M2794.007700 Smart Materials and Design

Stress – Strain in Anisotropic Material

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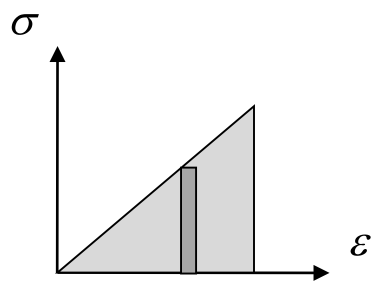
STRESS-STRAIN IN ANISOTROPIC MATERIAL

General anisotropic materials

- **Recap** - **isotropic stress, strain**
  - **generalized Hooke's law ; 81 elements**

- **Special material**



Work per unit volume

$$W = \frac{1}{2}\sigma_i\varepsilon_i = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j$$

$$(\sigma_i = C_{ij}\varepsilon_j)$$

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} = C_{ij}\varepsilon_j$$

also

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij}$$

$$W = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j \qquad \sigma_j = \frac{\partial W}{\partial \varepsilon_j} = C_{ij}\varepsilon_i$$

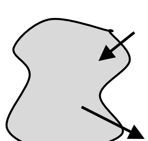
$$\frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = C_{ji}$$

$$C_{ij} = C_{ji}$$

$$S_{ij} = S_{ji}$$

21 constant

cf. Maxwell's reciprocal law

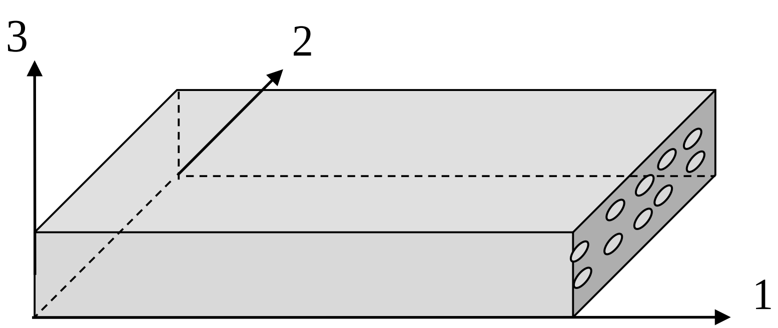


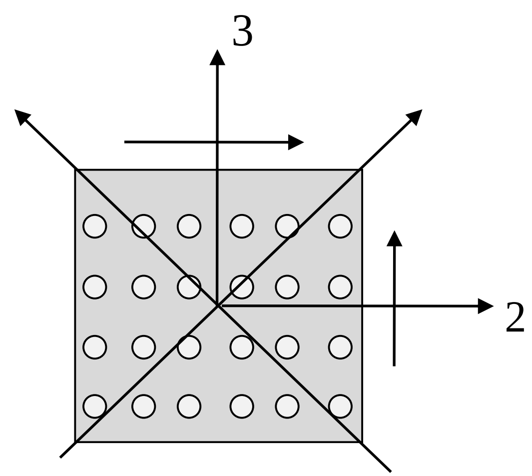
Orthotropic material

- **3 mutually perpendicular planes of material symmetry**

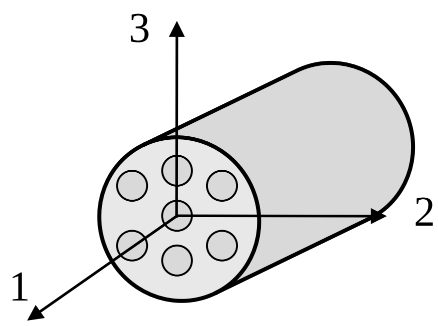
$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

⇨ 9 elements





Transversely isotropic material



- One of principal plane is isotropic

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{12} & C_{23} & C_{22} & & & \\ & & & \frac{C_{22}-C_{23}}{2} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

⇨ 5 constants

Isotropic material

$$
\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}
\begin{matrix} \\ \\ \\ = \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{matrix}
\begin{pmatrix}
C_{11} & C_{12} & C_{12} & & & \\
C_{12} & C_{22} & C_{23} & & & \\
C_{12} & C_{23} & C_{22} & & & \\
 & & & \frac{C_{22}-C_{23}}{2} & & \\
 & & & & C_{55} & \\
 & & & & & C_{66}
\end{pmatrix}
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}
$$

$$
\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}
=
\begin{pmatrix}
C_{11} & C_{12} & C_{12} & & & \\
C_{12} & C_{11} & C_{12} & & & \\
C_{12} & C_{12} & C_{11} & & & \\
 & & & \frac{C_{11}-C_{12}}{2} & & \\
 & & & & \frac{C_{11}-C_{12}}{2} & \\
 & & & & & \frac{C_{11}-C_{12}}{2}
\end{pmatrix}
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}
$$

