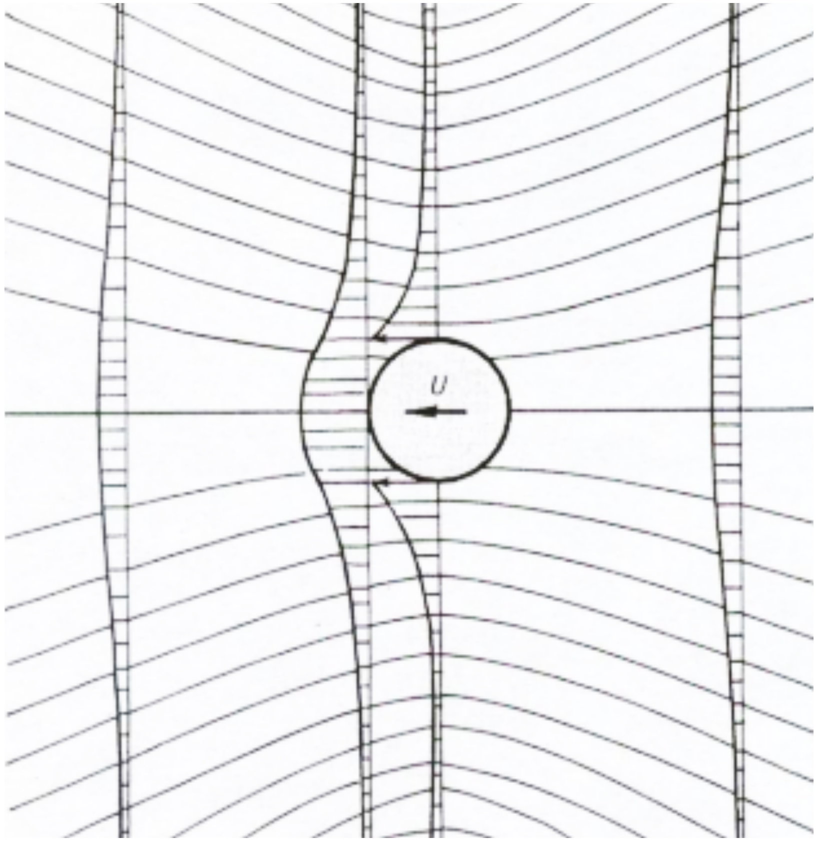
