M2794.007700 Smart Materials and Design

- Introduction to Composite Materials
- Review of Stress and Strain

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GENDER INNOVATIONS

Gender In Engineering Education

▪ Pregnant Crash Test Dummies:

- Rethinking Standards and Reference Models

Female crash test dummies

The Challenge

- Conventional seatbelts **do not** fit pregnant women properly (Weiss et al., 2001).
- Even a relatively minor crash at 56km/h (35 mph) can cause harm. With over **13 million women** pregnant (European Union and USA) each year, the use of seatbelts during pregnancy is a major safety concern (Eurostat, 2011; Finer et al., 2011).

Method: Rethinking Standards and Reference Models

- The male body is often defined as the norm and serves as the primary object of study.
- In this case, crash test dummies were first developed to model the U.S. 50th percentile man (taken as the norm).
- This means that other segments of the population were left out of the discovery phase in design.
- Inattention to humans of different sizes and shapes may result in unintended harm.

Gendered Innovations:

1. Taking both women and men as the norm may expand creativity in science and technology. From the start, devices should be designed for safety in broad populations.
2. Analyzing sex has led to the development of pregnant crash dummies and computer simulations.

Gender In Engineering Education

- **Cases**

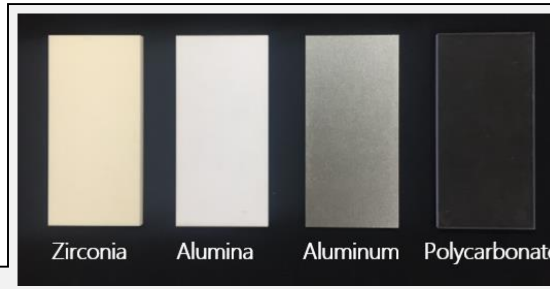
Imaging equipment for organ toxicity models

Materials for protective shielding

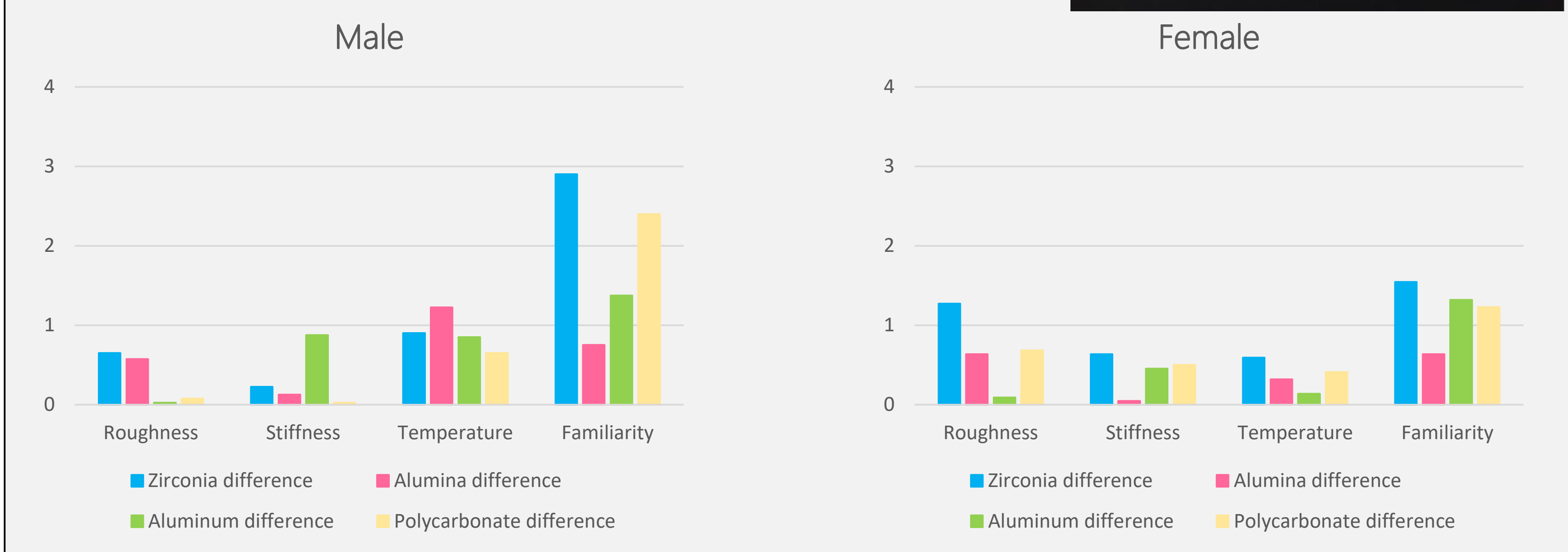
- **If your team project is related to Gendered Innovations, there will be a 2% of extra grade**

Emotional ICT industrial products

- Devices that are used in public
- There are various gender issues on the material of the product
- Grip, scratch, pattern of cover material, color, etc.



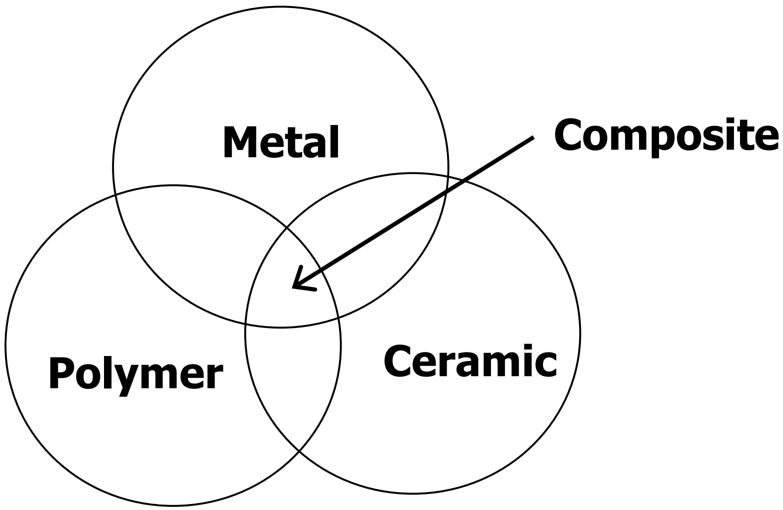
Difference between Blind touch and touch



INTRODUCTION TO COMPOSITE MATERIALS

Basic types of materials

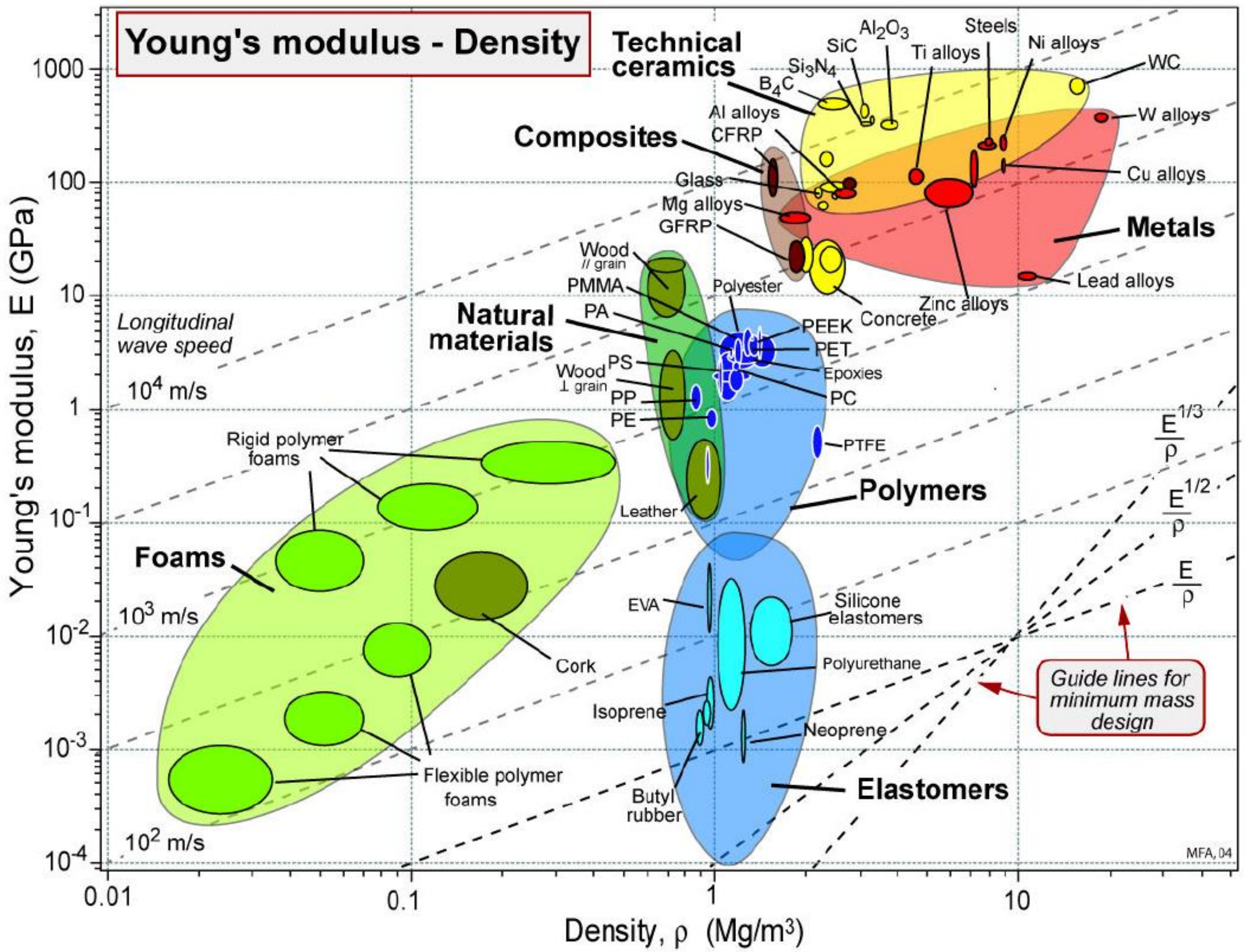
- **Metal**
- **Polymer**
- **Ceramic**



- **Where do we use engineering composites?**
  - Aerospace structures
    - Commercial 5 ~ 10 % weight
    - Business jet
    - Military
    - Satellite ~ 100% structural parts
  - Sports goods : tennis rackets, golf shaft, ski, …
  - Automobile, ship
  - Biomedical : hip joint

Why do we use composite?

- High E/ρ
- High σ/ρ
- Fatigue life
- Corrosion resistance



Ashby chart [1]

[1] The CES Edupack Resource Booklet 2 – Material and Process Charts, © Granta Design (2009)

What is composite material?

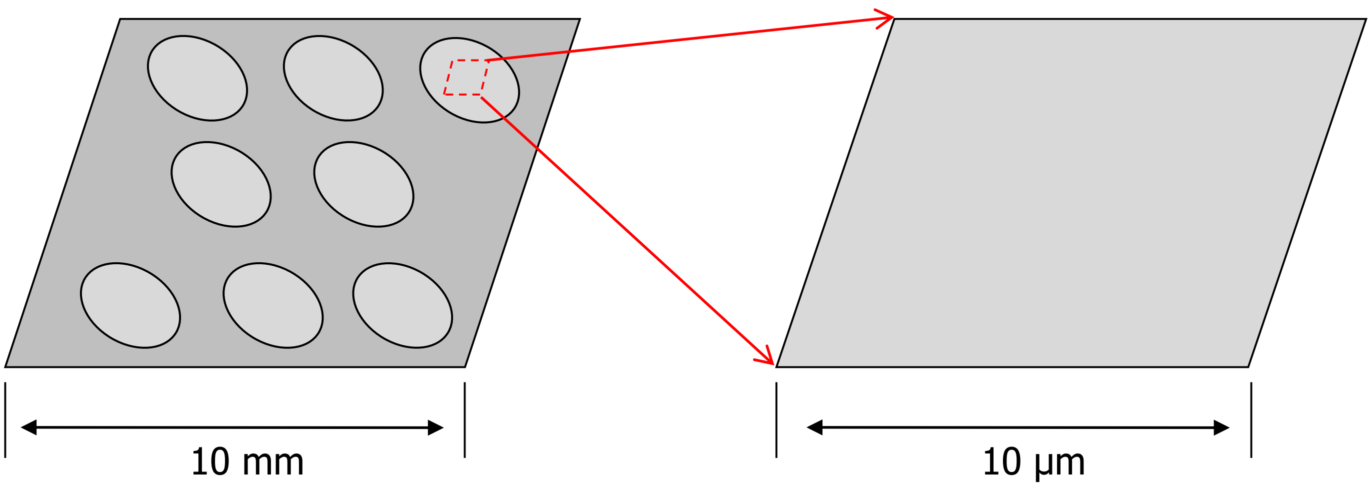
- **Two or more materials (metal, polymer and/or ceramic)**
- **Consist of two or more phases in macroscopic scale**
  - Mechanical performance and properties are designed to be superior to those of the constituent materials acting independently
    - Phase ~ Constituent : Alloys (X)
    - Macroscopic scale(~fiber dimension) > Grain size(molecular dimension)