⇨ 2 constants

More on the orthotropic material

- No interaction between $\sigma_1, \sigma_2, \sigma_3$ & $\gamma_1, \gamma_2, \gamma_3$
- No interaction between $\tau_1, \tau_2\ \tau_3$ & $\varepsilon_1, \varepsilon_2, \varepsilon_3$
- No interaction between τ's & γ's in different plane

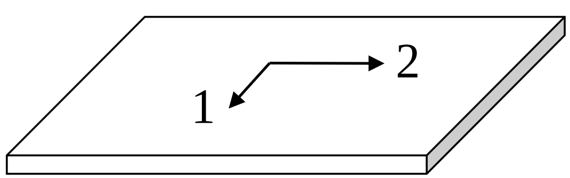
  eg. τ_1 & γ_2

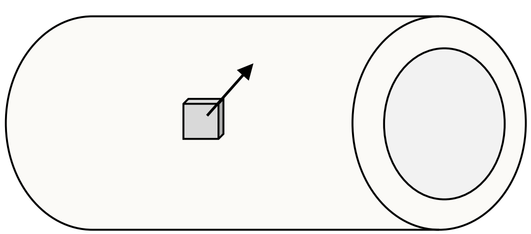
Under plane stress

$$\sigma_3 = \tau_{23}(\tau_1) = \tau_{13}(\tau_2) \cong 0$$

When stresses in thickness direction (3) assumed to zero

eg. Thin plate, (pressure vessel)





Normal component
very small

More on the orthotropic material

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3$$

$$\sigma_2 = C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3$$

$$0 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 \qquad \varepsilon_3 = -\frac{C_{13}\varepsilon_1}{C_{33}} - \frac{C_{23}\varepsilon_2}{C_{33}}$$

$$\gamma_1 = \gamma_2 = 0$$

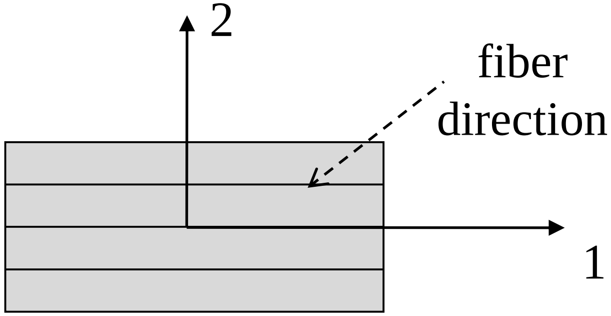
$$\tau_3 = C_{66}\gamma_3$$

$$\sigma_1 = \left(C_{11} - \frac{C_{13}C_{13}}{C_{33}}\right)\varepsilon_1 + \left(C_{12} - \frac{C_{23}C_{13}}{C_{33}}\right)\varepsilon_2 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2$$

$$\sigma_2 = \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_1 + \left(C_{22} - \frac{C_{23}C_{23}}{C_{33}}\right)\varepsilon_2 = Q_{12}\varepsilon_1 + Q_{22}\varepsilon_2$$

More on the orthotropic material

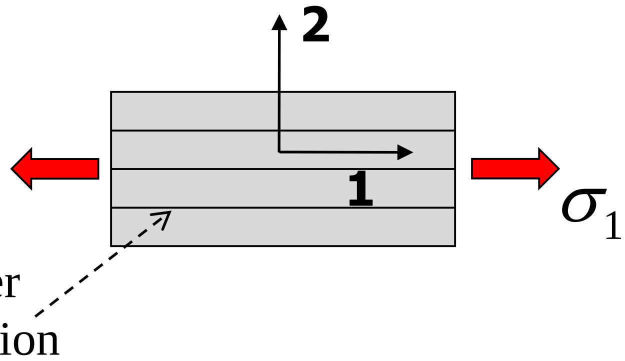
$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_3 \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_3 \end{pmatrix} \Leftrightarrow (\varepsilon) = [S](\sigma) \qquad \text{eq.1}$$

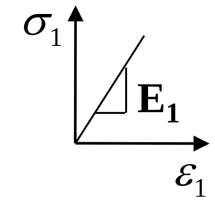


How to relate S_{ij} to engineering constants

- **Orthotropic (general case)**

Test 1.

