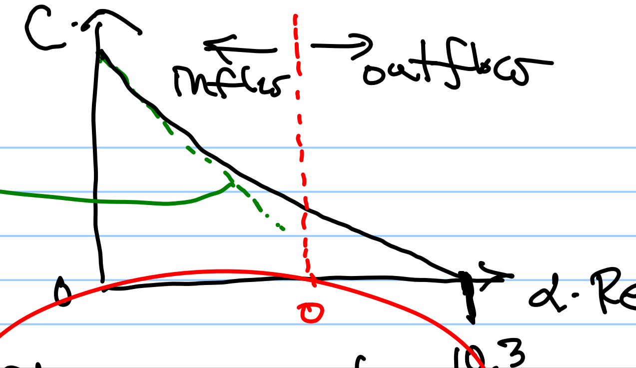
$\downarrow$

$$\left(\tfrac{2}{3}\alpha \cdot Re\right)^{1/2} = \int_0^1 \frac{1}{(1-f)(f^2+f+K)}\, df. \qquad K = \frac{c}{\tfrac{2}{3}\alpha \cdot Re}.$$

↳ for a given $K = \dfrac{c}{\tfrac{2}{3}\alpha \cdot Re}$, "c" is varying with respect to $\alpha \cdot Re$ → multiple solutions are possible.

if $Re < 0$ (inflow).

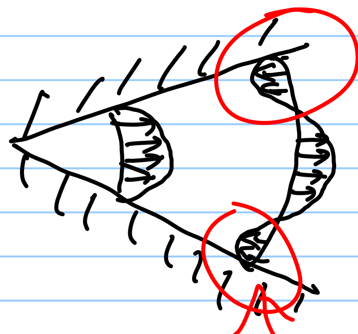
$C \simeq -\frac{4}{3}\cdot(\alpha\cdot Re)$



if $Re > 0$ (outflow)

$C = 0$ at $\underline{\alpha\cdot Re \simeq 10.3}$ $\Rightarrow$ flow separation occurs.

favorable press. grad.



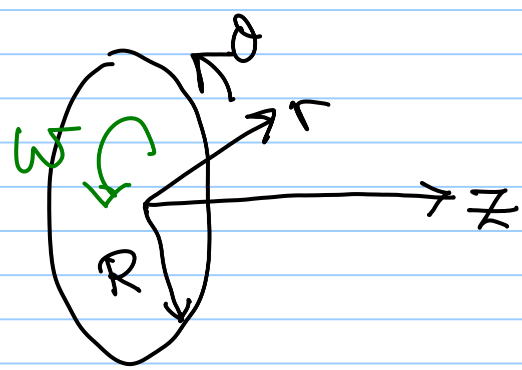
* Characteristics of wedge-shaped flow

① convergent (accelerating) case → thin viscous layer

② divergent (diffusing) case → at critical Re. ($\alpha \cdot Re \simeq 10.3$), flow separates. (everywhere viscous)

↳ similar to conventional boundary-layer behavior.

f-3) Flow near an infinite rotating DISC



$R \to \infty$

{ Von Karman viscous pump.

(refer to "White")

↳ viscous drag at disc sets up a

· axisymmetric: $\frac{\partial}{\partial \theta} = 0$. swirling (3D) flow toward the disc.

· steady, incompressible, constant ρ.

· from continuity & 3 mtm eqns $\rightarrow$ p, u_r, u_θ, u_z.

BC) @ $z = 0$, $u_r = u_z = 0$. $u_\theta = r\omega$, $p = \text{const}$.

@ $z \rightarrow \infty$, $u_r = 0$, $u_\theta = 0$ $\frac{\partial u_z}{\partial z} = 0$.

assume non-dimensional form of u_θ.

$$\frac{u_\theta}{r\omega} = G(z)$$

$\rightarrow$ u_r : induced by centrifugal force, should be scaled in the same way.

$$\frac{u_r}{r\omega} = F(z)$$

· since the viscous diffusion works in z-direction.

