State Feedback and State Observer

Chen Ch. 8
Friedland Ch. 6-11
Choset Ch. 8

Dongjun Lee (이동준)

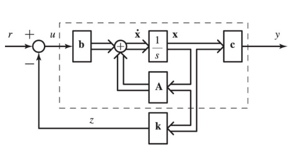
Department of Mechanical & Aerospace Engineering
Seoul National University

State Feedback with State Observer

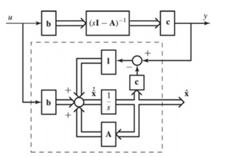
- Consider CT-LTI system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

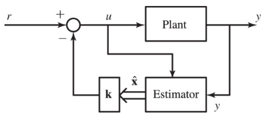
- Best you can get is all state x is available for control $\Rightarrow$ state feedback.
- State feedback (e.g., $u = Kx$) vs output feedback (e.g., $u = u(y)$).
- Static control (e.g., $u = Kx$) vs dynamic control (e.g., IMC).
- Yet, typically too expensive/noisy to measure all state $x \Rightarrow$ state observer.
- State observer: utilize known dynamics to propagate estimated state $\hat{x}$ with correction using real output y and estimated output $C\hat{x} + Du$.
- State feedback with estimated state $\Rightarrow$ observer-based state feedback.
- Regulation control, tracking control, IMC in state space.



state feedback control



Luenberger observer w/ feedback



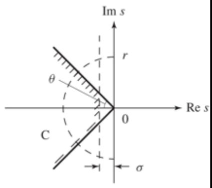
observer-based state-feedback

Pole Placement and CTRB Canonical Form

- Consider SISO CT-LTI system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

  - System's behavior depends on the eigenvalues of A.
  - Poles of $H(s) = C(sI - A)^{-1}B + D \subset$ eigenvalues of A.



- Consider state feedback control $u = -Kx + v$. Then,

$$\dot{x} = (A - BK)x + v = A_{cl}x + v$$

i.e., system dynamics changed by state feedback from A to A_{cl}.

- If (A, B) in CTRB canonical form, can assign arbitrary eigenvalues to A_{cl}.
- $H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o} \Rightarrow$ CTRB canonical form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix} x + Du$$

- With $u = -\begin{bmatrix} k_o & k_1 & k_2 \end{bmatrix} x + v \Rightarrow A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o - k_o & -a_1 - k_1 & -a_2 - k_2 \end{bmatrix}$

SI Pole Placement and CTRB

Th. 8-3: If SI CT-LTI system (A, B) is CTRB, we can assign arbtrary eigenvalues of A_{cl} by state feedback $u = -Kx + v$.

- **Th. 8-2:** Any CTRB (A, B) is equivalent to CTRB canonical form.
  - From proof of Th. 7-3M, define similarity-TF $x = P^{-1}\bar{x}$ with $P^{-1} = C\bar{C}^{-1}$, where

$$\bar{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 - a_1 \end{bmatrix}, \quad \bar{C}^{-1} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

  - Check if $PAP^{-1} = \bar{A} \Rightarrow AP^{-1} = P^{-1}\bar{A} \Rightarrow$

$$\begin{aligned} AP^{-1} &= \begin{bmatrix} (a_1 A + a_2 A^2 + A^3)B & a_2 AB + A^2 B & AB \end{bmatrix} \\ &= \begin{bmatrix} -a_o B & -a_2 AB + A^2 B & AB \end{bmatrix} \end{aligned}$$

which is the same as $P^{-1}\bar{A} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} -a_o & 0 & 0 \\ 0 & a_2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

  - Similarity-TF preserves eigenvalues $\Rightarrow$ if (A, B) CTRB, can assign any CL-eigenvalues via SSFB $u = \bar{K}\bar{x} = \bar{K}Px$, $\bar{K} = [\bar{k}_o, \bar{k}_1, \bar{k}_2]$.

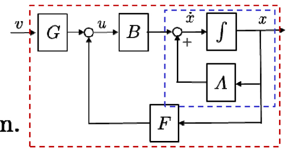
MI Pole Placement and CTRB

Th. 8-3M: All the eigenvalues of $A_{cl} = A - BK$ can be arbitrarily assigned by the state feedback $u = -Kx$ iff MI CT-LTI system (A, B) is CTRB.