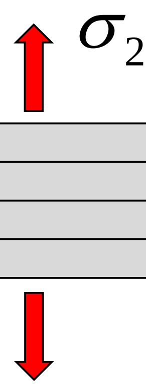




$$\varepsilon_1 = \frac{\sigma_1}{E_1}$$

$$\varepsilon_2 = -\upsilon_{12}\varepsilon_1 = -\upsilon_{12}\frac{\sigma_1}{E_1}$$

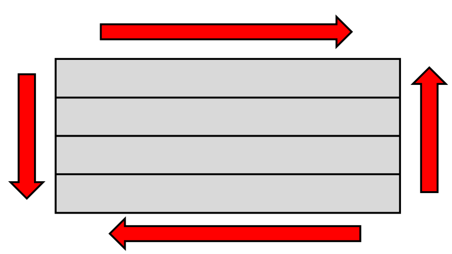
Test 2.



$$\varepsilon_2 = \frac{\sigma_2}{E_2}$$

$$\varepsilon_1 = -\upsilon_{21}\varepsilon_2 = -\upsilon_{21}\frac{\sigma_2}{E_2}$$

Test 3.



$$\tau_{12} = G_{12}\gamma_{12}$$

How to relate S_{ij} to engineering constants

- **In matrix form**

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_3\end{pmatrix}=\begin{pmatrix}S_{11} & S_{12} & 0\\ S_{21} & S_{22} & 0\\ 0 & 0 & S_{66}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_3\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{1}{E_1} & -\dfrac{\upsilon_{12}}{E_1} & \\ -\dfrac{\upsilon_{21}}{E_2} & \dfrac{1}{E_2} & \\ & & \dfrac{1}{G_{12}}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_3\end{pmatrix}$$

$$\frac{\upsilon_{12}}{E_1}=\frac{\upsilon_{21}}{E_2}$$

$$Q_{11}=\frac{E_1}{1-\upsilon_{12}\upsilon_{21}}$$

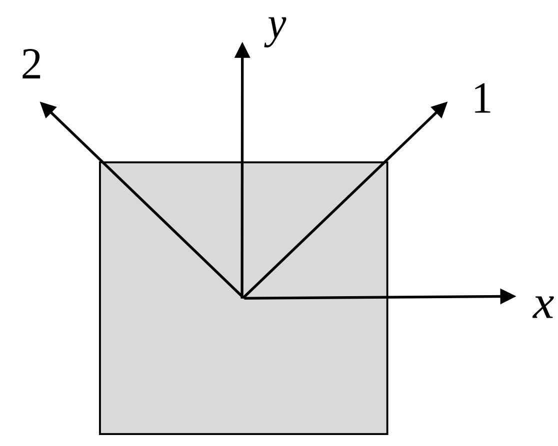
$$Q_{22}=\frac{E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{12}=\frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}}=\frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{66}=G_{12}$$

Transformation of stress & strain

- **Two coordinate systems**
  - Loading axes (x,y)
  - Principal material axes (1,2)



$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_3\end{pmatrix}=[T]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_s\end{pmatrix} \qquad \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\gamma_3\end{pmatrix}=[T]\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\gamma_s\end{pmatrix}$$

$$[T]=\begin{bmatrix} m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & m^2-n^2\end{bmatrix} \qquad \begin{aligned} m&=\cos\theta\\ n&=\sin\theta\end{aligned}$$

$$[T^{-1}]=[T(-\theta)]=\begin{bmatrix} m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & m^2-n^2\end{bmatrix}$$

Transformation of elastic parameters

- **recap – simplification of material constants**

General anisotropic (81)

orthotropic (9)

Transverseley isotropic (5)

isotropic (2) – (E,υ)

Plane stress (4)