  - cf) Characteristic dimensions (general)

	100 nm		100 μm		1 mm	
Atomic	————	Micro scale	————	Meso scale	————	Macro scale

Classifications of materials

- **Homogeneous**
  - Independent of location
  - Scale/characteristic volume dependent concept
- **Heterogeneous/inhomogeneous**
  - Properties vary from point to point

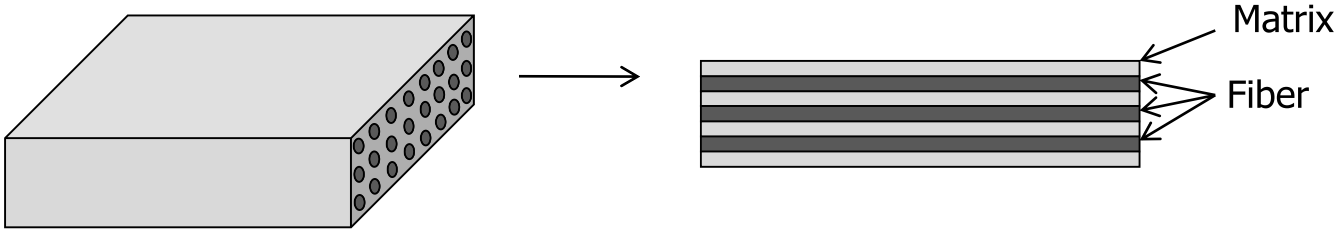


Heterogeneous

Homogeneous/Quasi Homogeneous

Classifications of materials

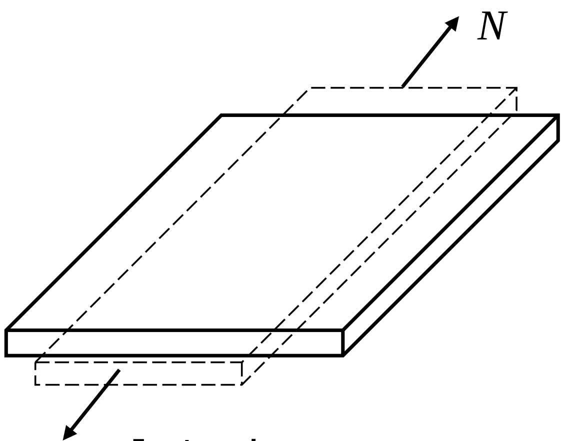
- **Isotropic**
  - Properties are the same in all direction
  - Infinite number of plane of material symmetry
- **Anisotropic**
  - Properties vary with direction
    - Zero plane of material symmetry(general anisotropy)
- **Orthotropic**
  - 3 mutually perpendicular plane of symmetry
- **Most metal, ceramic and polymer : homogeneous and isotropic**
- **Continuous fiber reinforced composite : (Quasi) homogeneous and anisotropic material**



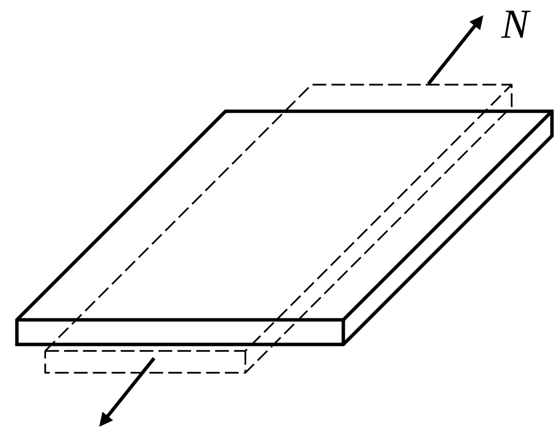
- **cf) Functionally graded material (FGM)**
  - Continuously varied properties

Behavior of general anisotropic material under loading

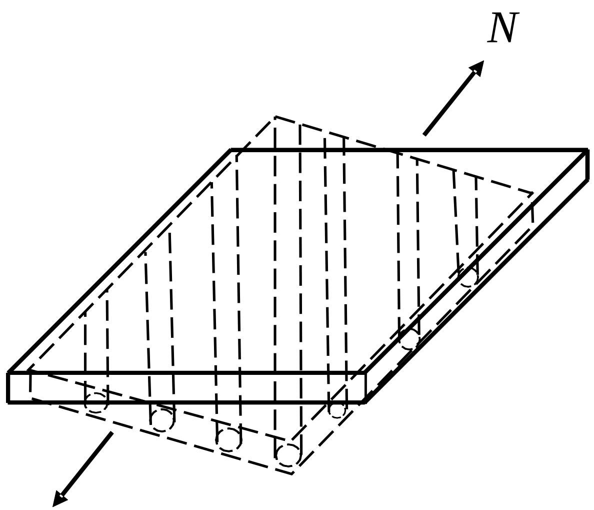
- **Response of various materials under uniaxial normal and pure shear loading**

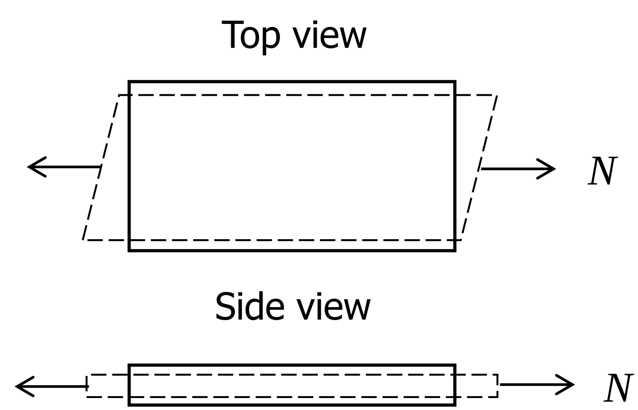


Isotropic



Orthotropic (loaded along principal material directions)

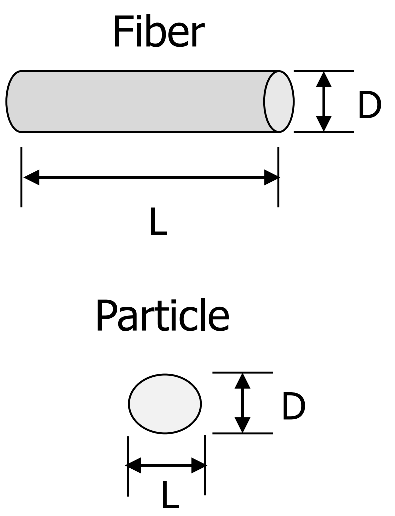




Anisotropic (or orthotropic material loaded along nonprincipal directions)

Types of composite materials

- **Reinforcement**
  - Strengthen
  - Fibers L/D >> 1 : "fiber reinforced composite"
  - Particles L/D ~ 1 : "particulate composite"
- **Matrix**
  - Polymer, metal and ceramic

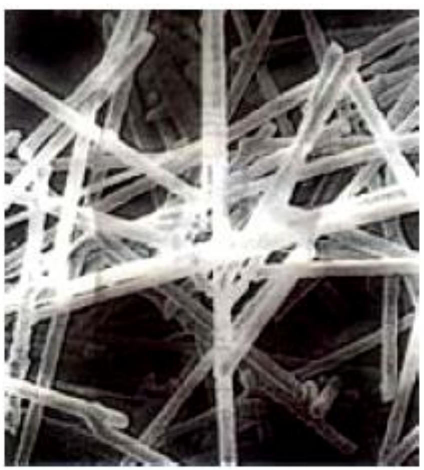


- **cf) "Laminated composite"**
  - Thin layers bonded together
  - Bimetal, clad metal and plywood

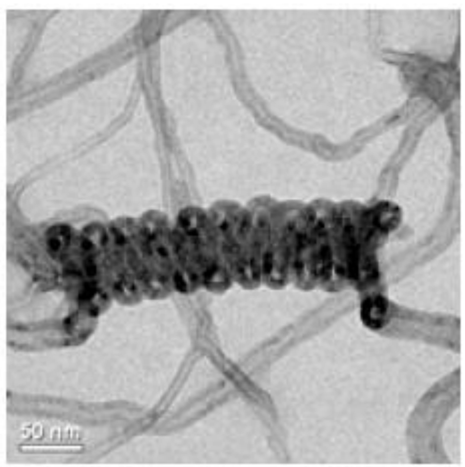
Why fibers?

- **Stronger than bulk**
  - Griffith's experiment (1920)
    - Glass fiber
    - Less (surface cracks) that would induce failure

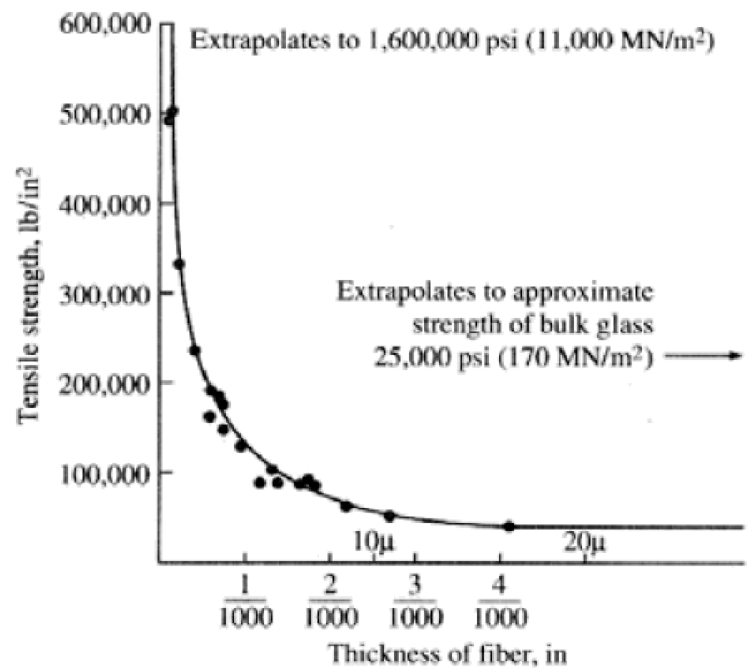
- **Size effect**



SiC whisker
(0.5 micro meter dia.)

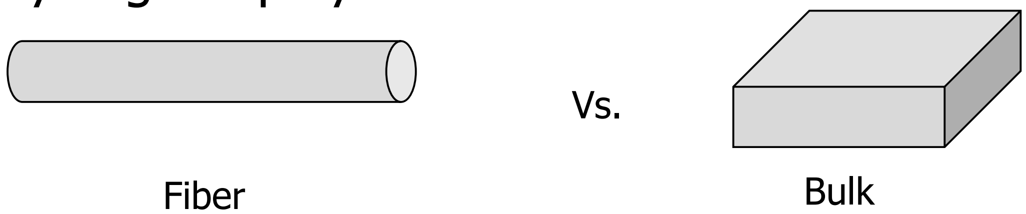


Multiwall CNT
(20 nano meter dia.)