, introduce viscous length $\sqrt{\nu/\omega}$.

to make $z^* = z/\sqrt{\nu/\omega}$.

· from the Continuity, $H(z) = u_z/\sqrt{\nu\omega}$, also

define, $P^*(z) = P/\rho\nu\omega$.

$$\Rightarrow \begin{cases} H' + 2F = 0 \\ F^2 + F'H - G^2 = F'' \\ FG + HG' + FG = G'' \\ 2HF - 2F' = p^{*\prime} \end{cases}$$

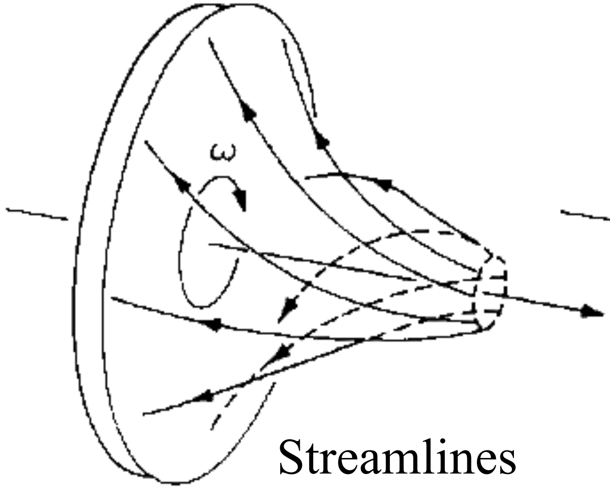
$\downarrow$ numerical integration.

@ $z^* = 0$,
$u_\theta = r\omega \rightarrow G(0) = 1$
$u_r = 0 \rightarrow F(0) = 0$
$u_z = 0 \rightarrow H(0) = 0$.
$p = 0 \rightarrow p^*(0) = 0$

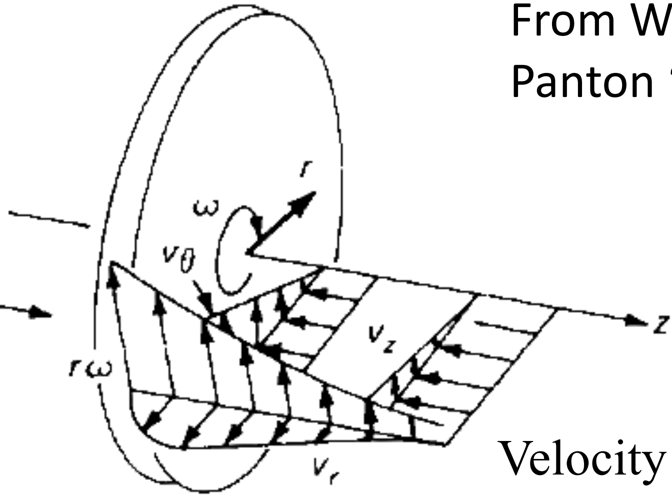
@ $z^* \rightarrow \infty$,
$u_r = 0 \rightarrow F(\infty) = 0$
$u_\theta = 0 \rightarrow G(\infty) = 0$.

Von Karman's Viscous Pump Principle

From White "Viscous Fluid Flow" and Panton "Incompressible Flow"