$Q_{11}, Q_{22}, Q_{12}, Q_{66}$

$\sigma_3 = \tau_{23} = \tau_{13} = 0$

$(\tau_1)\ (\tau_2)$

Material axes (1, 2) – on axes
Loading axes (x, y) – off axes

Transformation of elastic parameters

- **In simple form**

$$[\sigma]_{x,y} = [Q]_{x,y}[\varepsilon]_{x,y}$$

To obtain $[Q]_{x,y}$ in loading axes (off-axes) using $[Q]_{1,2}$ in material axes & transformation

$$\begin{aligned}[\sigma]_{x,y} &= [T^{-1}][\sigma]_{1,2} &\longleftarrow\quad [\sigma]_{1,2} = [T][\sigma]_{x,y}\\ &= [T^{-1}][Q]_{1,2}[\varepsilon]_{1,2} &\longleftarrow\quad [\sigma]_{1,2} = [Q]_{1,2}[\varepsilon]_{1,2}\\ &= \underline{[T^{-1}][Q]_{1,2}[T]}[\varepsilon]_{x,y} &\longleftarrow\quad [\varepsilon]_{1,2} = [T][\varepsilon]_{x,y}\end{aligned}$$

Known value

$$\begin{cases}[Q]_{1,2} \rightarrow f(E_1, E_2, \upsilon_{12}, G_{12})\\ [T],[T^{-1}] \rightarrow f(\cos\theta, \sin\theta)\end{cases}$$

Transformation of elastic parameters

- **Actual matrix form**

  Eg. (3.31) & (3.35) are for ten/comp along principal axes – no shear strain
  Similarly, under pure shear, τ_3, along principal axes(1,2), only a pure strain γ_3

∴ No coupling between normal stresses & shear deformation and between shear stress & normal strains

Not true if along arbitrary axes(x,y)

- **Stress - Strain**

$$[\sigma]_{x,y} = [Q]_{x,y} [\varepsilon]_{x,y}$$

$$i.e \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{bmatrix}$$

Match with previous page's eq

(3.63)

Coupled terms

A

Transformation of elastic parameters

- **rewrite**

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & 2Q_{xs} \\ Q_{yx} & Q_{yy} & 2Q_{ys} \\ Q_{sx} & Q_{sy} & 2Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_s \end{bmatrix} \qquad (3.64)$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{pmatrix} = [T^{-1}] \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_3 \end{pmatrix} = [T^{-1}] \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_3 \end{bmatrix}$$

on axis

$$= [T^{-1}] \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_3 \end{bmatrix} = [T^{-1}] \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_s \end{bmatrix}$$

to use eq.60. (3.65)

B

Transformation of elastic parameters

$$\begin{bmatrix} Q_{xx} & Q_{xy} & 2Q_{xs} \\ Q_{yx} & Q_{yy} & 2Q_{ys} \\ Q_{sx} & Q_{sy} & 2Q_{ss} \end{bmatrix} = \left[T^{-1} \right] \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} [T]$$

A=B

Different from $[Q]_{1,2}$ Matrix

- **Result**

$$\begin{cases} Q_{xx} = m^4 Q_{11} + n^4 Q_{22} + 2m^2 n^2 Q_{12} + 4m^2 n^2 Q_{66} \\ Q_{yy} = n^4 Q_{11} + m^4 Q_{22} + 2m^2 n^2 Q_{12} + 4m^2 n^2 Q_{66} \\ Q_{xy} = m^2 n^2 Q_{11} + m^2 n^2 Q_{22} + (m^4 + n^4) Q_{12} - 4m^2 n^2 Q_{66} \\ Q_{xs} = m^3 n Q_{11} - mn^3 Q_{22} + (mn^3 - m^3 n) Q_{12} + 2(mn^3 - m^3 n) Q_{66} \\ Q_{ys} = mn^3 Q_{11} - m^3 n Q_{22} + (m^3 n - mn^3) Q_{12} + 2(m^3 n - mn^3) Q_{66} \\ Q_{ss} = m^2 n^2 Q_{11} + m^2 n^2 Q_{22} - m^2 n^2 Q_{12} + (m^2 - n^2)^2 Q_{66} \end{cases}$$

Transformation of elastic parameters

- **Strain - Stress**

$$[\varepsilon]_{x,y} = [\mathrm{S}]_{x,y}[\sigma]_{x,y}$$

$$= \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_s \end{bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{sx} \\ S_{yx} & S_{yy} & S_{ys} \\ \frac{1}{2}S_{sx} & \frac{1}{2}S_{sy} & \frac{1}{2}S_{ss} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}$$

Similarly

C

$$[\varepsilon]_{x,y} = [\mathrm{S}]_{x,y}[\sigma]_{x,y}$$

$$= [\mathrm{T}]^{-1}[\varepsilon]_{1,2}$$

$$= [\mathrm{T}]^{-1}[\mathrm{S}]_{1,2}[\sigma]_{1,2} = [\mathrm{T}]^{-1}[\mathrm{S}][\mathrm{T}][\sigma]_{x,y}$$