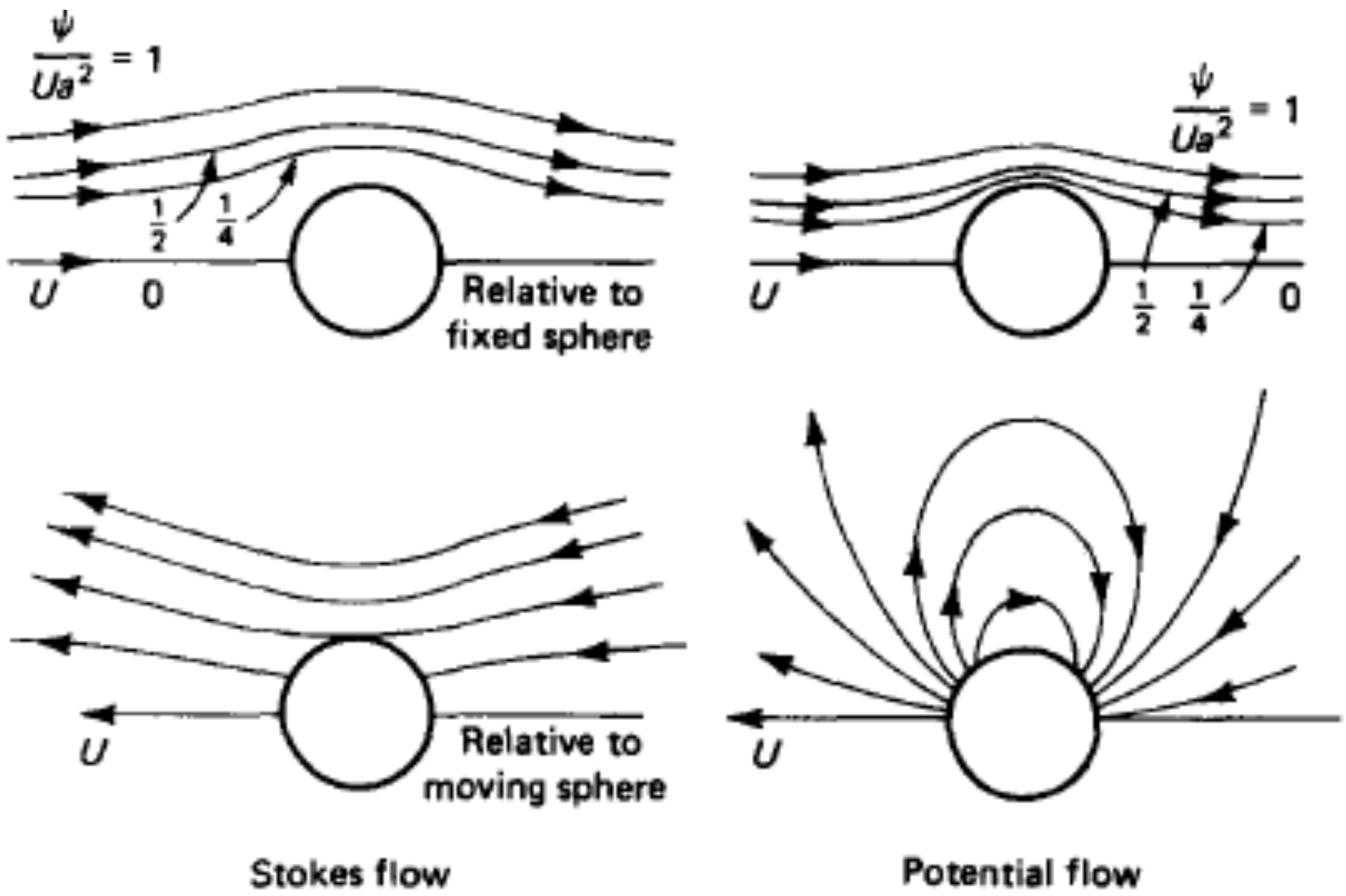
Creeping Flow around a Sphere