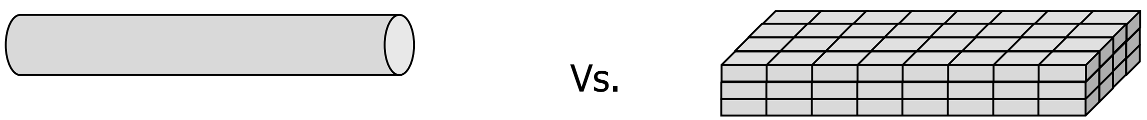


Other materials

- **Other materials showing similar behavior**
  - Polymer fibers : highly aligned polymer chains



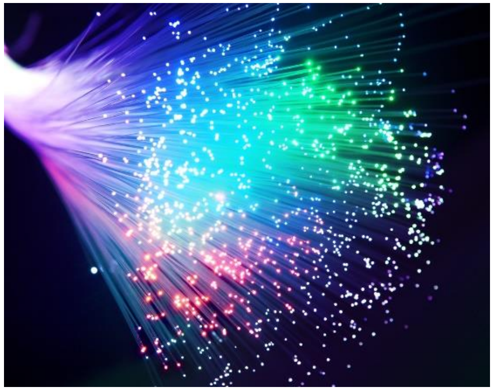
  - Whisker
    - Single crystal : lower dislocation density than a polycrystalline solid

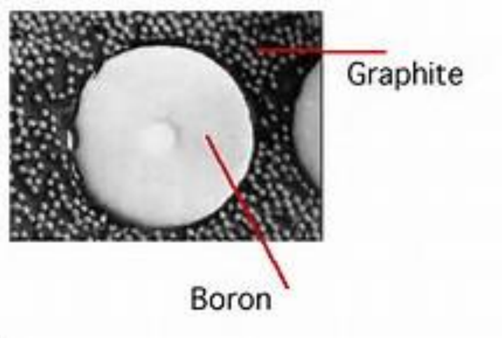


    - L/D ~ 100
    - Strongest reinforcing material
    - Eg) Iron crystal theoretical strength : 200GPa
      Iron whisker : 13GPa
      Iron bulk : 0.5 ~ 0.7 Gpa

Typical fiber materials

- **Typical fiber materials**
  - Glass
    - Cheap
    - Strong
    - Low modulus
  - Boron
    - Tungsten/carbon substrate
  - Carbon (graphite/pitch)
    - Aerospace
    - Cost is an important factor
  - Kevlar, aramid, polyamid
  - SiC (Silicone Carbide)
    - Very high temperature

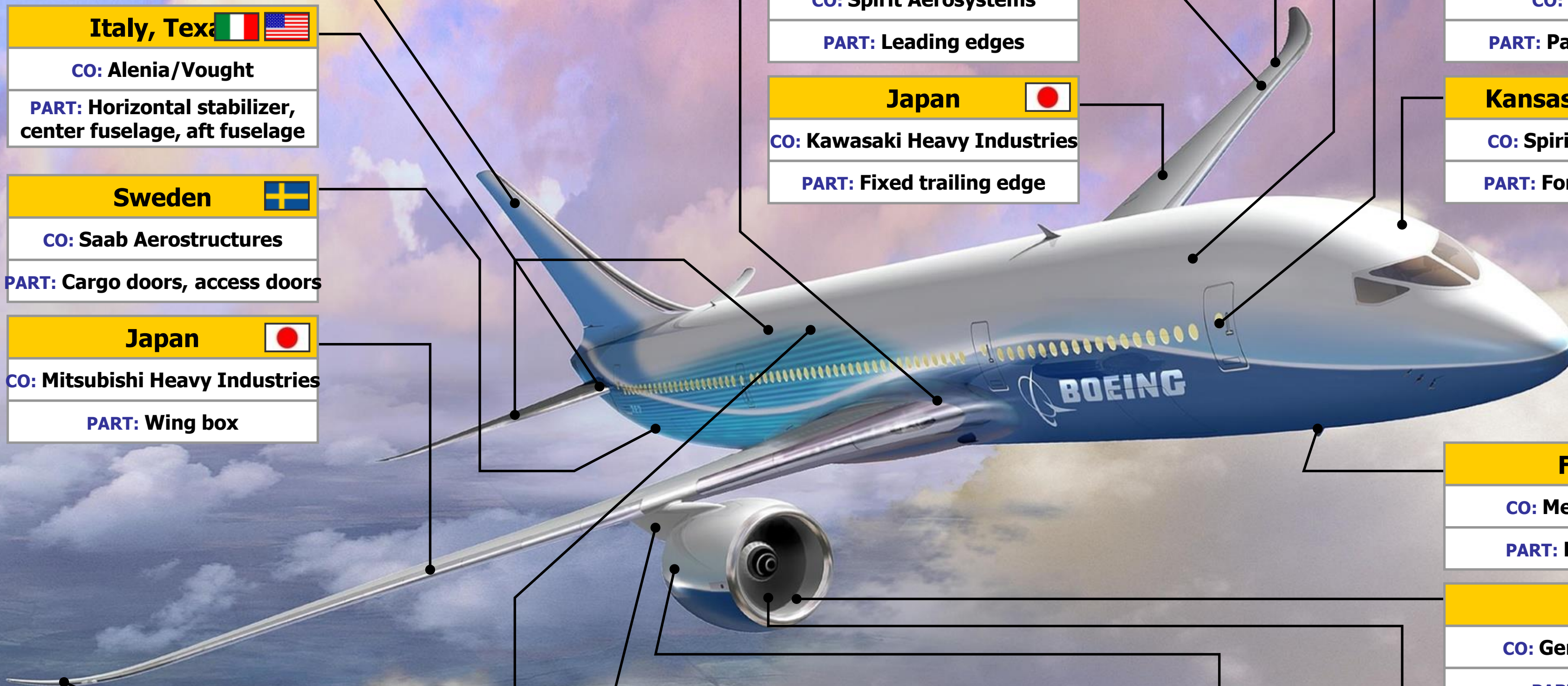




Applications of reinforced plastic

Boeing 787

Global collaboration
US design, manufactured around the world
Higher efficiency – composite materials (40~55% weight)



Washington, Canada, Australia
CO: Boeing Fredrickson
PART: Vertical tail assembly

Washington, Canada, Australia
CO: Boeing Winnipeg
PART: Wing-to-body fairing

Australia
CO: Hawker de Havilland
PART: Movable trailing edges

Japan
CO: Kawasaki Heavy Industries
PART: Fuselage, wheel well

Italy, Texa
CO: Alenia/Vought
PART: Horizontal stabilizer, center fuselage, aft fuselage

Kansas, Oklahom
CO: Spirit Aerosystems
PART: Leading edges

France
CO: Latecoere
PART: Passenger doors

Japan
CO: Kawasaki Heavy Industries
PART: Fixed trailing edge

Kansas, Oklahom
CO: Spirit Aerosystems
PART: Forward fuselage

Sweden
CO: Saab Aerostructures
PART: Cargo doors, access doors

Japan
CO: Mitsubishi Heavy Industries
PART: Wing box

France
CO: Messier-Dowty
PART: Landing gear

Ohio
CO: General Electric
PART: Engines

Korea
CO: Korean Airlines-Aerospace Division
PART: Wingtips

Japan
CO: Fuji Heavy Industries
PART: Center wing box

Kansas, Oklahom
CO: Spirit Aerosystems
PART: Engine Pylons

North Carolina
CO: Goodrich
PART: Naoelles

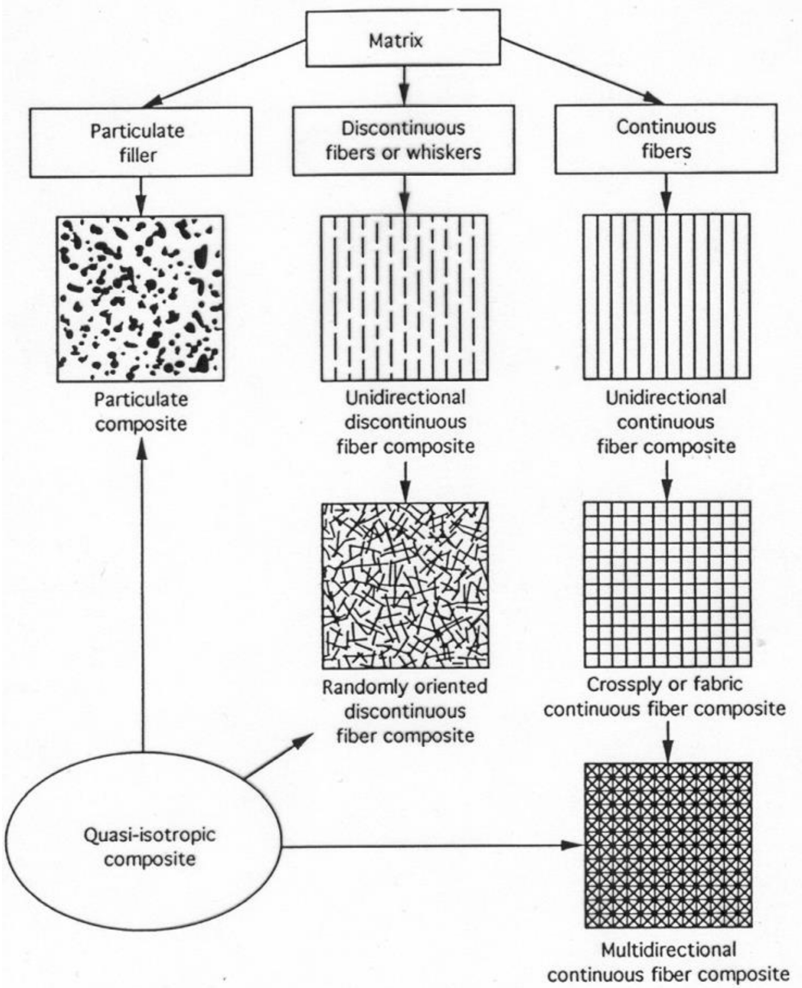
U. K.
CO: Rolls-Royce
PART: Engines

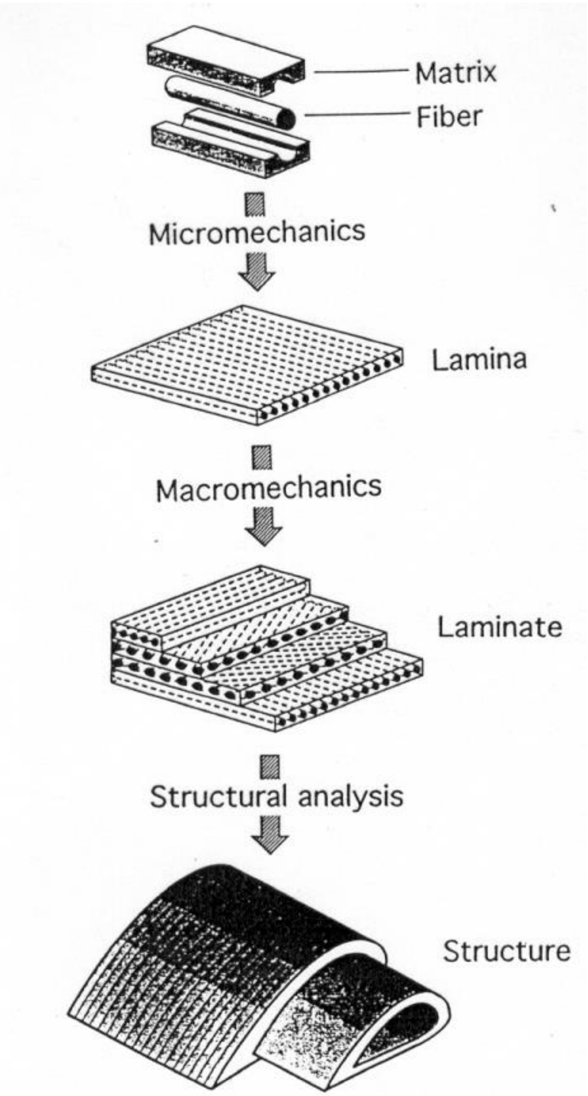
Thermoset and thermoplastics

Thermoset	Thermoplastic
Epoxy	Polyethylene
Polyester	Polystylene
Phenolics	Polypropylene
Bismaleimide (BMI)	Polyetherether ketone (PEEK)
Polyimide	Polyethersulfone
Undergo chemical change	Non-reacting
Process irreversible	Post-formable
Low viscosity	High viscosity
Long cure (~ 2hr)	Short process time
Lower process temperature	High process temperature
Good filler wetting	Rapid processing
Long process time	High process temperature
Restricted storage (Refrigeration)	Less chemical solvent (Resistance)

Classification by reinforcement

- **Particulate filler**
- **Discontinuous fibers/whiskers**
- **Continuous fibers**





Matrices

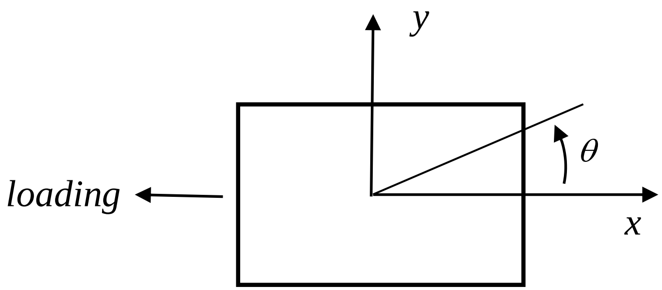
- **Functions of matrix**
  - Fiber alone can not support longitudinal compression
  - Weight reduction (usually low density than fiber)
  - Cost reduction (cheaper than fiber, $m_{total}=m_{fiber}+m_{matrix}$)
  - Protect fiber from environmental attack
    - eg. Ultra violet degradation, chemical attack