- **Heymann's lemma:** Let (A, B) CTRB and $b_i \in \Re^n$, $i = 1, .., p$, be a column vector of B. Then, $\forall b_i$, $\exists F_i \in \Re^{p\times n}$ s.t., $(A - BF_i, b_i)$ is CTRB. Equivalently, $\forall b \in \mathcal{R}(B) \subset \Re^n$, $\exists F \in \Re^{p\times n}$ s.t., $(A - BF, b)$ is CTRB.
  - Heymann's lemma says that a MI CTRB system, with a preliminary SSFB, can be made a CTRB SI system.
  - For MI CT-LTI system $\dot{x} = Ax + Bu$, $B \in \Re^{n\times p}$, $u \in \Re^p$, consider

$$u = -Fx + Gv, \quad G \in \Re^p, \quad b = BG \in \Re^n, \quad v \in \Re$$

where $b \in \mathcal{R}(B)$ and $(A - BF, b)$ CTRB.

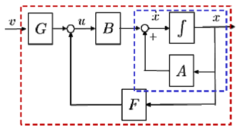


- $\dot{x} = (A - BF)x + BGv = (A - BF)x + bv$.
- Here, $(A - BF, b)$ CTRB $\Rightarrow$ CTRB canonical form.
- $\exists$ P s.t., $x = P^{-1}\bar{x}$ with pole placement $v = -\bar{K}\bar{x}$.
- $u = -Fx - G\bar{K}Px + u' \Rightarrow \dot{x} = A_{cl}x + Bu'$.
- Given n eigenvalues, $F + G\bar{K}P$ not unique with different eigenvectors.

MI Pole Placement: Example

Th. 8-3M: All the eigenvalues of $A_{cl} = A - BK$ can be arbitrarily assigned by the state feedback $u = -Kx$ iff MI CT-LTI system (A, B) is CTRB.

- $u = -Fx + Gv = -Fx - G\bar{K}Px + u'$.




```
A = [-1 1 0;1 -3 0;0 1 -1]; B=[0 0;1 1;0 1];

rank(ctrb(A,B))

F = [1 0 1;0 0 1]; G = [1; 0];

Ap = A-B*F; b = B*G;

a = charpoly(Ap);
ao = a(4); a1 = a(3); a2 = a(2);
ko = 24 - ao; k1 = 26 - a1; k2 = 9 - a2;
Kbar = [ko k1 k2];

cC = [b Ap*b Ap*Ap*b];

barCinv = [a1 a2 1; a2 1 0; 1 0 0];

P=inv(cC*barCinv);

Acl = Ap - B*G*Kbar*P;

eig(Acl)
```

```
>> rank(ctrb(A,B))

ans =

     3
```

```
>> rank(ctrb(A+B*F,B*G))

ans =

     3
```

```
>> eig(Acl)

ans =

   -2
   -4
   -3
```

```
K = F + G*Kbar*P;

eig(Acl)

K1 = place(A,B,[-2 -3 -4])

K1 =

    7.0000    2.0000   -1.0000
   -0.0000    1.0000    1.0000
```

```
K =

     7     3    -1
     0     0     1
```

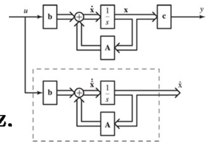
- SSFB cannot change CTRB, but can change OBSV, i.e., w/ SSFB, still in same CTRB canonical form, yet, new pz-cancelation possible:

$$H_{ssfb}(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2' s^2 + a_1' s + a_o'}$$

State Observer

- Consider CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx$. Again, it is typically expensive/noisy to measure all the state x.

- **Open-loop observer:** may simply propagate the state according to the dynamics with observability map to match IC:



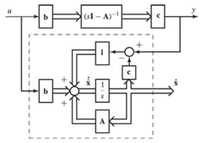
$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \text{with estimated } \hat{x}(0)$$

  - OBSV dynamics: $\dot{e} = Ae$, $e = x - \hat{x} \Rightarrow e \to 0$ if A is Hurwitz.
  - $e(t) \not\to 0$ if unstable A, not robustifiable against disturbance/noise, etc.

- **Luenberger observer:** incorporate feedback for correction, thereby, improve robustness:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L(y - C\hat{x})$ is the corrective feedback of error between real output and estimated output.



  - OBSV dynamics: with $y = Cx$,

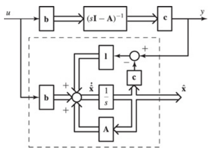
$$\dot{e} = (A - LC)e$$

  - If we can set $A - LC$ Hurwitz $\Rightarrow$ exponentially stable $\Rightarrow$ robust against non-Hurwitz A and disturbance.