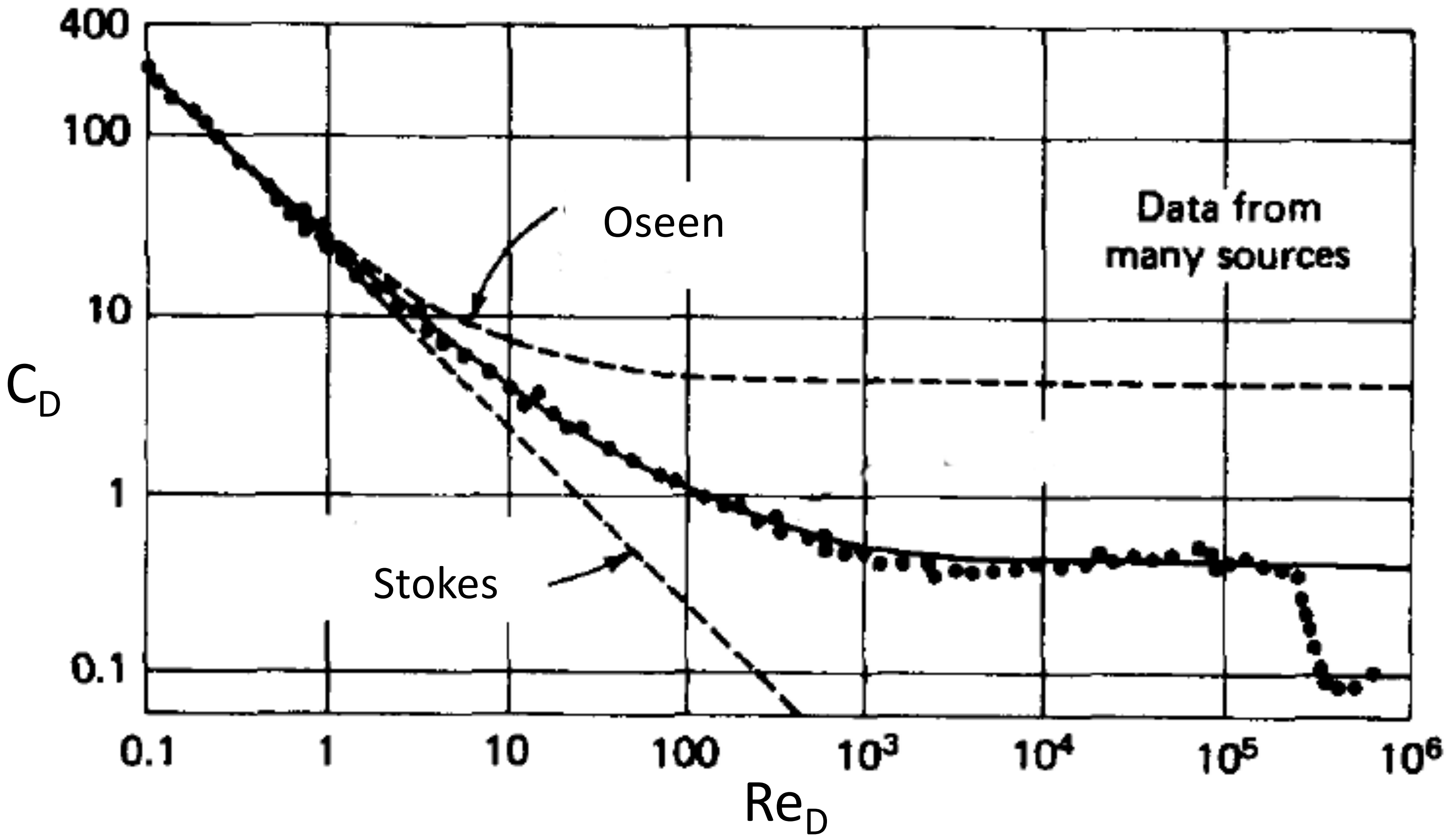
- Streamlines and velocity distributions in Stokes' solution of creeping flow due to a moving sphere
- This varies with Re_D
- There is no vortex shedding in the wake
- Viscous effects are felt everywhere