Type of matrices (tempertature)

- **Polymer**
  - Thermoset
    - Epoxy
      - 120°C/250°F for sporting goods, 175°C/350°F for aerospace
    - Polyimide
      - 370°C/700°F
  - Thermoplastic
    - PEEK
    - Polysulfone (400°C/750°F)
- **Metal**
  - Aluminum, Magnesium (~ 800°C/1500°F)
  - Titanium (900°C/1700°F)
- **Ceramic**
  - Glass, Si_3N_4, Al_2O_3, SiC (1000~1500°C)
  - Carbon
    - C-C : Aircraft brake (~2600°C/4700°F)

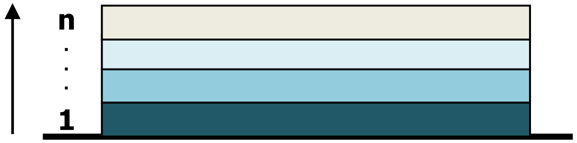
Lamina and Laminate

- **Lamina (ply)**
  - A layer of unidirectional fibers or woven fabric
  - Orthotropic material
- **Laminate**
  - Made up of two or more laminas (plies)
- **Ply orientation**
  - Angle between reference x-axis & fiber orientation in a counter clockwise direction



Lamina and Laminate

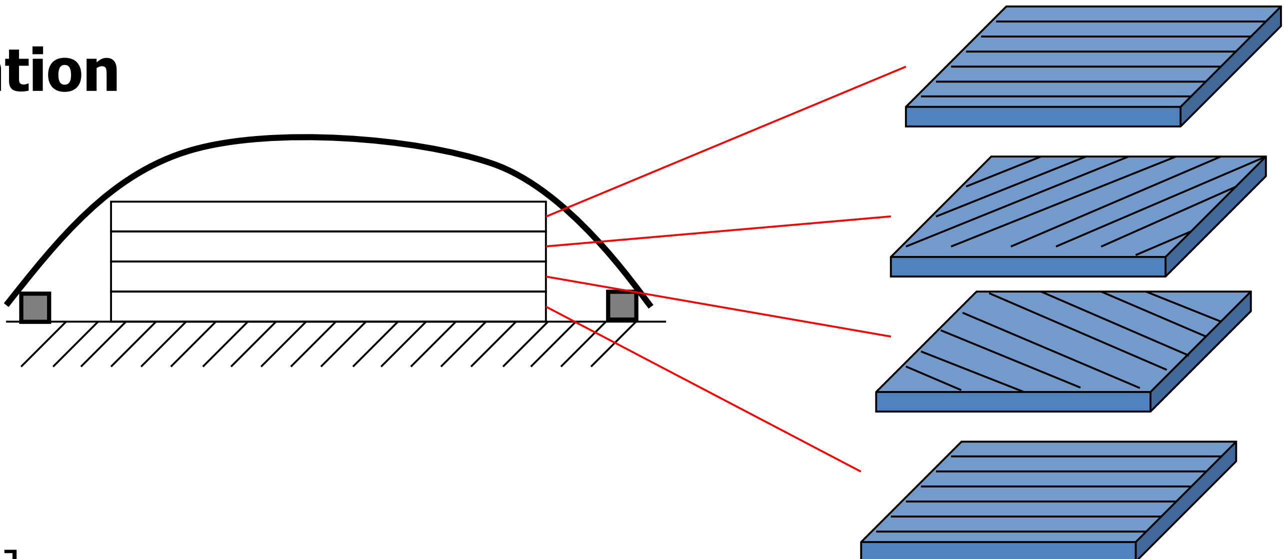
- **Stacking sequence**
  - Configuration indicating the exact location or sequence of each ply
  - From the bottom ply



- **Example of laminate designation**
  - Unidirectional : [0]
  - Angle : θ [θ]
  - # (number) : repeated group of plies
  - s : symmetric
  - T : total (to distinguish from symmetric)
  - \- (over bar) : symmetric about mid-plane ply
  - Ex : $[(0/\pm45)_2]_s$

Stacking sequence

- **Notation**



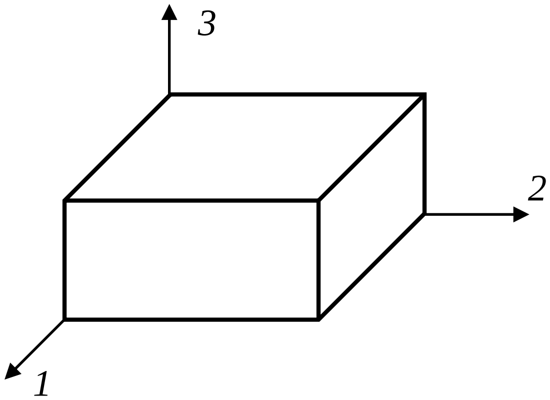
[0°/45°/-45°/0°]

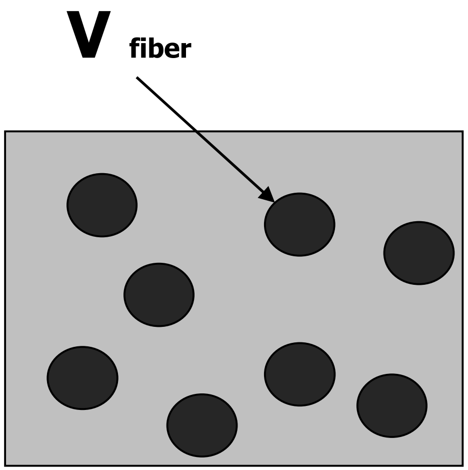
  - [0]
  - [θ]
  - Number : repetition of group
  - s : symmetric
  - T : total
- **Q:**
  - $[(0/45)_2] \rightarrow [0/45/0/45]$
  - $[(0/45)_2]_s \rightarrow [0/45/0/45/45/0/45/0]$
  - $[(0/\pm45)_2]_s \rightarrow [0/+45/-45/0/+45/-45/-45/+45/0/-45/+45/0]$

General material properties

- **General material properties**
  - Material constant : 1, 2, 3 direction
  - Fiber volume ratio

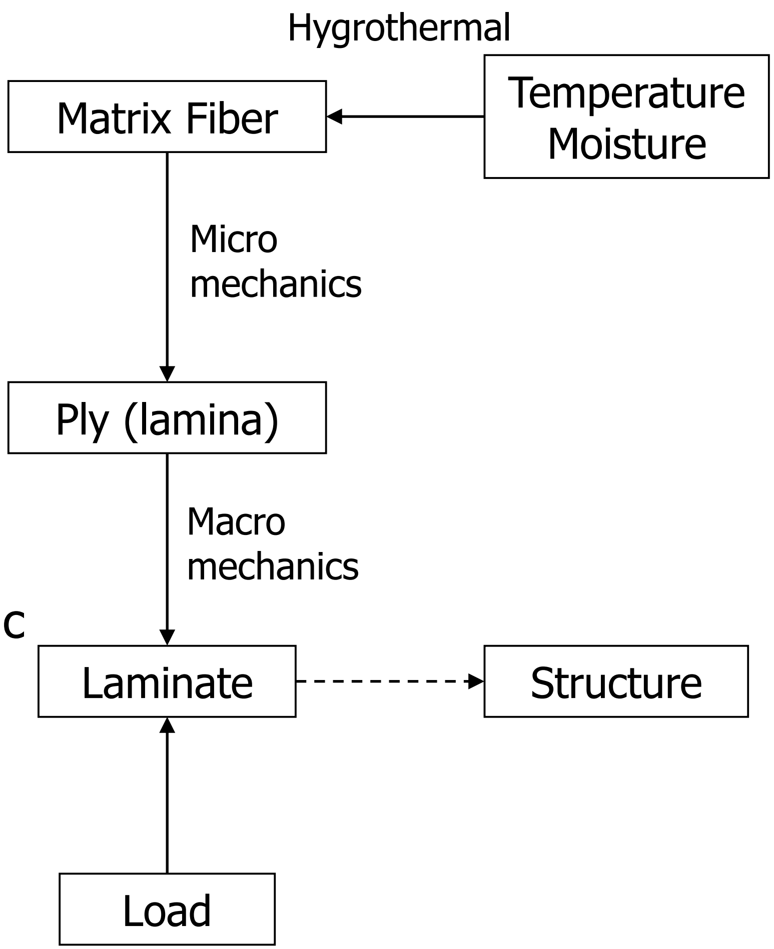
$$v_f = \frac{v_{fiber}}{v_{comp}}$$





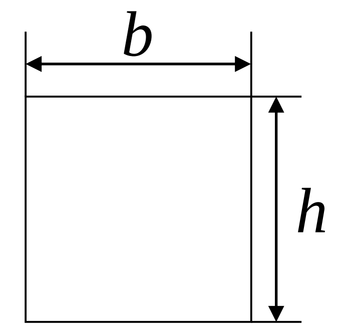
Analysis level

- **Analysis level**
  - Micro mechanics
    - Fiber & matrix
    - Fiber failure
      - Tensile, buckling and spliting
      - Local matrix failure (tensile/compression)
      - Interface failure
  - Macro mechanics
    - Lamina level
    - Consider as homogeneous and anisotropic
    - Average stress, strength
  - Lamination theory
    - Laminate level
    - CLT (Classical Lamination theory)
    - Structure level

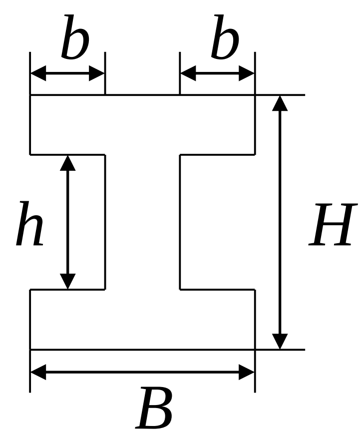


Manufacturing processes

1. **Prepreg**
    1. Autoclave : 20 ~ 100 psi (138 ~ 689 kPa)
    2. Vacuum : 1 atm (14.7 psi, 101 kPa)
    3. Hot press : plate
2. **Filament winding**
    1. Pressure vessel
    2. Shaft
3. **Pultrusion**
    1. Simple cross section
4. **Molding**
    1. SMC (Sheet Mold Compound)
    2. BMC (Bulk Mold Compound)
    3. RTM (Resin Transfer Molding)
5. **Automation of layups**
    1. Tape layup
    2. 3DP with short fiber
6. **Honeycomb**
    1. High stiffness by geometric design



$$I = \frac{bh^3}{12}$$

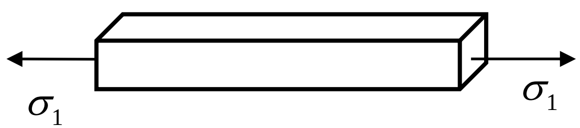


$$I = \frac{BH^3 - bh^3}{12}$$

Elastic behavior of unidirectional lamina

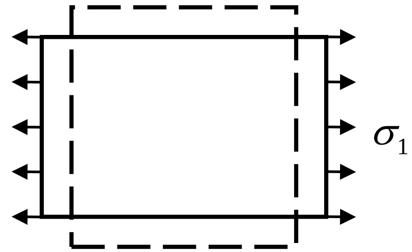
- **Objectives**
  - Isotropic E, v : anisotropic elastic constant ?
  - Transformation of σ, ε

- **Behavior of isotropic materials**
  - Axial loading

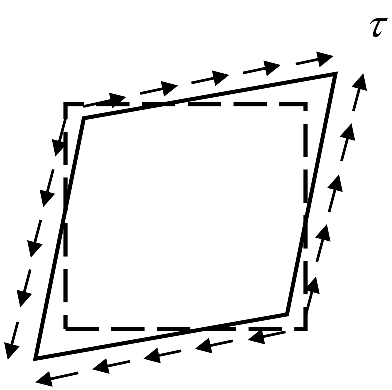


$$\varepsilon = \frac{\sigma}{E}$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3)$$



  - Shear loading



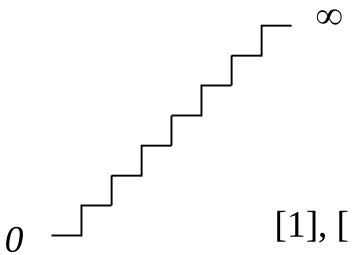
$$\gamma = \frac{\tau}{G}$$

$$G = \frac{E}{2(1+\upsilon)}$$

➡ *E, v : only 2 elastic constant*

Scalar, vector and tensor

- **Scalar (Scalae) – "stairs"**
  - Tensor of order zero
  - A pure number
  - Mass of car, temperature of body….