Transformation of elastic parameters

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_s \end{pmatrix} = \left[T^{-1}\right] \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_3 \end{pmatrix} = \left[T^{-1}\right] \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & \frac{1}{2}S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_3 \end{bmatrix}$$

(3.61) (3.33)

$$= \left[T^{-1}\right] \underbrace{\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & \frac{1}{2}S_{66} \end{bmatrix} [T]}_{D} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}$$

(3.57)

D

C=D

$$\begin{bmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ \frac{1}{2}S_{sx} & \frac{1}{2}S_{sy} & \frac{1}{2}S_{ss} \end{bmatrix} = \left[T^{-1}\right] \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & \frac{1}{2}S_{66} \end{bmatrix} [T]$$

Transformation of elastic parameters

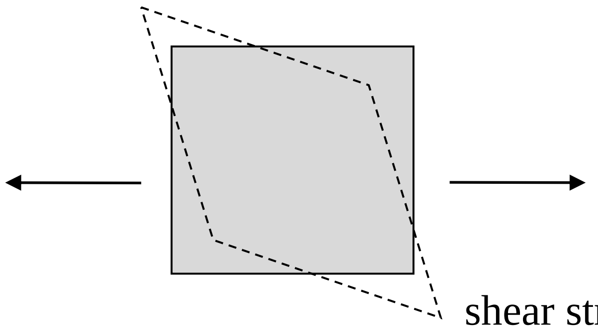
- **Result**

$$
\begin{cases}
S_{xx} = m^4 S_{11} + n^4 S_{22} + 2m^2 n^2 S_{12} + m^2 n^2 S_{66} \\
S_{yy} = n^4 S_{11} + m^4 S_{22} + 2m^2 n^2 S_{12} + m^2 n^2 S_{66} \\
S_{xy} = m^2 n^2 S_{11} + m^2 n^2 S_{22} + (m^4 + n^4) S_{12} - m^2 n^2 S_{66} \\
S_{xs} = 2m^3 n S_{11} - 2mn^3 S_{22} + 2(mn^3 - m^3 n) S_{12} + (mn^3 - m^3 n) S_{66} \\
S_{ys} = 2mn^3 S_{11} - 2m^3 n S_{22} + 2(m^3 n - mn^3) S_{12} + (m^3 n - mn^3) S_{66} \\
S_{ss} = 4m^2 n^2 S_{11} + 4m^2 n^2 S_{22} - 8m^2 n^2 S_{12} + (m^2 - n^2)^2 S_{66}
\end{cases}
$$

Transformation of elastic parameters

$[S]_{x,y}$ in terms of engineering constant

" shear coupling coefficient"



$$n_{xs} = \frac{\gamma_s}{\varepsilon_x}$$

Normal loading ↗ ↖ Shear strain

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\upsilon_{yx}}{E_y} & \frac{n_{sx}}{G_{xy}} \\ -\frac{\upsilon_{xy}}{E_x} & \frac{1}{E_y} & \frac{n_{sy}}{G_{xy}} \\ \frac{n_{xs}}{E_x} & \frac{n_{ys}}{E_y} & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{pmatrix} \qquad (eq.\, 3.77)$$

NOTE)

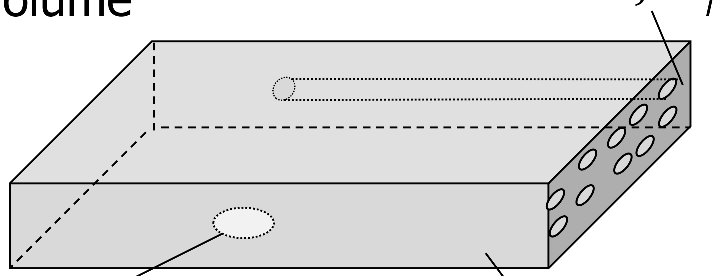
Micromechanics – E_1, E_2, G_{12}, υ_{12} are assumed to be known from direct experiment of unidirectional (properties vary greatly)

Transformation of elastic parameters

- **Micromechanics**

$$C_{ij} = f\left(E_f, E_m, \upsilon_f, \upsilon_m, V_f, V_m, S, A\right)$$

- Volume



when $V_v \cong 0,\ V_f + V_m = 1$

Volume

$$\overline{V} = \overline{V_f} + \overline{V_m} + \overline{V_v}$$

Volume fraction

$$V_i = \frac{\overline{V_i}}{\overline{V}} \qquad i = f, m, v$$

$$V_f + V_m + V_v = 1$$

How to get ?