Creeping Flow around a Sphere



From White "Viscous Fluid Flow"

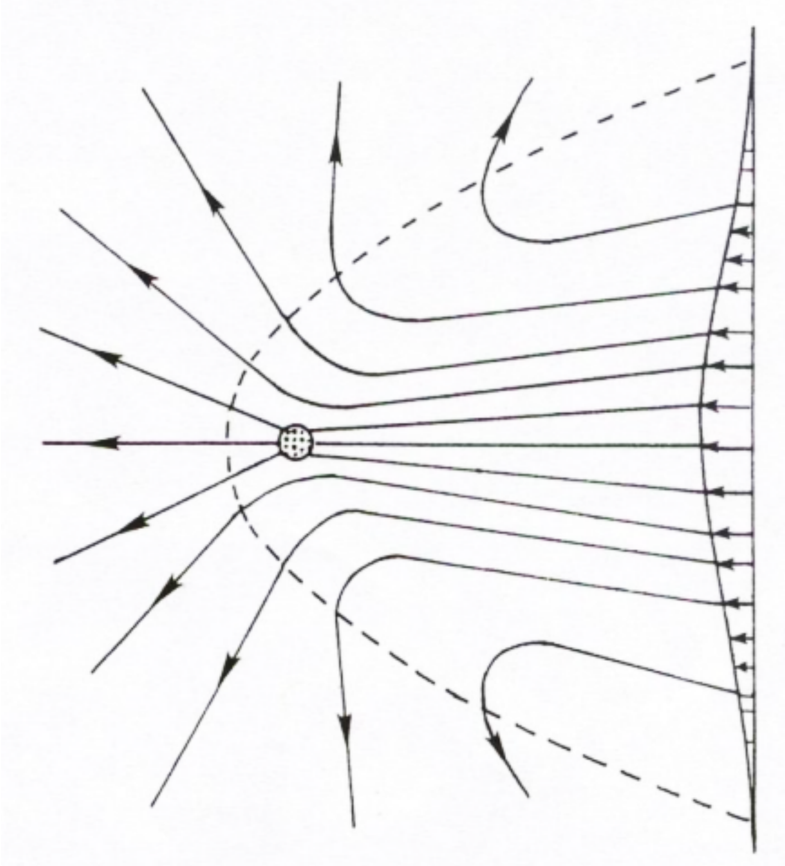
Creeping Flow around a Sphere

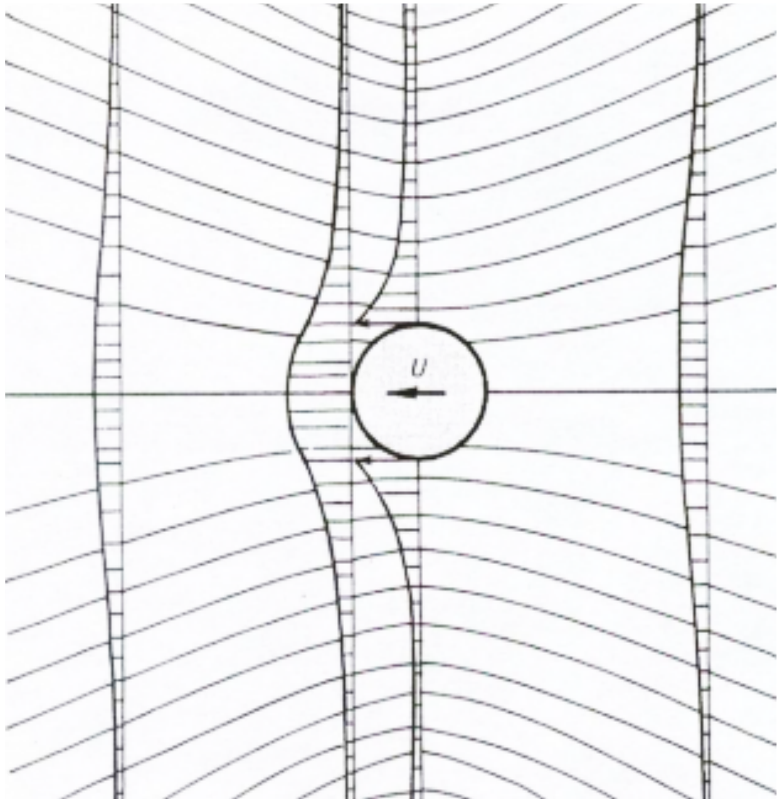


From White "Viscous Fluid Flow"

Creeping Flow around a Sphere

- Streamlines and velocity distributions in Oseen's solution of creeping flow due to a moving sphere.





- Note the upper stream and downstream asymmetry, a result of partial accounting for advection in the far field

* Non-uniformity of Stokes Solution & Oseen's Improvement.

- as $r \to \infty$ (far away from the sphere) : Stokes sol. is NOT valid.

- $\dfrac{\text{inertia (advection)}}{\text{viscosity}} \sim Re \cdot \dfrac{r}{a}$.

↳ even if $\underline{Re \ll 1}$
if r/a is large enough
to have $r/a > \dfrac{1}{Re}$
$\left(Re \cdot \dfrac{r}{a} \gg 1\right)$

- $\underline{\bar{u} \cdot \nabla \bar{u} = -\nabla p + \mu \nabla^2 \bar{u}}$. ← $\circledast$

→ inertia becomes important.
(at far-field) → limits of Stokes's solution.