State Estimation and OBSV

Th. 8-O3: All the eigenvalues of $A - LC$ can be arbitrarily assigned if and only if (A, C) is OBSV.

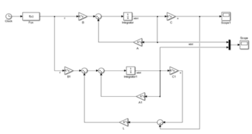
- Consider Luenberger observer:

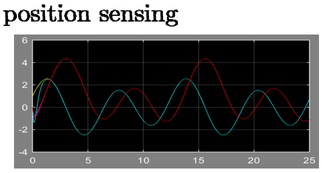


$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

with the OBSV dynamics: $\dot{e} = (A - LC)e$.

- Since $\lambda_i(A - LC) = \lambda_i(A^T - C^T L^T)$, this problem is the same as CTRB of (A^T, C^T), which is OBSV of (A, C).
- Observer gain L should be set s.t., SSOBV is faster than SSFB, yet, slower than noise bandwidth.
- Observer can be used to replace noisy sensor (or be fused together).
- (Ex) $\ddot{x} + x = u$, $u = \sin 0.5t + 1 \Rightarrow A = [0 -1; 10]$, $B = [1; 0]$, $C = [01] \Rightarrow$ estimate velocity from position sensing

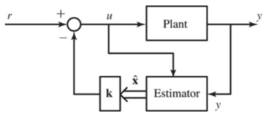




```
>> A=[0 -1;1 0]

A =

     0    -1
     1     0

>> L = place(A', C', [-5+i -5-i])

L =

   25.0000   10.0000
```

Separation Principle

- CT-LTI system: $\dot{x} = Ax + Bu$, $y = Cx$.
- Luenberger state observer: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - Cx)$.
- Feedback of estimated state: $u = -K\hat{x} + v$.



- Controller-observer dynamics: with $e := x - \hat{x}$,

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

- Observer dynamics: $\dot{e} = (A - LC)e \Rightarrow$ unCTRB & not affected by control.
- If $e \to 0$ fast enough, dynamics reduced to $\dot{x} = (A - BK)x + Bv$, $y = Cx$.
- IO-dynamics: $H(s) = \frac{Y(s)}{V(s)} = C(sI - A + BK)^{-1}B$ (with $(x(0), e(0)) = 0$).
- **Separation principle:**

$$\{\lambda(A_{\text{total}})\} = \{\lambda(A - BK)\} \cup \{\lambda(A - LC)\}$$

i.e., we can design state-feedback $(A - BK)$ and state-observer $(A - LC)$ individually and separately.

Reference Tracking Control

- So far, we have mainly considered state stabilization using $u = -Kx$.
- Consider the reference tracking problem: $y(t) \to r(t)$. Then, even just with $u = -Kx + r$,

$$Y(s) = [C(sI - A + BK)^{-1}B + D]R(s) = H(s)R(s)$$

i.e., if $H(s)$ strictly stable with $H(0) \approx I$ (e.g., high gain K), $y(t) \to\approx r(t)$ if $r(t)$ is constant or slowly varying.

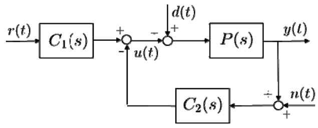
- It would however be more economical or less aggressive if we can incorporate **feedforward action** into the feedback control.
- Consider CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du \Rightarrow$ from linearity, for steady-state, we would have:

$$x_{ss} = N_x r, \quad u_{ss} = N_u r, \qquad N_x \in \Re^{n \times m}, \quad N_u \in \Re^{p \times m}$$

which should satisfy, e.g., for SISO system:

$$0 = Ax_{ss} + Bu_{ss} = (AN_x + BN_u)r$$
$$r = Cx_{ss} + Du_{ss} = (CN_x + DN_u)r$$

Reference Tracking Control



- CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$.
- For SISO system:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} N_x \\ N_u \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Reference tracking control: $u = -K(x - x_{ss}) + N_u r = -Kx + (KN_x + N_u)r$.
- CL-dynamics:

$$\dot{x} = (A - BK)x + B(KN_x + N_u)r = (A - BK)x + B\bar{N}r$$
$$y = Cx + Du$$

- Transfer function relation:

$$Y(s) = C(sI - A + Bk)^{-1}B\bar{N}R(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o}\bar{N}R(s)$$

- Reference tracking may further improved w/ pre-compensator: with

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o}\bar{N}P(s)R(s)$$

  although still under the same limitations of the model matching (i.e., relative degree, non-minimum phase zeros, etc.).
- (Ex) $\dot{x} = -x + u$ with $u = kx + r$ vs $u = -kx + (kn_x + n_u)r$.