- Mass $M = M_f + M_m + M_v \quad M_v \cong 0$

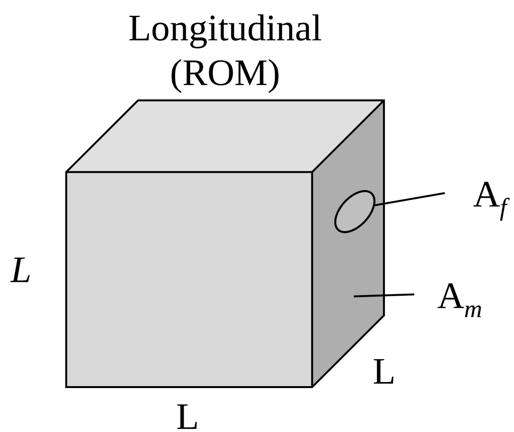
$$M = \rho_f \overline{V_f} + \rho_m \overline{V_m}$$

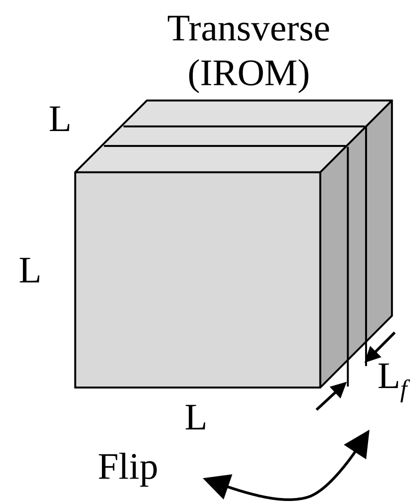
$$\rho = \frac{M}{\overline{V}} = \rho_f \frac{\overline{V_f}}{\overline{V}} + \rho_m \frac{\overline{V_m}}{\overline{V}}$$

$$\rho = \rho_f V_f + \rho_m V_m$$

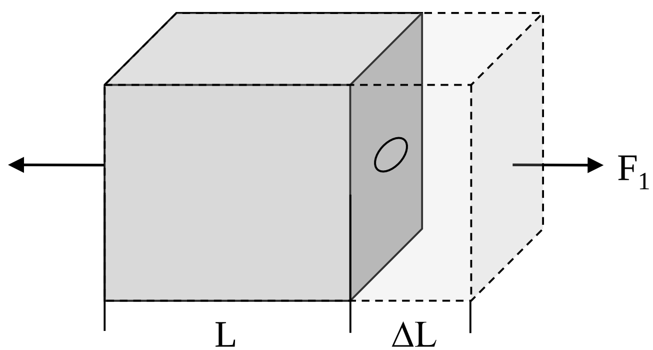
Transformation of elastic parameters

- Representative elements for elastic constants





- Longitudinal Young's modulus



$$\Delta L = \varepsilon_1 L = \varepsilon_{f1} L = \varepsilon_{m1} L$$

$$(\text{Assume}: \varepsilon_1 = \varepsilon_{f1} = \varepsilon_{m1})$$

$$F_1 = F_f + F_m$$

$$\frac{F_1}{A} = \frac{F_f}{A} + \frac{F_m}{A}$$

$$= \frac{F_f}{A_f}\frac{A_f}{A} + \frac{F_m}{A_m}\frac{A_m}{A}$$

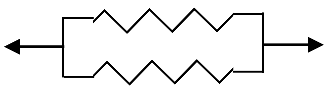
$$\sigma_1 = \sigma_f V_f + \sigma_m V_m$$

