Precision Metrology2

σ_n vs σ_{n-1}

σ_n : Standard deviation when the deviation is calculated from an assigned value (not the average value of the measurement data)

No. of Independent terms (degree of freedom) =n

Standard deviation, $\sigma_n = \sqrt{\sum(X_i-X_o)^2/n}$

where X_o=assigned value and not mathematically related to the measurement data

σ_{n-1} :Standard deviation when the deviation is calculated from the average value of the measurement data.

No. of Independent terms(degree of freedom) =n-1

Standard deviation, $\sigma_{n-1} = \sqrt{\sum(X_i-\underline{X})^2/n-1}$

where $\underline{X}$=average value=$\sum X_i/n$

Behaviour of Single value and Average value

Set of Measurement Values: X_1, $X_2 \ldots X_n$ → can be represented as ($\underline{X}$, σ)

Range of X_i (Range of Single measurement):

$$| X_i - \underline{X} | \leq k\sigma$$

$$\rightarrow \underline{X} - k\sigma \leq X_i \leq \underline{X} + k\sigma,$$

(k=3 @99% probability)

Range of X_m (Range of average of m repeat measurement):

$$| X_m - \underline{X} | \leq k\sigma / \sqrt{m}$$

$$\rightarrow \underline{X} - k\sigma / \sqrt{m} \leq X_m \leq \underline{X} + k\sigma / \sqrt{m},$$

(k=3 @99% probability)

∴ Measurement repeatability can be greatly reduced

"Beauty of Average"

Evaluation of Systematic Error and Random Error

True Value=Nominal Value + Error

Error=True Value – Nominal Value

$= T_i - N_i = X_i$, $i=1,2\ldots n$, n=No. of repeat measurement

Systematic Error = Average of Error = $\underline{X} = \sum X_i / n$

Random Error (Random uncertainty) = $\pm 3\sigma$

(@99% Probability)

Repeatability can be expressed as the Random uncertainty.

Error=Systematic Error + Random Error (uncertainty)

$= \underline{X} \pm 3\sigma = \Delta \pm R$

HW1’)Diametre Measurement of a Cylinder using a Vernier Calipers

Nominal Value=14.5mm

10 Measured Values

14.9; 14.6; 14.8; 14.9; 14.6; 14.7; 14.7; 14.8; 14.9; 14.8

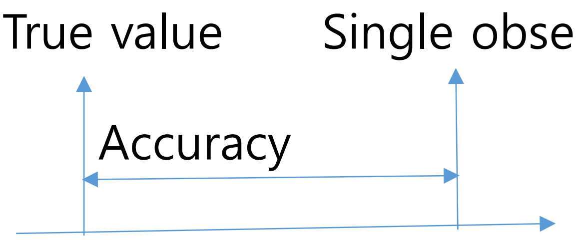
Questions:

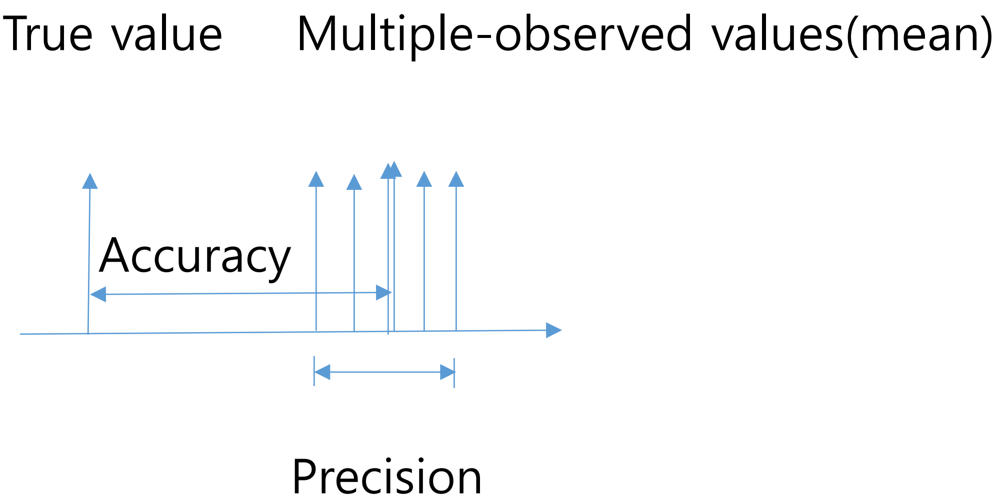
0) Resolution, Spread,
1) Average of 10 measured values
2) Standard deviation from the average
3) Standard deviation from the nominal value, 14.5
4) Systematic error
5) Random error (random uncertainty)
6) Repeatability
7) Range of a single measurement value
8) Range of average of 5 repeat measurement
9) Range of average of 10 repeat measurement