Streamlines



Velocity Profiles

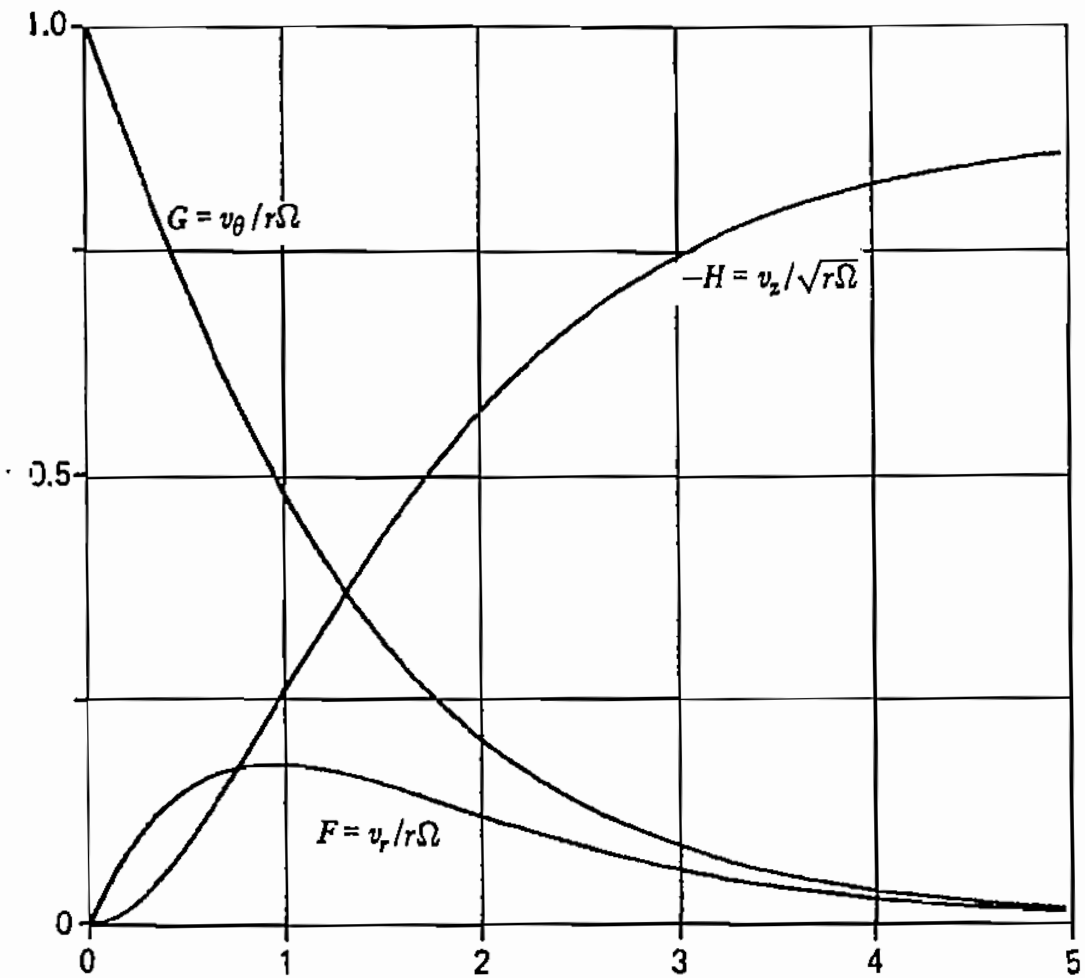
$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0$$

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right)$$

$$v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{1}{r}v_r v_\theta = \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2}\right)$$