[1], [10]

- **Vector (Vectus) – "carried"**
  - Tensor of order 1
  - Magnitude & direction
  - Position, velocity, force

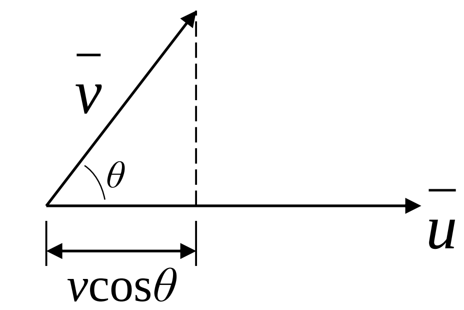
- **Tensor (Tensus) – "tension", "stretched"**
  - Hamilton, Irish
  - Tensor of order 2 and higher
  - Stress inside solid, fluid, moment of inertia ($I_{xx}\ I_{xy}$)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Vector algebra

- **Dot (scalar) product**

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$$

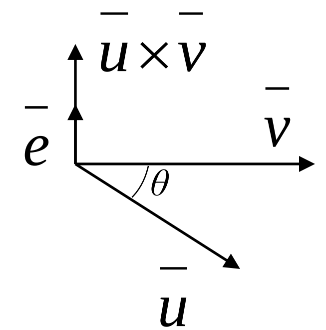


  - $\bar{u}\,\cdot$: map a vector to scalar

- **Cross (vector) product**

$$\bar{u} \times \bar{v} = (\|\bar{u}\| \, \|\bar{v}\| \sin\theta)\bar{e}$$

$$\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \, \|\bar{v}\| \sin\theta$$



  - $\bar{u}\,\times$: map a vector to vector

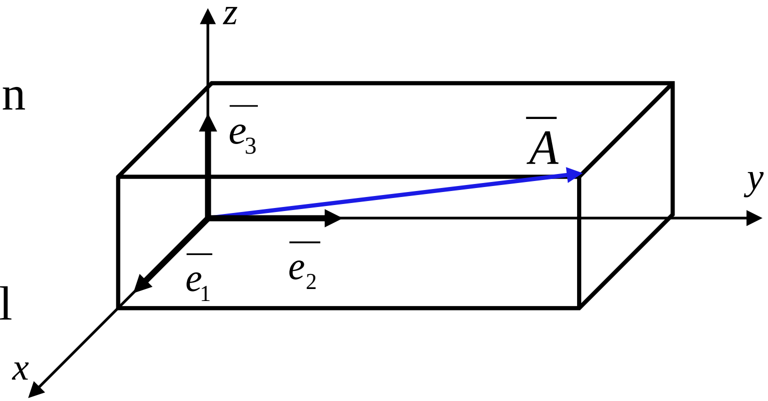
- **Dyad (direct) product**

$$[T] = \bar{a} \otimes \bar{b} \qquad \underbrace{(\bar{a} \otimes \bar{b})}_{Tensor} \cdot \bar{v} = \underbrace{(\bar{a}\bar{b})}_{Tensor} \cdot \bar{v} = \bar{a}(\bar{b} \cdot \bar{v}) = (\bar{b} \cdot \bar{v})\bar{a}$$

  - Map $\bar{v}$ into a vector parallel to $\bar{a}$

Indicial notation

- **Indicial notation (simple way to represent tensors)**
  - A vector in 3D space



$$\overline{A} = A_1\overline{e}_1 + A_2\overline{e}_2 + A_3\overline{e}_3$$ Symbolic notation

$\overline{e}_1, \overline{e}_2, \overline{e}_3$: unit vector

$\Leftrightarrow$ $\overline{A} = A_i\overline{e}_i$ Symbolic indicial

$\Leftrightarrow$ $\overline{A} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} or \begin{Bmatrix} A_1 & A_2 & A_1 \end{Bmatrix}$ Matrix notation

$\Leftrightarrow$ $\overline{A} = A_i$ Indicial notation

Convention

- **An unrepeated subscript has a range over which it takes all values**
  - eg) i=1, 2, 3

- **A repeated subscript implies summation over the range of permissible values**
  - eg) dyadic product

$$\overline{A} = A_1\overline{e_1} + A_2\overline{e_2} + A_3\overline{e_3} \qquad \overline{B} = B_1\overline{e_1} + B_2\overline{e_2} + B_3\overline{e_3}$$

$$\begin{aligned}\overline{A}\overline{B} &= (A_1\overline{e_1} + A_2\overline{e_2} + A_3\overline{e_3})(A_1\overline{e_1} + A_2\overline{e_2} + A_3\overline{e_3}) \\ &= A_1B_1\overline{e_1}\overline{e_1} + A_1B_2\overline{e_1}\overline{e_2} + A_1B_3\overline{e_1}\overline{e_3} \\ &+ A_2B_1\overline{e_2}\overline{e_1} + A_2B_2\overline{e_2}\overline{e_2} + A_2B_3\overline{e_2}\overline{e_3} \\ &+ A_3B_1\overline{e_3}\overline{e_1} + A_3B_2\overline{e_3}\overline{e_2} + A_3B_3\overline{e_3}\overline{e_3}\end{aligned}$$

$$\Leftrightarrow \quad A_iB_j\overline{e_i}\overline{e_j} \quad \Leftrightarrow \quad A_iB_j \quad i,j=1,2,3$$

indicial

$$\Leftrightarrow \quad \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}$$

Matrix

  - eg) dot product

$$\overline{v}\cdot\overline{e_i} = v_i \qquad \overline{u}\cdot\overline{v} = u_iv_i$$

  - eg) cross product

$$\overline{u}\times\overline{v} = \varepsilon_{ijk}u_iv_j\overline{e_k}$$

Permutation symbol $\varepsilon_{ijk} = \begin{cases} 1 & \text{if (i, j, k) has even permutation} \\ -1 & \text{if (i, j, k) has odd permutation} \\ 0 & \text{if two or more indices are zero} \end{cases}$

Coordinate transformation

Vector $\overline{A}$

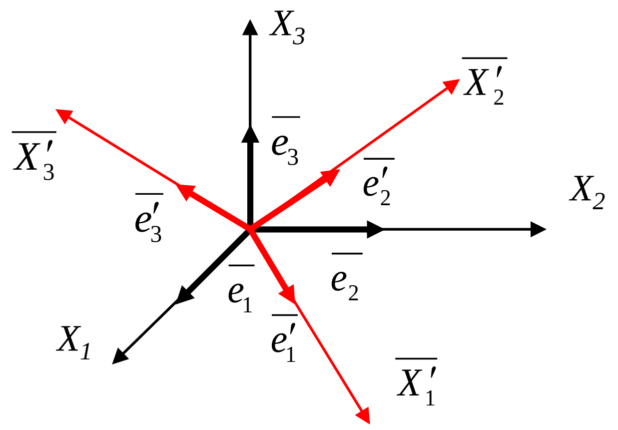
$$\overline{A} = A_1\overline{e_1} + A_2\overline{e_2} + A_3\overline{e_3}$$

$$A_1' = A_1(\overline{e_1'}\cdot\overline{e_1}) + A_2(\overline{e_1'}\cdot\overline{e_2}) + A_3(\overline{e_1'}\cdot\overline{e_3})$$

$$\Leftrightarrow A_1' = \alpha_{11}A_1 + \alpha_{12}A_2 + \alpha_{13}A_3$$

$$where\ \ \alpha_{ij} = e_i' \cdot e_j$$

$$A_i' = \alpha_{ij}A_j$$

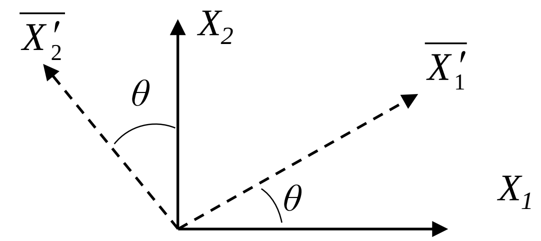


α_{ij} = Cosine of the angle between the i-th primed axis and the j-th unprimed axis

Coordinate transformation

1. 2-D transformation

$$\alpha_{11} = \cos(X_1', X_1) = \cos\theta$$
$$\alpha_{12} = \cos(X_1', X_2) = \cos(90-\theta) = \sin\theta$$
$$\alpha_{21} = \cos(X_2', X_1) = \cos(90+\theta) = -\sin\theta$$
$$\alpha_{22} = \cos(X_2', X_2) = \cos\theta$$



$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

2. Transformation of 2nd order tensor

$$A_{ij}' = \alpha_{ik}\alpha_{jl}A_{kl}$$

Stress-strain in anisotropic material (3D)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad C_{ijkl} : \text{Stiffness matrix} \qquad S_{ijkl} : \text{Compliance matrix}$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad i,\ j, k, l = 1, 2, 3$$

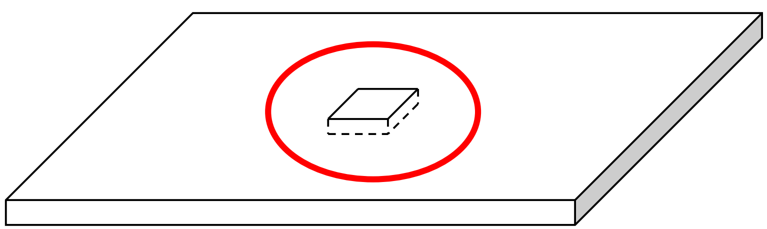
$$\left[C_{ijkl}\right] = \left[S_{ijkl}\right]^{-1}$$

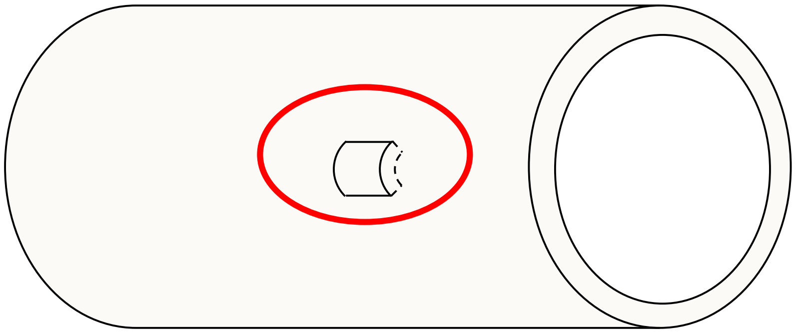
Number of elements in stiffness/compliance matrix	
81 general	
36 symmetry of stress-strain	$\sigma_{ij} = \sigma_{ji}\ \&\ \varepsilon_{ij} = \varepsilon_{ji}$
21 symmetry of material constant	$C_{ij} = C_{ji}\ \&\ S_{ij} = S_{ji}$
9 orthotropic case 3 mutually perpendicular planes of material symmetry	$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \underline{C_{11}} & \underline{C_{12}} & \underline{C_{13}} & & & \\ C_{12} & \underline{C_{22}} & \underline{C_{23}} & & & \\ C_{13} & C_{23} & \underline{C_{33}} & & & \\ & & & \underline{C_{44}} & & \\ & & & & \underline{C_{55}} & \\ & & & & & \underline{C_{66}} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$
5 Transversely isotropic	$C_{12} = C_{13}\,, C_{22} = C_{33}\,, C_{44} = \dfrac{C_{22} - C_{23}}{2}\,, C_{55} = C_{66}$
2 isotopic	$C_{11} = C_{22} = C_{33}\,, C_{44} = C_{55} = C_{66} = \dfrac{C_{11} - C_{12}}{2} \qquad E\ \&\ \nu$