Accuracy and Precision

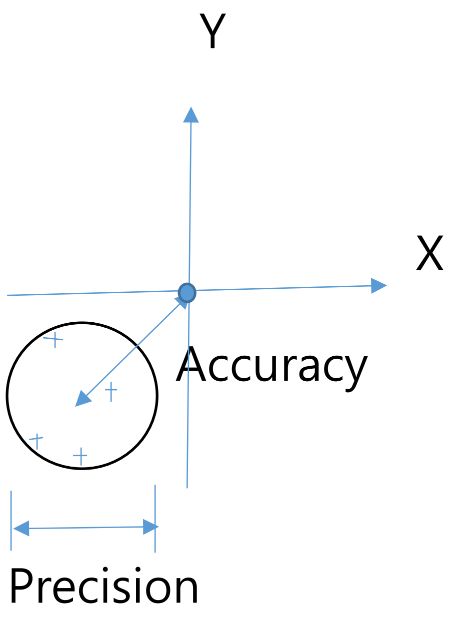
Accuracy: The closeness of an observed quantity to the true value (by ISO). It can be evaluated as the systematic error.

Precision: The closeness of agreement between the repeat measurement results (by ISO). It can be evaluated as the random error(random uncertainty), or repeatability.





Accuracy and Precision in 2D plane



"The smaller the random error, the more precise, and higher repeatability"

"The smaller the systematic error, the more accurate"

Ex) Precision Machine, Precision Engineering, Machine Accuracy

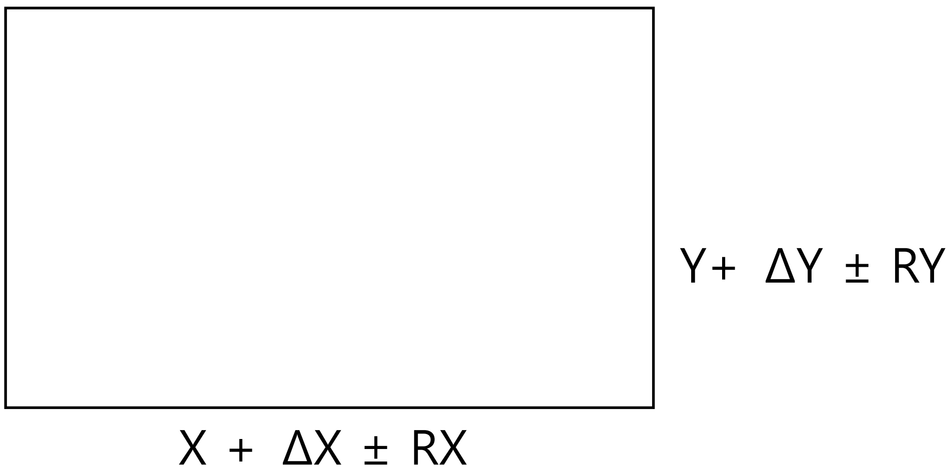
True Value= Nominal Value + Δ $\pm$ R

$X_T = X + \Delta X \pm RX$

Error Propagation

: The influence of individual errors on the total error

Ex) Area of Rectangle, A = XY



When systematic and random error are introduced to X and Y length, the influence to the Area of Rectangle?

Area of Rectangle, A -> $A + \Delta A \pm RA$

Systematic error, ΔA??

Random error, RA??

1. Systematic Error, ΔA

True Value, $X_T = X+\Delta X$

True Value, $Y_T = Y+\Delta Y$

1) First order approximation

$A_T=F(X_T,Y_T)=F(X+\Delta X,\ Y+\Delta Y)$

$=F(X,Y)+\ \partial F/\partial X\cdot\Delta X+\partial F/\partial Y\cdot\Delta Y$

$=F(X,Y)+(\partial F/\partial X,\partial F/\partial Y)\cdot(\Delta X,\Delta Y)$

$=A+\nabla F\cdot(\Delta X\ ,\Delta Y)$

$\therefore\ \Delta A=\nabla F\cdot(\Delta X\ ,\Delta Y)=\ \partial F/\partial X\cdot\Delta X+\partial F/\partial Y\cdot\Delta Y$

2) Mth order approximation

$A_T=F(X_T,Y_T)=F(X+\Delta X,\ Y+\Delta Y)$

$=F(X,Y)+\ \Delta X\cdot\partial F/\partial X+\ \Delta Y\cdot\partial F/\partial Y$

$+1/2(\Delta X^2\partial^2F/\partial X^2+2\Delta X\Delta Y\partial^2F/\partial X\partial Y+\ \Delta Y^2\partial^2F/\partial Y^2)+\ldots$