Rule of mixture

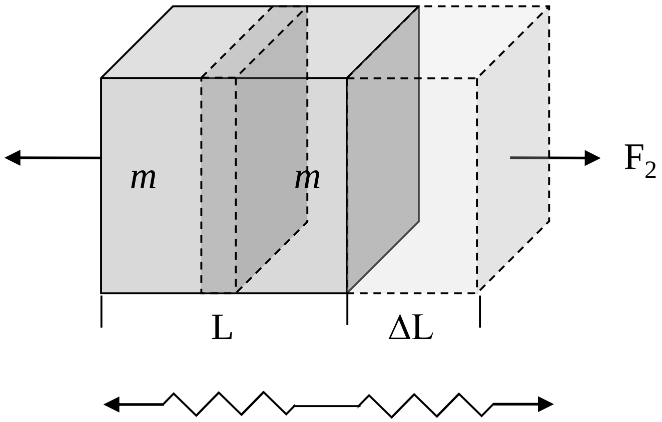
$$\begin{cases} \sigma_{f1} = \varepsilon_{f1} E_{f1} \\ \sigma_{m1} = \varepsilon_{m1} E_{m1} \end{cases}$$

$$\varepsilon_1 E_1 = \varepsilon_{f1} E_{f1} V_f + \varepsilon_{m1} E_{m1} V_m$$

$$E_1 = E_{f1} V_f + E_{m1} V_m$$ ROM



- Transverse Young's modulus



$$F_2 = \sigma_2 A = E_2 \varepsilon_2 A$$

$$\varepsilon_2 = \frac{\Delta L}{L}$$

$$\Delta L = 2 L_m \varepsilon_{m2} + L_f \ \varepsilon_{f2}$$

Rule of mixture

(IROM-continued)

$$\sigma_2 = E_2\varepsilon_2 = E_2\left(2\frac{L_m}{L}\varepsilon_{m2} + \frac{L_f}{L}\varepsilon_{f2}\right)$$

$$\varepsilon_{m2} = \frac{\sigma_{m2}}{E_{m2}}, \qquad \varepsilon_{f2} = \frac{\sigma_{f2}}{E_{f2}}$$

$$(\text{Assume } \sigma_2 = \sigma_{f2} = \sigma_{m2})$$

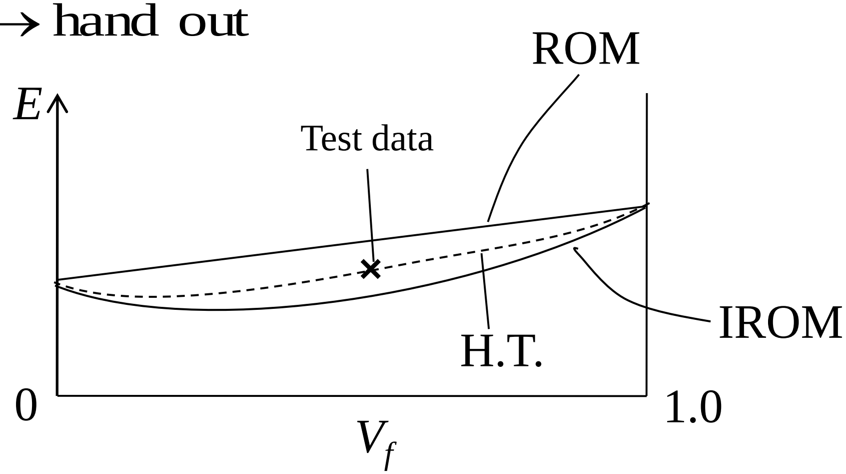
$$\sigma_2 = E_2\left(V_m\frac{\sigma_{m2}}{E_{m2}} + V_f\frac{\sigma_{f2}}{E_{f2}}\right)$$

$$\frac{1}{E_2} = \frac{V_m}{E_{m2}} + \frac{V_f}{E_{f2}} \qquad \text{IROM}$$

$$G_{12}, G_{23}, \upsilon_{12}, \upsilon_{23} \rightarrow \text{hand out}$$

Halpin-Tsai

- Semi empirical model

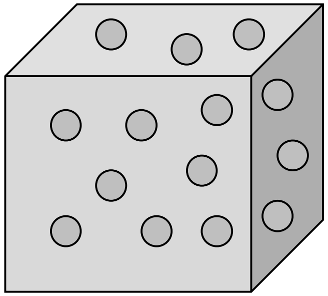


Rule of mixture

1) $$E_f = 30\times10^6 \ psi$$
$$E_m = 0.4\times10^6 \ psi$$
$$V_f = 0.6$$
$$E_1 = 30\times10^6 \times(0.6) + 0.4\times10^6 \times(0.4)$$
$$= 18.16\times10^6 \ psi$$

2) $$\frac{1}{E_2} = \frac{0.6}{30\times10^6} + \frac{0.4}{0.4\times10^6}$$
$$E_2 = 0.98\times10^6 \ psi$$

What about particulate composites



ROM, IROM