· Oseen's solution.

$$u = U_0 + u', \quad v = v', \quad w = w'. \quad \text{—} (**)$$

U_0: free-stream.

u', v', w': perturbation velocity ($\ll 1$) (due to sphere)

mtm conservation

(*) + (**) and linearize.

$$\therefore \rho U_0 \frac{\partial u_i'}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 \bar{u}_i \quad : \text{Oseen's eq.}$$

$\bar{u} \cdot \nabla \bar{u}$ (linearized form)

BC) @ $r = a$. $u' = -U_0$, $v' = w' = 0$.

@ $r \to \infty$, $u' = v' = w' = 0$.

$\hookrightarrow C_D = \frac{24}{Re}\left(1 + \frac{3}{16}Re\right)$: Osean's sol.

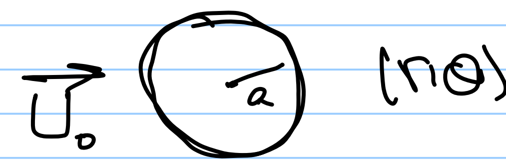
cf) Stokes's sol : $C_D = \frac{24}{Re}$.

MIDTERM. 10/31 (THU). 15:30 – 2HOURS, 302-508.

(3) Flow past a circular cylinder(?)

- Do we have a counter part of Stokes' solution for a sphere (3D) in 2D (i.e., circular cylinder)? No.

- $\nabla^2 \bar{\omega} = 0$. in r-θ coordinates.



↳ we seek $\psi(r,\theta) = f(r)\cdot \sin\theta$.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2 \psi = 0$$

$$\Rightarrow f(r) = Ar^3 + Br\cdot\log r + Cr + \frac{D}{r}$$

BC: @ $r \to \infty$, $f(r) = U_0 r$: $A = B = 0$, $C = U_0$.

@ $r = a$, no-slip $\left(\frac{\partial \psi}{\partial r} = 0, \ \frac{\partial \psi}{\partial \theta} = 0 \right)$.

↳ No "D" can satisfy this BC!

⇒ "Stokes' Paradox".

· if we re-visit stokes' approximation.

- valid @ $Re \to 0$.

- $\psi = \psi_0 + \psi_1 Re + \psi_2 Re^2 + \psi_3 Re^3 + \cdots$.

(1) in 3D, ψ_0 exist. → Stokes sol. ↓ NOT valid as $r \to \infty$

(sphere)

ψ_1 was added.

(Oseen's improvement)

② in 2D (circular cylinder), ψ_0 does not exist
(∵ inertia term can't be neglected at all)

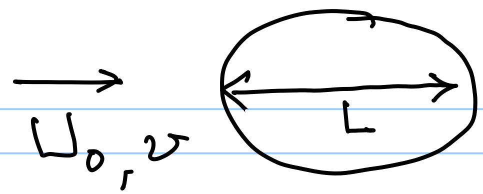
MIDTERM
HW#2, etc.

BOUNDARY LAYERS

· As Re ↑, convection of vorticity (inertia) >> diffusion of vorticity (inertia)

⇒ viscous region becomes thin.

↳ outer : inviscid, irrotational ⇒ free-stream
inner : viscous, rotational ⇒ bdry.



$$\begin{cases} \text{residence time} : L/U_0 \\ \text{diffusion} \;\;\text{“}\;\; : \sqrt{\dfrac{\nu L}{U_0^3}} \end{cases}$$

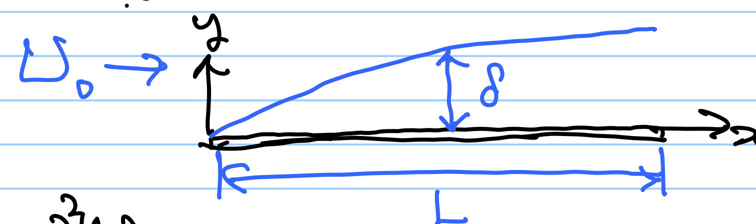
if diffusion time is much shorter than residence time.

$$\sqrt{\frac{\nu L}{U_0^3}} \ll \frac{L}{U_0} \longrightarrow \sqrt{\frac{U_0 L}{\nu}} = \sqrt{Re} \gg 1 .$$

⓪ order of magnitude analysis (Refer to Schlichting)

① 2D, ② incompressible, ③ flat-plate boundary-layer flow

④ thin boundary layer



$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \end{cases}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

w/ reference velocity, U_0, $\hat{u} \equiv u/U_0$.