$+1/m!(\ \Delta X^m\partial^mF/\partial X^m+m\Delta X^{m-1}\Delta Y\partial^mF/\partial X^{m-1}\partial Y+\ldots)+\ldots$

$=\ F(X,Y)+\sum\ 1/m!\ (\Delta X\cdot\partial/\partial X+\Delta Y\cdot\partial/\partial Y)^mF$

$\therefore\ \Delta A=\sum\ 1/m!\ (\Delta X\cdot\partial/\partial X+\Delta Y\cdot\partial/\partial Y)^mF$

Similarly for A=F(X,Y,Z)

$\Delta A=\sum\ 1/m!\ (\Delta X\cdot\partial/\partial X+\Delta Y\cdot\partial/\partial Y+\Delta Z\cdot\partial/\partial Z)^mF$

$=\ \Delta X\cdot\partial/\partial X+\Delta Y\cdot\partial/\partial Y+\Delta Z\cdot\partial/\partial Z$ (up to 1st order)

$+1/2(\Delta X\cdot\partial/\partial X+\Delta Y\cdot\partial/\partial Y+\Delta Z\cdot\partial/\partial Z)^2F$ (up to 2nd order)

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2. Random Part, RA?

RMS Propagation of Variance, V_F

V_{xx}=Variance of X=$1/n-1\sum(X_i-\underline{X})^2 = \sigma_{xx}^2$

V_{yy}=Variance Y=$1/n-1\sum(Y_i-\underline{Y})^2 = \sigma_{yy}^2$

V_{xy}=Covariance of X,Y=$1/n-1\sum(X_i-\underline{X})(Y_i-\underline{Y})$

$=\rho_{xy}\sqrt{V_{xx}V_{yy}} \quad = \quad \rho_{xy}\sigma_{xx}\sigma_{yy}$

where, ρ_{xy}

$=V_{xy}/\sigma_{xx}\sigma_{yy} = \sum(X_i-\underline{X})(Y_i-\underline{Y})/\sqrt{\sum(X_i-\underline{X})^2}\sqrt{\sum(Y_i-\underline{Y})^2}$

=Correlation Coefficient between [0,1]

0: Not Correlated, or Independent

1: Fully Correlated, or Dependant

Variance of F, V_F can be expressed using the covariance matrix,

$$V_F=[\partial F/\partial X,\ \partial F/\partial Y]\begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix}[\partial F/\partial X,\ \partial F/\partial Y]^T$$

$$=V_{xx}(\partial F/\partial X)^2+V_{yy}(\partial F/\partial Y)^2 + 2V_{xy}(\partial F/\partial X)(\partial F/\partial Y)$$

$\therefore \sigma^2{}_F$

$=\sigma^2{}_{xx}(\partial F/\partial X)^2+\sigma^2{}_{yy}(\partial F/\partial Y)^2+2\rho_{xy}\sigma_{xx}\sigma_{yy}(\partial F/\partial X)(\partial F/\partial Y)$

$=\sigma^2{}_{xx}(\partial F/\partial X)^2+\sigma^2{}_{yy}(\partial F/\partial Y)^2$, if X, Y are independent

Thus random error of A, RA=$3\sigma_F$ (@99% probability)

True Value after Error Propagation = A + ΔA ± RA

Covariance matrix formulation for X,Y,Z variables

$V_F=$

$$[\partial F/\partial X,\partial F/\partial Y,\partial F/\partial Z]\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix}[\partial F/\partial X,\partial F/\partial Y,\partial F/\partial Z]^T$$

$=V_{xx}(\partial F/\partial X)^2+V_{yy}(\partial F/\partial Y)^2+V_{zz}(\partial F/\partial Z)^2$

$+2V_{xy}(\partial F/\partial X)(\partial F/\partial Y)+2V_{yz}(\partial F/\partial Y)(\partial F/\partial Z)$

$+2V_{zx}(\partial F/\partial Z)(\partial F/\partial X)$

$=\sigma^2{}_{xx}(\partial F/\partial X)^2+\sigma^2{}_{yy}(\partial F/\partial Y)^2+\sigma^2{}_{zz}(\partial F/\partial Z)^2$

$+2\rho_{xy}\sigma_{xx}\sigma_{yy}(\partial F/\partial X)(\partial F/\partial Y)$

$+2\rho_{yz}\sigma_{yy}\sigma_{zz}(\partial F/\partial Y)(\partial F/\partial Z)$

$+2\rho_{zx}\sigma_{zz}\sigma_{xx}(\partial F/\partial Z)(\partial F/\partial X)$

$=\sigma^2{}_{xx}(\partial F/\partial X)^2+\sigma^2{}_{yy}(\partial F/\partial Y)^2+\sigma^2{}_{zz}(\partial F/\partial Z)^2$

(for Independent X,Y,Z)

“Combined Standard Uncertainty”