Internal Model Control

- Consider CT-LTI system $\dot{x} = Ax + B(u + d)$, $y = Cx$, where d is a disturbance, only known to satisfy differential equation noise model:

$$\dot{\xi}_d = A_d\xi, \quad d = C_d\xi$$

- Disturbance model: $A_d = 0$, $C_d = 1$ (constant d w/ unknown magnitude); $A_d = [0, -w; w, 0]$, $C_d = [1, 0]$ (sinusoid d with known frequency w).
- Total (augmented) system dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

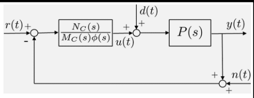
- ξ is not CTRB, yet, may still be OBSV from the output $y \Rightarrow$ estimate ξ and use to cancel the disturbance $d = C_d\xi$.
- Control design:

$$u = -Kx - \hat{d} + v = -\begin{bmatrix} K & C_d \end{bmatrix}[x; \xi] + v$$

  with state observer:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{\xi}} \end{pmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{\xi} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y - C\hat{x})$$

Internal Model Control



- Control design:

$$u = -Kx - \hat{d} + v = -\begin{bmatrix} K & C_d \end{bmatrix} [x;\xi] + v$$

with state observer:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{\xi}} \end{pmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{\xi} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y - C\hat{x})$$

which can be rewritten as, with $v = 0$:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{\xi}} \end{pmatrix} = \begin{bmatrix} A - BK - L_1C & 0 \\ -L_2C & A_d \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$$

- Controller TF $C_{y \to u}(s)$ is given by

$$C_{y\to u}(s) = -\begin{bmatrix} K & C_d \end{bmatrix} \begin{bmatrix} sI - A + BK + L_1C & 0 \\ L_2C & sI - A_d \end{bmatrix}^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

showing that the control contains characteristic polynomial of the disturbance $D_d(s) \Rightarrow$ IMC.

- IMC will not work if OL-TF contains the same zeros of (sustained) disturbance *and* reference $\Rightarrow$ unstable pz-cancelation (cf. Th. 8.5).

Reduced Order Observer

- Consider MO CT-LTI system: $\dot{x} = Ax + Bu$, $y = Cx$. Then, from $y = Cx$, we may directly extract some state information $\Rightarrow$ no need to estimate.
- Suppose (A, C) detectable. Also, $y \in \Re^p$ with $\text{rank}(C) = p$.
- Then, $\exists\, P \in \Re^{n\times n}$ s.t., $CP = [I_p,\ 0_{p\times(n-p)}]$ (i.e., from $CR = [C_1,\ 0_{p\times(n-p)}]$, $P := R \cdot \text{diag}[C_1^{-1}, I_{n-p}]$).
- Define $x = Pz$, $z = [z_a; z_b] \Rightarrow y = Cx = CPz = [z_a; 0]$, i.e., $y = z_a \in \Re^p$ and only z_b is needed to be estimate.
- Then, $\bar{A} = P^{-1}AP = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$, $\bar{B} = P^{-1}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$ with:

$$\dot{z}_b = \bar{A}_{21}y + \bar{A}_{22}z_b + \bar{B}_2u$$
$$\dot{y} = \bar{A}_{11}y + \bar{A}_{12}z_b + \bar{B}_1u$$

where we may consider the first one state equation to estimate z_b, whereas the second one output equation (with $\dot{y} - \bar{A}_{11}y$ known).

- Define z_b-observer:

$$\dot{\hat{z}}_b = \bar{A}_{22}\hat{z}_b + \bar{A}_{21}y + \bar{B}_2u + L(\dot{y} - \bar{A}_{11}y - \bar{A}_{12}\hat{z}_b - \bar{B}_1u)$$

with observation dynamics: $\dot{e} = (\bar{A}_{22} - L\bar{A}_{12})e \Rightarrow (\bar{A}_{22}, \bar{A}_{12})$ OBSV?