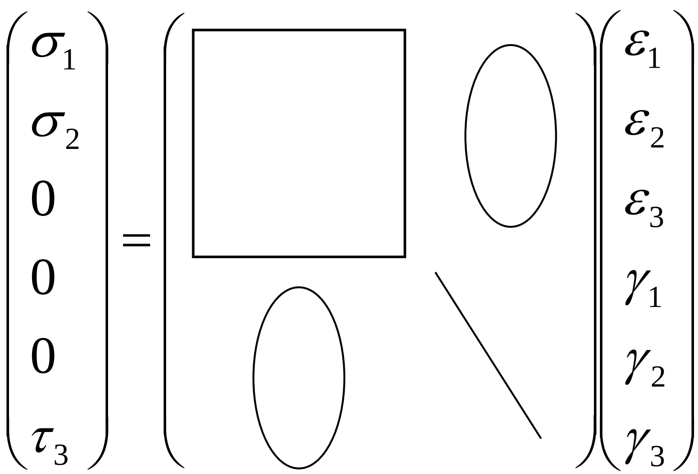
Stress-strain in anisotropic 2D

- **Plane stress**
  - no stress in thickness direction

$$\sigma_3 = \tau_{23}(\tau_1) = \tau_{13}(\tau_2) \cong 0$$







Stress-strain in anisotropic 2D

$$\begin{aligned}\sigma_1 &= \left(C_{11} - \frac{C_{13}C_{13}}{C_{33}}\right)\varepsilon_1 + \left(C_{12} - \frac{C_{23}C_{13}}{C_{33}}\right)\varepsilon_2 \\ &= \qquad Q_{11}\varepsilon_1 \qquad + \qquad Q_{12}\varepsilon \\ \sigma_2 &= \left(C_{12} - \frac{C_{13}C_{23}}{C_{23}}\right)\varepsilon_1 + \left(C_{22} - \frac{C_{23}C_{23}}{C_{33}}\right)\varepsilon_2 \\ &= \qquad Q_{12}\varepsilon_1 \qquad + \qquad Q_{22}\varepsilon_2 \\ \tau_3 &= \mathrm{C}_{66}\gamma_3\end{aligned}$$

$$\varepsilon_3 = -\left(\frac{C_{13}}{C_{33}}\varepsilon_1\right) - \left(\frac{C_{23}}{C_{33}}\varepsilon_2\right)$$

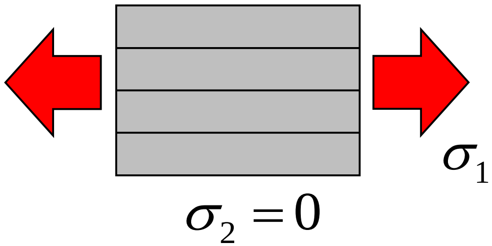
In matrix form

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_3\end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_3\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_3\end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0\\ S_{21} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_3\end{pmatrix}$$

How to measure the material constants?

- **Test 1**

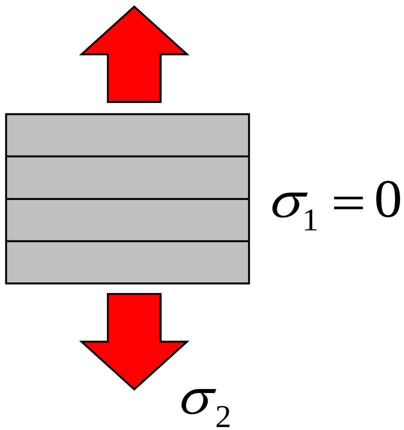


$$\varepsilon_1 = \frac{1}{E_1}\sigma_1$$

$$\varepsilon_2 = -\upsilon_{12}\varepsilon_1 = \frac{-\upsilon_{12}}{E_1}\sigma_1$$

$$\left(\frac{1}{E_1} = S_{11} \quad , \quad \frac{-\upsilon_{12}}{E_1} = S_{12}\right)$$

- **Test 2**



$$\varepsilon_2 = \frac{1}{E_2}\sigma_2$$

$$\varepsilon_1 = -\upsilon_{21}\varepsilon_2 = \frac{-\upsilon_{21}}{E_2}\sigma_2$$

$$\left(\frac{1}{E_2} = S_{22} \quad , \quad \frac{-\upsilon_{21}}{E_2} = S_{21}\right)$$

$$S_{12} = S_{21} \Rightarrow \frac{\upsilon_{12}}{E_1} = \frac{\upsilon_{21}}{E_2}$$

$$Q_{11} = \frac{E_1}{1-\upsilon_{12}\upsilon_{21}}$$

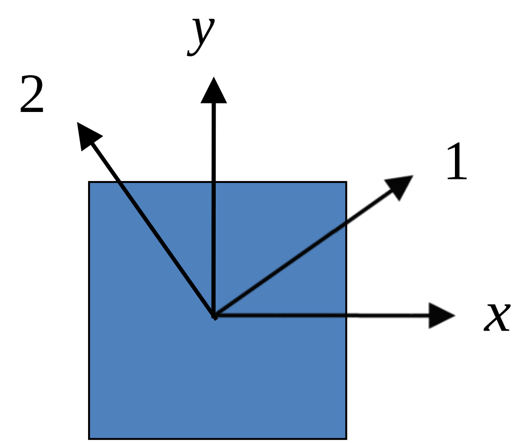
$$Q_{22} = \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{12} = \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} = \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{66} = G_{12}$$

Transformation of stress and strain

- **Loading axes** (x, y)
- **Principal mat. Axes** $(1, 2)$



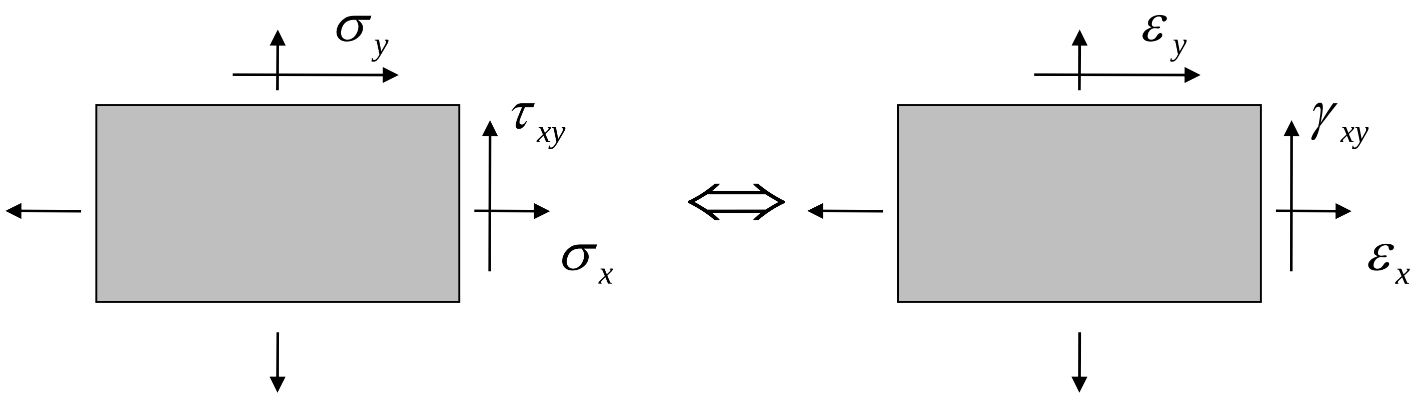
$$[T]=\begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix}$$

$$m=\cos\theta$$
$$n=\sin\theta$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_3 \end{pmatrix} = [T] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{pmatrix} \qquad \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_3 \end{pmatrix} = [T] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_s \end{pmatrix}$$

Transformation of stress and strain

- **Q : How to get off-axis stress-strain relation?**



$$
\begin{aligned}
\lfloor \sigma_{xy} \rfloor &= \lfloor Q_{xy} \rfloor \lfloor \varepsilon_{xy} \rfloor \\
&= \left[T^{-1}\right] \left[\sigma_{12}\right] &&\Leftarrow \left[\sigma_{12}\right] = [T] \left[\sigma_{xy}\right] \\
&= \left[T^{-1}\right] \left[Q_{12}\right] \left[\varepsilon_{12}\right] &&\Leftarrow \left[\sigma_{12}\right] = \left[Q_{12}\right] \left[\varepsilon_{12}\right] \\
&= \left[T^{-1}\right] \left[Q_{12}\right] [T] \left[\varepsilon_{xy}\right] &&\Leftarrow \left[\varepsilon_{12}\right] = [T] \left[\varepsilon_{xy}\right]
\end{aligned}
$$

$$\varepsilon_{12} = \frac{1}{2}\gamma_3$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\gamma_s$$

$$Q_{xx} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta$$

$$Q_{xy} = \cdots \;\; \cdots \;\; \cdots$$

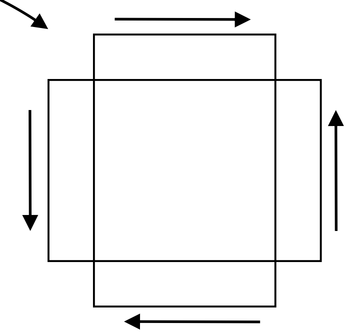
$$\vdots$$

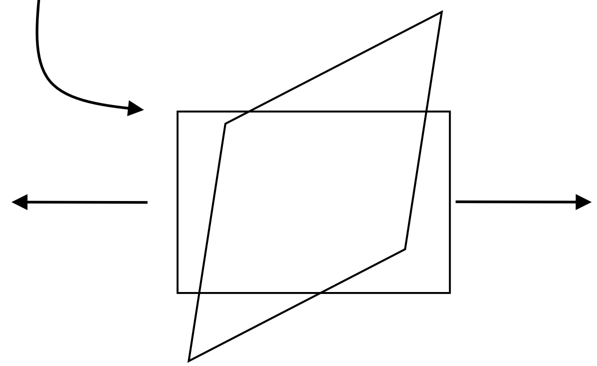
$$Q_{ss} = \cdots \;\; \cdots \;\; \cdots$$

Transformation of stress and strain

Coupled terms

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_s\end{bmatrix}=\begin{bmatrix}Q_{xx} & Q_{xy} & Q_{xs}\\ Q_{yx} & Q_{yy} & Q_{ys}\\ Q_{sx} & Q_{sy} & Q_{ss}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_s\end{bmatrix}$$

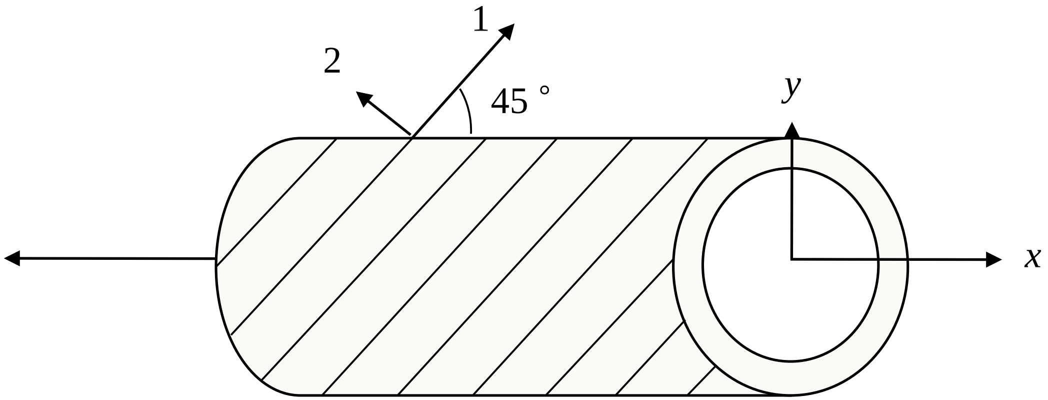




$$[\varepsilon_{xy}]=[S_{xy}][\sigma_{xy}]$$

Smart materials and design H.W. #3

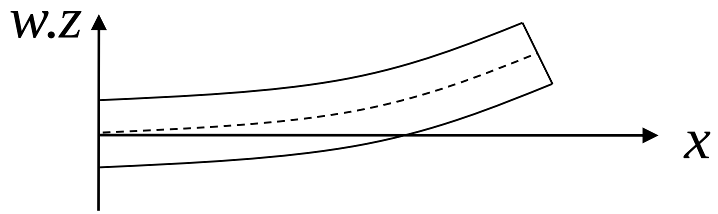
1) For a high strength (HS) carbon fiber/epoxy lamina, calculate the on-axis stiffness matrix, $[Q]_{1,2}$ (E_{11} =131.0 GPa, E_{22} = 11.2 GPa, ν_{12} = 0.28, G_{12} = 6.55 GPa)
2) Assuming it is isotropic lamina, calculate the on-axis stiffness matrix, $[Q]_{1,2}$ (E_{11} = E_{22} = 131.0 GPa, ν_{12} = ν_{21} = 0.30, G_{12} = G_{21} = 6.50 GPa)
3) Using the $[Q]_{1,2}$ matrix in problem 1), calculate the off-axis stiffness matrix $[Q]_{x,y}$ of the cylinder where the fiber angle is 45° from the x-axis