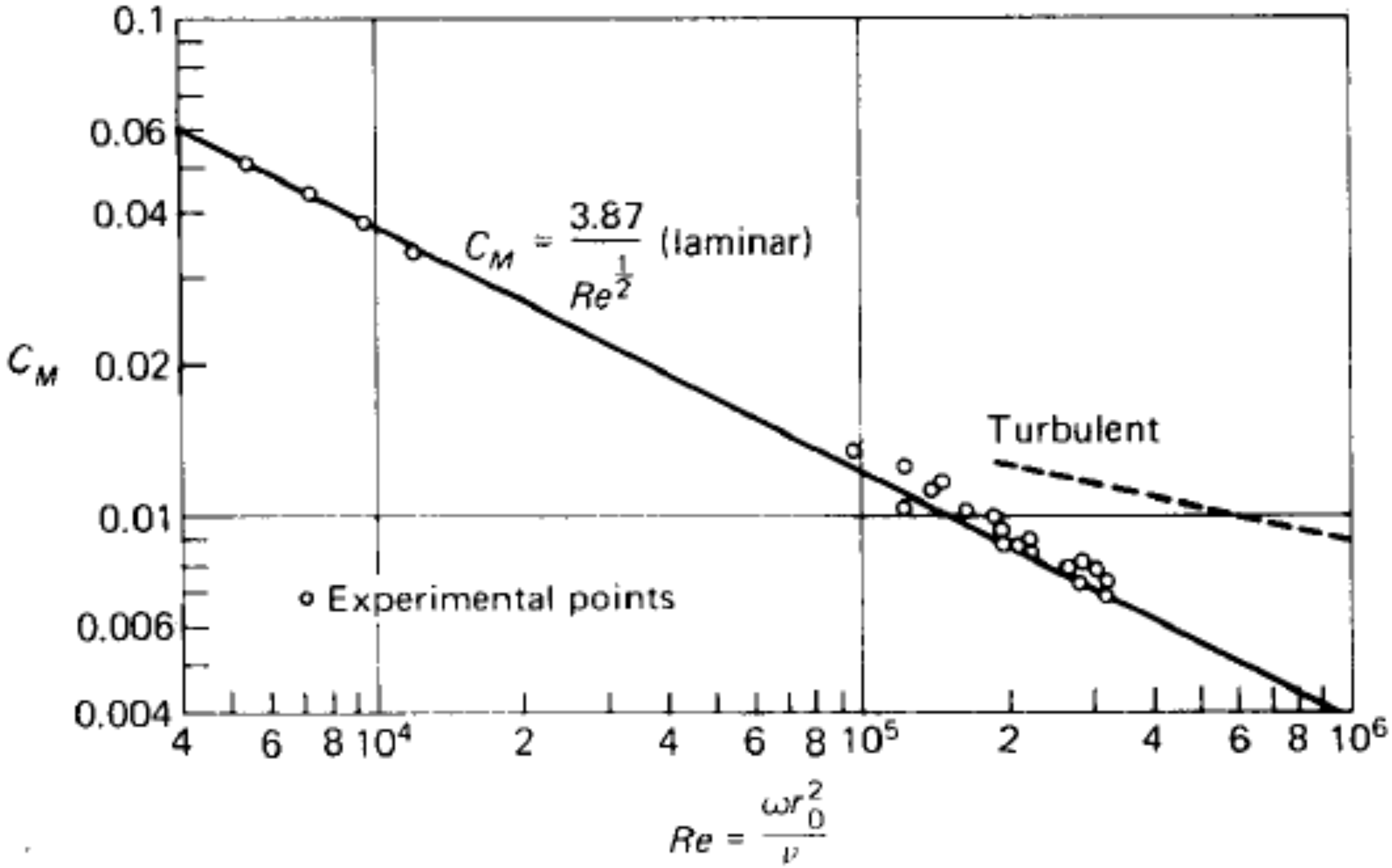
$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Von Karman's Viscous Pump Principle



$$z^* = \frac{z}{\sqrt{\nu / \omega}}$$

Von Karman's Viscous Pump Principle



Let's define the thickness, δ (where the swirling flow near the disk is confined) as the point where u_θ drops 1% of its wall value ; $G \equiv u_\theta / r\Omega = 0.01$
value at the

$\rightarrow \delta \simeq 5.5 \sqrt{\frac{\nu}{\omega}}$.

· Circumferential wall shear stress, $\tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z} \Big|_{z=0}$.

$\rightarrow$ ~~total~~ torque required to turn a disk of radius, r_o.

$$M = \int_0^{r_o} \tau_{z\theta} \cdot r \cdot dA = \int_0^{r_o} \tau_{z\theta} \cdot r \, (2\pi r \cdot dr).$$