$\hat{p} \equiv p/\rho U_0^2$, $\hat{t} \equiv t/\tau$. $\hat{x} \equiv x/L$.

$$\Rightarrow \underset{1}{\frac{\partial \hat{u}}{\partial \hat{x}}} + \underset{1}{\frac{\partial \hat{v}}{\partial \hat{y}}} = 0$$

$O(1)$

$O(\delta) \sim O(0)$

$$\left(\frac{L}{U_0 \tau}\right)\underset{1}{\frac{\partial \hat{u}}{\partial \hat{t}}} + \underset{1}{\hat{u}}\,\underset{1}{\frac{\partial \hat{u}}{\partial \hat{x}}} + \underset{\delta}{\hat{v}}\,\underset{1/\delta}{\frac{\partial \hat{u}}{\partial \hat{y}}} = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{Re_L}\left(\underset{1}{\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}} + \underset{1/\delta}{\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}}\right)$$

$$\left(\frac{L}{U_0 \tau}\right)\underset{\delta}{\frac{\partial \hat{v}}{\partial \hat{t}}} + \underset{1}{\hat{u}}\,\underset{\delta}{\frac{\partial \hat{v}}{\partial \hat{x}}} + \underset{\delta}{\hat{v}}\,\underset{1}{\frac{\partial \hat{v}}{\partial \hat{y}}} = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{1}{Re_L}\left(\underset{\delta}{\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}} + \underset{1/\delta}{\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}}\right), \quad Re_L \equiv \frac{U_0 L}{\nu}.$$

Select L so that $\delta/L \ll 1$, ($\delta \ll L$) and $\frac{\partial \hat{u}}{\partial \hat{x}} \sim O(1)$.

- $\hat{u} \equiv u/U_0 \sim O(1)$. $\rightarrow$ $\hat{x} \sim O(1)$, $\hat{y} \equiv y/L \sim O(\delta) \sim O(\epsilon)$.

$\rightarrow \dfrac{\partial \hat{u}}{\partial \hat{y}} \sim O(1)$. $\rightarrow$ $\hat{v} \sim O(\delta)$.

$\rightarrow \dfrac{\partial \hat{v}}{\partial \hat{x}} \sim O(\delta)$, $\dfrac{\partial^2 \hat{v}}{\partial \hat{x}^2} \sim O(\delta)$, $\dfrac{\partial^2 \hat{u}}{\partial \hat{x}^2} \sim O(1)$.

$\dfrac{\partial \hat{u}}{\partial \hat{y}} \sim O(1/\delta)$, $\dfrac{\partial^2 \hat{u}}{\partial \hat{y}^2} \sim O(1/\delta^2)$.

If $L/U_0\tau \gg 1$: Rayleigh BL.

$\ll 1$: Quasi-steady BL

$\tau \to \infty$: steady BL.

· Assume steady flow.

· from x-mtm : $\frac{1}{Re_L} \cdot \frac{1}{\delta^2} \sim O(1) \rightarrow Re_L \sim \frac{1}{\delta^2}$. $(Re_L \gg 1)$

· BL concept is valid at higher Re.

· from y-mtm : $\frac{1}{Re_L} \cdot \frac{1}{\delta} \sim O(\delta)$, $-\frac{\partial \hat{p}}{\partial \hat{y}} \sim O(\delta)$

$$\therefore \frac{\partial \hat{p}/\partial \hat{x}}{\partial \hat{p}/\partial \hat{y}} \sim O(\delta) \qquad \therefore \frac{\partial \hat{p}}{\partial \hat{x}} \gg \frac{\partial \hat{p}}{\partial \hat{y}}.$$

$$\left(\begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \end{array} \right.$$

: Prandtl's Boundary-Layer eqn.