4) Write the transforming code (Matlab) from $[Q]_{1,2}$ to $[Q]_{x,y}$

Classical lamination theory

- **Composite beam**



- **N-plane displacement**

$$u = u_0 - z\frac{\partial w}{\partial x}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x^0 - z\kappa_x$$

$$v = v_0 - z\frac{\partial w}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \varepsilon_y^0 - z\kappa_y$$

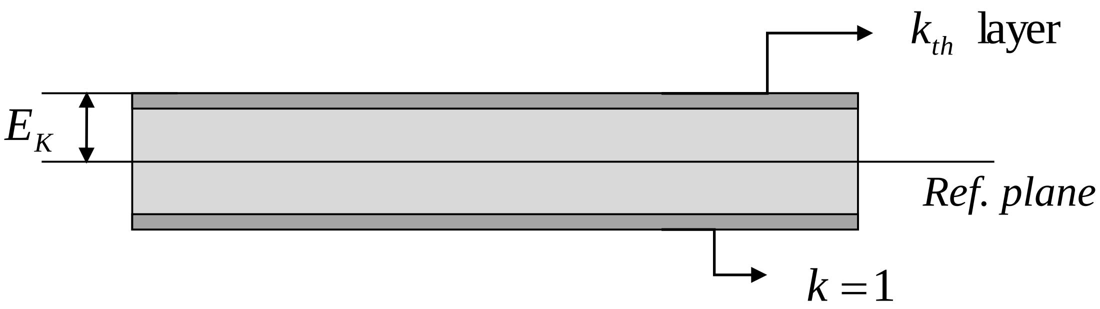
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial x} = \gamma_s^0$$

$$2\frac{\partial^2 w}{\partial x \partial y} = \kappa_s$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{pmatrix} + z\begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{pmatrix}$$

Classical lamination theory



for k_{th} layer

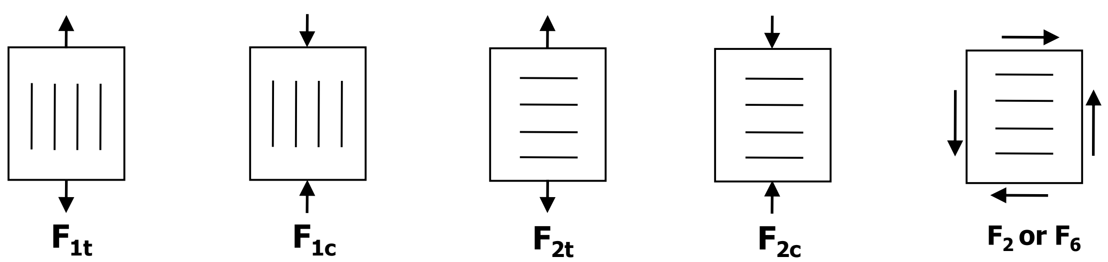
$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{pmatrix}_k = \begin{pmatrix} Q_{xx} & Q_{xy} & \cdots \\ \vdots & & \\ & & \end{pmatrix}_k \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{pmatrix}_k$$

$$= \begin{pmatrix} & & \\ & Q & \\ & & \end{pmatrix}_k \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{pmatrix} + z \begin{pmatrix} & & \\ & Q & \\ & & \end{pmatrix}_k \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{pmatrix}$$

Failure

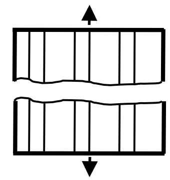
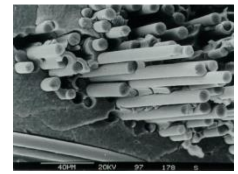
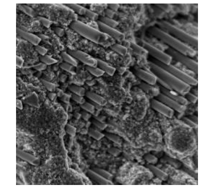
- **Failure theories : How to define failure of composite?**
    - Max. stress
    - Max strain
    - Tsai – Hill
    - Tsai - Wu

- **Parameters**

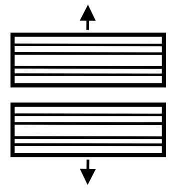
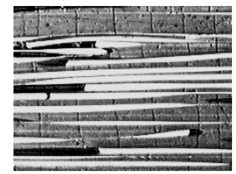
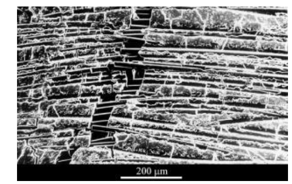


Failure

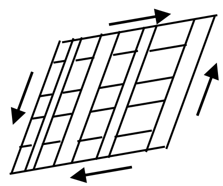
- **Tension along the fiber**
  - fiber breakage & matrix failure

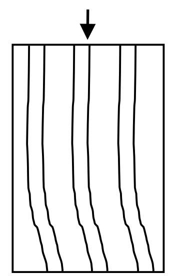
- **Tension normal to the fiber**
  - matrix cracking

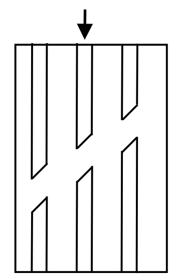
- **Shear**
  - matrix shear out



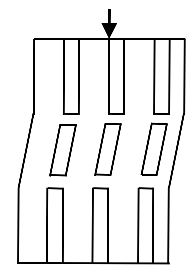

- **Compression along the fiber**



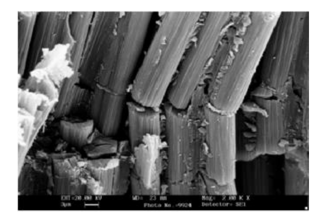
Micro buckling



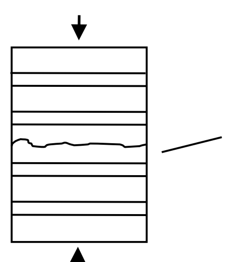
shear out in fibers



kinking



- **Compression normal to the fiber**



Matrix failure

REVIEW OF STRESS AND STRAIN

Review of isotropic stress analysis

- **Isotropic** ⟹ **2 material constant,** $E \,\&\, \nu$

Failure

⇑

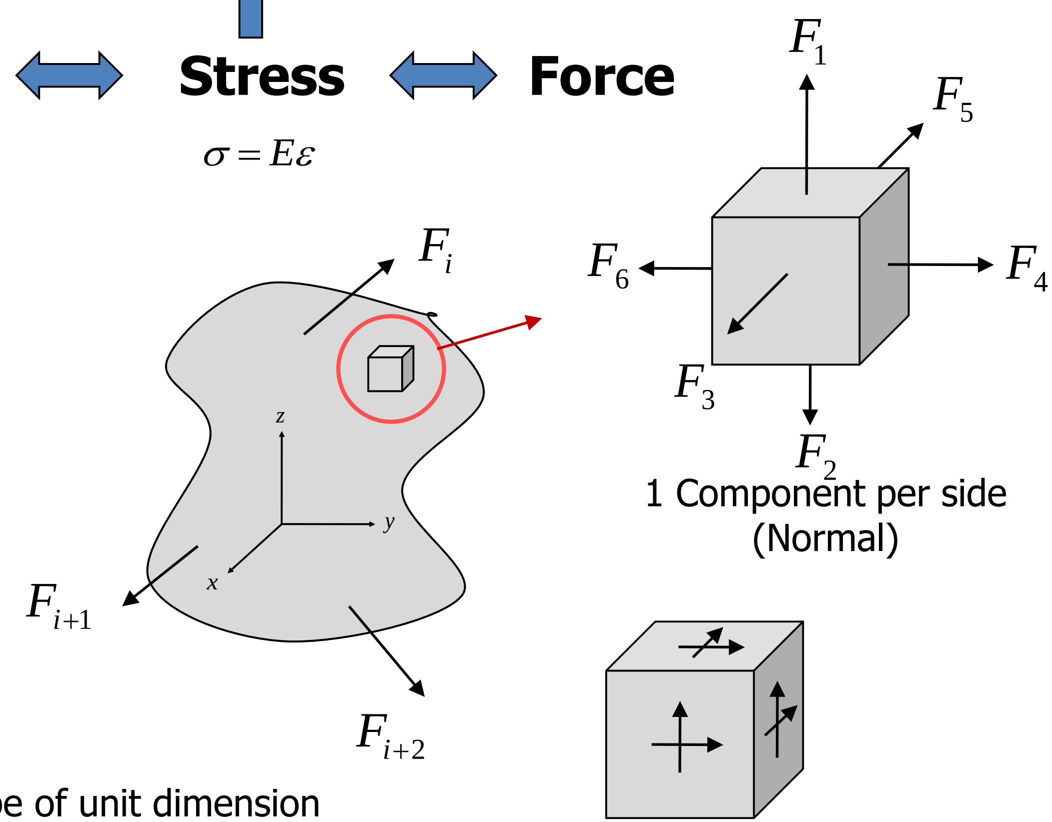
- **Displacement** ⟺ **Strain** ⟺ **Stress** ⟺ **Force**

u $\qquad \varepsilon = \dfrac{\partial u}{\partial x} \qquad \sigma = E\varepsilon$

- **Stress**

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$$

  - 6 components



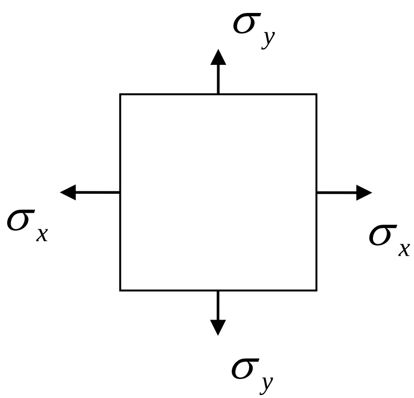
For cube of unit dimension

2 Component per side
(Shear)

Review of isotropic stress analysis

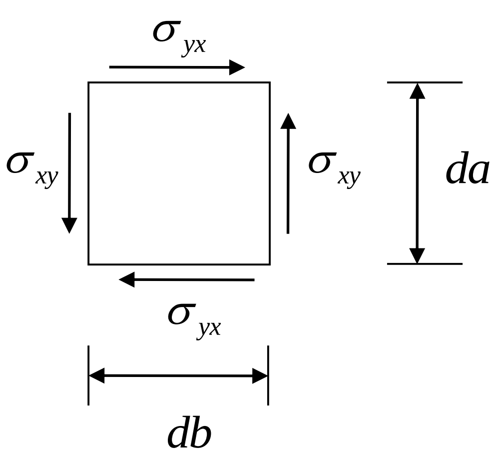
- **Known relations**
  - For Normal stresses equilibrium of forces & moments

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$
$$\Rightarrow \text{3 Components}$$
$$(\sigma_x, \sigma_y, \sigma_z)$$



  - For shear stresses

$$\text{Force} \quad \sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$



Review of isotropic stress analysis

$$\sum F_x = 0$$

Moment $\sum F_y = 0$

$$\sum F_z = 0$$

$$M_c = \sigma_{yx} db - \sigma_{xy} da = 0 \qquad da = db$$

$$\sigma_{yx} = \sigma_{xy}$$

$$\sigma_{zx} = \sigma_{xz}$$

$$\sigma_{yz} = \sigma_{zy}$$

$\Rightarrow$ 3 Components

$$(\sigma_x, \sigma_y, \sigma_z)$$

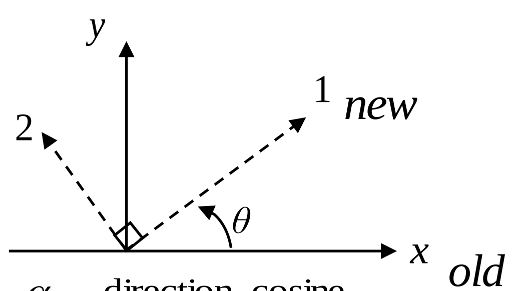
$$= (\tau_{xy}, \tau_{yz}, \tau_{zx})$$

- **Transformation**

Vector $\underset{new}{A'_i} = \alpha_{ij} \underset{old}{A_j}$

Tensor $A'_{ij} = \alpha_{ik} \alpha_{jl} A_{kl}$

$$\underset{new}{\sigma'_{ij}} = \alpha_{ik} \alpha_{jl} \underset{old}{\sigma_{kl}}$$