Reduced Order Observer

- Transformed z-dynamics:

$$\dot{z}_b = \bar{A}_{21}y + \bar{A}_{22}z_b + \bar{B}_2 u$$
$$\dot{y} = \bar{A}_{11}y + \bar{A}_{12}z_b + \bar{B}_1 u$$

with the z_b-observer: $\dot{\hat{z}}_b = \bar{A}_{22}\hat{z}_b + \bar{A}_{21}y + \bar{B}_2 u + L(\dot{y} - \bar{A}_{11}y - \bar{A}_{12}\hat{z}_b - \bar{B}_1 u)$.

- Observation dynamics: $\dot{e} = (\bar{A}_{22} - L\bar{A}_{12})e \Rightarrow (\bar{A}_{22}, \bar{A}_{12})$ OBSV? $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$
- (A, C) OBSV iff $(\bar{A}, \bar{C})$ OBSV iff $(\bar{A}_{22}, \bar{A}_{12})$ OBSV:

$$\text{rank}\begin{bmatrix} sI_p - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_{n-p} - \bar{A}_{22} \\ I_p & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & -\bar{A}_{12} \\ 0 & sI_{n-p} - \bar{A}_{22} \\ I_p & 0 \end{bmatrix} = n$$

iff $\text{rank}\begin{bmatrix} sI_{n-p} - \bar{A}_{22} \\ -\bar{A}_{12} \end{bmatrix} = n - p$, i.e., $(\bar{A}_{22}, \bar{A}_{12})$ OBSV.

- z_b-observer contains (possibly) noisy $\dot{y}$. Define $w := \hat{z}_b - Ly$. Then, we can show that the z_b-observer can be rewritten as:

$$\dot{w} = (\bar{A}_{22} - L\bar{A}_{12})w + (\bar{B}_2 - L\bar{B}_1)u + (\bar{A}_{21} - L\bar{A}_{11} + \bar{A}_{22}L - L\bar{A}_{11}L)y$$

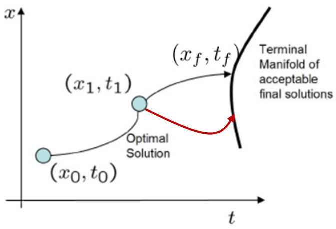
with $\bar{A}_{22} - L\bar{A}_{12}$ Hurwitz $\Rightarrow$ solve w & extract $\hat{z}_b = w + Ly \Rightarrow \hat{x} = P[y; \hat{z}_b]$.

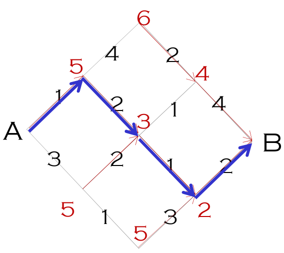
Principle of Optimality

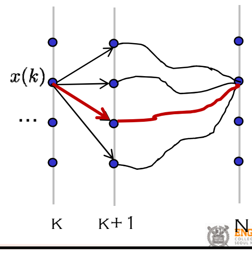
- **Principle of optimality (Bellman):** Suppose the path (x_o, t_o)-(x_1, t_1)-(x_f, t_f) is the optimal path for a problem. Then, the path (x_1, t_1)-(x_f, t_f) is also the optimal path for the same problem starting from (x_1, t_1).

  (Pf) Suppose not. Then, the original path is not optimal $\Rightarrow$ contraction.

- (Ex) Optimal path from A to B with minimum travel time, fuel cost, etc.
  1. Start from $B = x(N)$ (i.e., $k = N$).
  2. Given all $\{x(k+1), J^*_{k+1,N}(x(k+1))\}$, compute $\{u(x(k)), J^*_{k,N}(x(k))\}$ and construct optimal segment from each $x(k)$.
  3. Once hits $A = x(0)$, the optimal path is completed from $x(0)$ to $x(N)$.





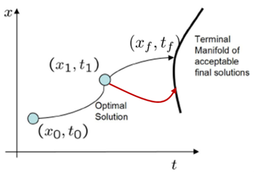


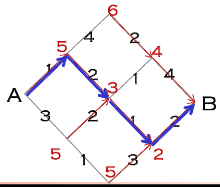
Dynamic Programming

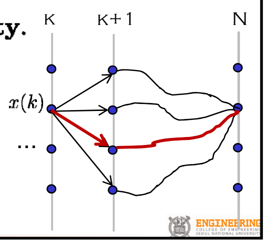
- **Dynamic programming** via principle of optimality:
  1. Start from $k = N$ and work backward.
  2. Given $\{x(k+1), J^*_{k+1,N}(x(k+1))\}$, for all $x(k)$, solve