α_{ij} – direction cosine

Review of isotropic stress analysis

- **Biaxial stress transformation**

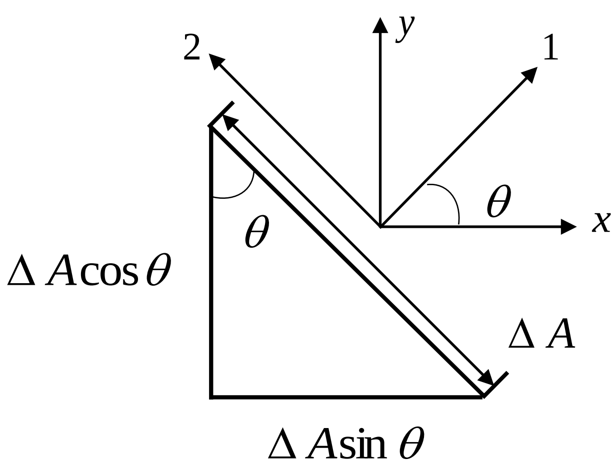
$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

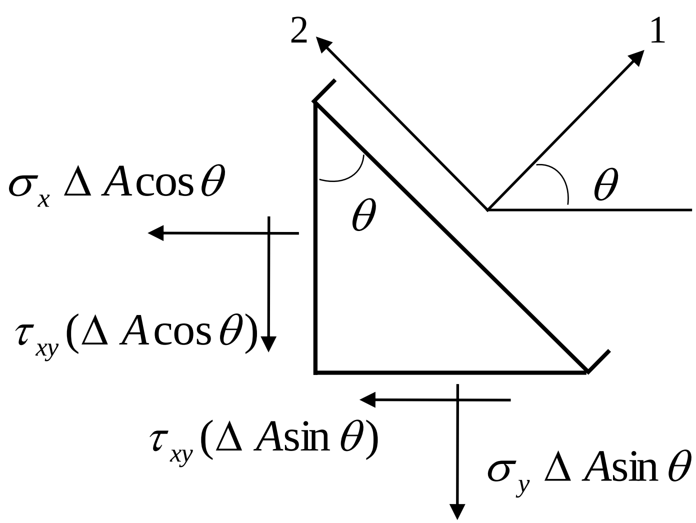
$new \qquad [T] \qquad old$

$$m = \cos\theta \qquad n = \sin\theta$$

Cf.) reverse direction (1,2) →(x,y) $\quad \sigma_{x,y} = [T]^{-1}\sigma_{1,2}$

$$[T(\theta)]^{-1} = [T(-\theta)] \qquad \because T(\theta)T(-\theta) = I$$





$$\sum F_1 = 0$$

$$\sum F_2 = 0$$

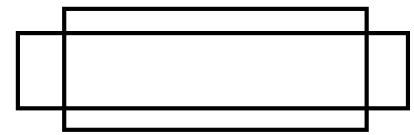
Review of isotropic stress analysis

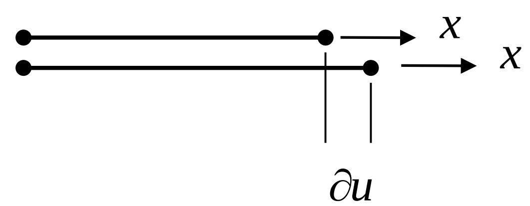
- **Strains**

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

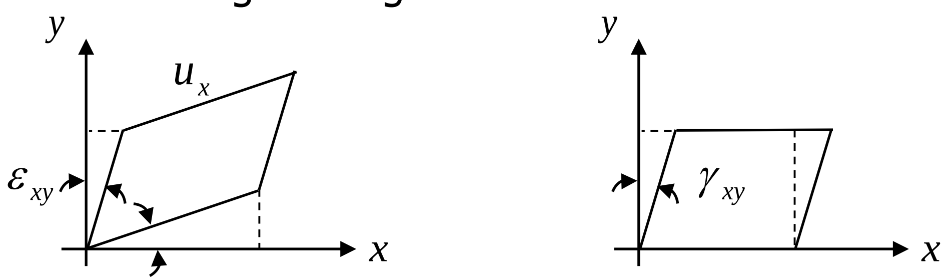
$$\varepsilon_y = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$





Engineering shear strain



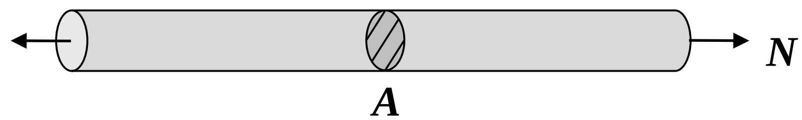
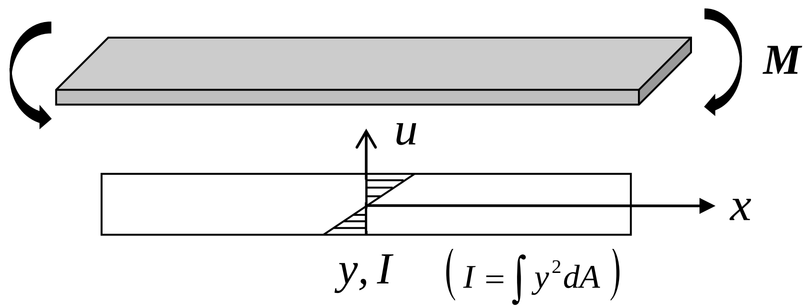
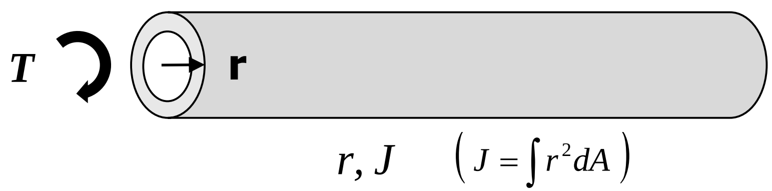
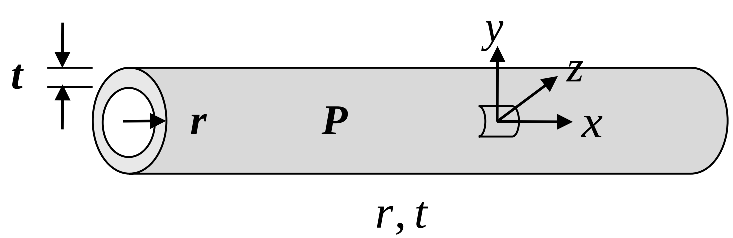
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

Tensorial shear strain

- **Biaxial stress transformation**

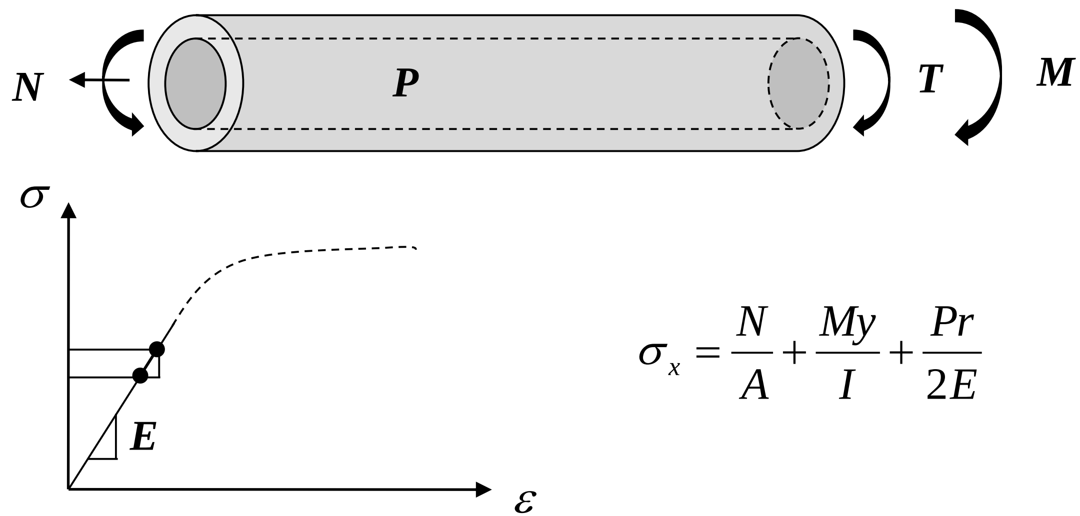
$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dfrac{\gamma_{12}}{2} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \dfrac{\gamma_{xy}}{2} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix}$$

Stress analysis of simple geometrics

loading	Geometry	stress
Axial load		$\sigma_x = \pm \frac{N}{A}$
Bending		$\sigma_x = \pm \frac{My}{I}$ $\sigma_{\max} = \pm \frac{Mc}{I}$
Torsion		$\tau_{xy} = \pm \frac{Tr}{J}$ $\tau_{\max} = \pm \frac{Tc}{J}$
Pressure		$\sigma_x = \pm \frac{Pr}{2t}$ $\sigma_y = \pm \frac{Pr}{t}$ $\sigma_z = \pm P$

Stress analysis of simple geometrics

- **For linear elastic case, the stresses can be added together**



$$\sigma_x = \frac{N}{A} + \frac{My}{I} + \frac{Pr}{2E}$$

- **Failure**

Yielding (Ductile)
Fracture (Brittle)

$\sigma' \gtrless S_y$ Yield strength

Strain energy $\gtrless$ Surface energy to propagate crack

Stress analysis of simple geometrics

- **Von Mises**

Principal direction
(No shear)

$$\frac{1}{\sqrt{2}}\left[\left(\sigma_1^p-\sigma_2^p\right)^2+\left(\sigma_2^p-\sigma_3^p\right)^2+\left(\sigma_3^p-\sigma_1^p\right)^2\right]^{\frac{1}{2}}=\sigma'$$

$$\frac{1}{\sqrt{2}}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+\left(\sigma_z-\sigma_x\right)^2+6\tau_{xy}{}^2+6\tau_{yz}{}^2+6\tau_{zx}{}^2\right]^{\frac{1}{2}}=\sigma'$$

General σ & ε (anisotropic case)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad i,j,k,l = 1,2,3$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad [S_{ijkl}] = [C_{ijkl}]^{-1}$$

General elastic constant C_{ijkl} , S_{ijkl}

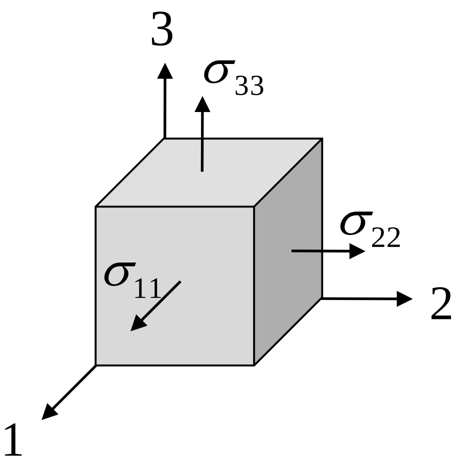
$(3)^4 = 81$ elements

from symmetry of stress, strain as in isotropic case

$$\left.\begin{array}{l}\sigma_{ij} = \sigma_{ji} \\ \varepsilon_{ij} = \varepsilon_{ji}\end{array}\right. \Rightarrow \begin{array}{c}\text{36 element} \\ (6\times 6)\end{array}$$

- **Simplified notation**

$$\begin{cases}\sigma_{11} = \sigma_1 \\ \sigma_{22} = \sigma_2 \\ \sigma_{33} = \sigma_3 \\ \sigma_{23} = \tau_{23} = \sigma_4 = \tau_1 \\ \sigma_{31} = \sigma_5 = \tau_2 \\ \sigma_{12} = \sigma_6 = \tau_3\end{cases}$$



General σ & ε (anisotropic case)

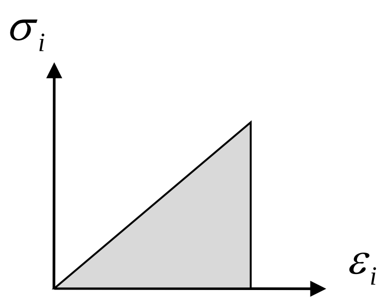
$$\begin{cases} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ 2\varepsilon_{23} = \gamma_{23} = \varepsilon_4 = \gamma_1 \\ 2\varepsilon_{31} = \varepsilon_5 = \gamma_2 \\ 2\varepsilon_{12} = \varepsilon_6 = \gamma_3 \end{cases}$$

Engineering shear strain!!

$$C_{1111} = C_{11} \qquad C_{1122} = C_{12} \qquad C_{1133} = C_{13}$$

$$C_{1123} = C_{14} \qquad C_{1131} = C_{15} \qquad \cdots$$

- **Symmetry of elastic constant**



Work per unit volume $\quad W = \frac{1}{2}\sigma_i\varepsilon_i = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j$

$$\left(\sigma_i = C_{ij}\varepsilon_j\right)$$

By differentiation $\quad \sigma_i = \frac{\partial W}{\partial \varepsilon_i}$