$$J^*_{kN}(x(k)) = \min_{u(k)} \left[J_{k,k+1}(x(k), u(k)) + J^*_{k+1,N}(x(k+1)) \right]$$

     where $x(k+1) = a(x(k), u(k), k)$ and $J_{k,k+1}$ is incremental cost.
  3. Once hits $x(0)$, construct the optimal path from $x(0)$ to $x(N)$.
- Find global optimum w/o full search (DP $\sim (n+1)^2 - 1$, FS $\sim (2n)!/(n!)^2$).
- Need to store $(u(k), J^*_{k,N})$ for all $x(k)$.
- Scale badly with $\dim(x(k)) \Rightarrow$ **curse of dimensionality.**







Hamilton-Jacobi-Bellman Equation

- Consider $\dot{x} = a(x, u, t)$ with the objective function to minimize:

$$J(x(t_o), t_o, u[t_o, t_f]) := J = h(x(t_f), t_f) + \int_{t_o}^{t_f} g(x(s), u(s), s) ds$$

- Consider $J(x(t), t, u([t, t_f])) := h(x(t_f), t_f) + \int_t^{t_f} g(x(s), u(s), s) ds$ and denote its optimum (with specific optimal control applied) by

$$J^*(x(t), t) = \min_{u([t,t_f])} \left[\int_t^{t_f} g(x(s), u(s), s) ds + h(x(t_f), t_f) \right]$$

- **Principle of optimality** then requires:

$$J^*(x(t), t) = \min_{u[t,t+\delta]} \left[\int_t^{t+\delta} g(x(s), u(s), s) ds + J^*(x(t+\delta), t+\delta) \right]$$

- Taking $\delta \to 0$ with $\delta x = a(x, u, t)\delta$, we then obtain **HJB equation**:

$$\left.\frac{\partial J^*}{\partial t}\right|_{(x(t),t)} + \min_{u(t)} \left[g(x(t), u(t), t) + \left.\frac{\partial J^*}{\partial x}\right|_{(x(t),t)} a(x(t), u(t), t) \right] = 0$$

HJB Equation w/ Hamiltonian

- For $\dot{x} = a(x,u,t)$ with cost function $J = h(x(t_f),t_f) + \int_{t_o}^{t_f} g(x(s),u(s),s)ds$, **HJB equation** is given by:

$$\frac{\partial J^*}{\partial t} + \min_{u(t)} \left[g(x(t),u(t),t) + \frac{\partial J^*}{\partial x} a(x(t),u(t),t) \right] = 0$$

- Define Hamiltonian: with $J_x^* := \frac{\partial J^*}{\partial x}$ and $J_t^* := \frac{\partial J^*}{\partial t}$,

$$\mathcal{H}(x(t),u(t),J_x^*(x(t),t),t) := g(x(t),u(t),t) + J_x^*(x(t),t) \cdot a(x(t),u(t),t)$$

- We may then rewrite HJB equation by

$$J_t^*(x(t),t) + \min_{u(t)} \mathcal{H}(x(t),u(t),J_x^*(x(t),t),t) = 0$$

  or, with optimizer $u^*(t)$, $J_t^*(x(t),t) + \mathcal{H}(x(t),u^*(t),J_x^*(x(t),t),t) = 0$.

- HJB equation defines a backward PDE w/ BC $J^*(x(t_f),t_f) = h(x(t_f),t_f)$; the state x integrated forward in time from $(x(t_o),t_o)$ with $u(J_x^*(x(t),t),x(t),t)$.

- HJB is **sufficient & necessary** condition for the global optmal control! Difficult to apply in practice though.

HJB Equation - Example

- For $\dot{x} = a(x,u,t)$ with cost function $J = h(x(t_f),t_f) + \int_{t_o}^{t_f} g(x(s),u(s),s)ds$, **HJB equation** is given by:

$$J_t^*(x(t),t) + \min_{u(t)} \mathcal{H}(x(t),u(t),J_x^*(x(t),t),t) = 0$$

  where $\mathcal{H}(x(t),u(t),J_x^*(x(t),t),t) := g(x(t),u(t),t) + J_x^*(x(t),t) \cdot a(x(t),u(t),t)$.

- (Ex 3.11-1) For $\dot{x} = x + u$, find optimal control to minimize

$$J = \tfrac{1}{4}x^2(T) + \int_0^T \tfrac{1}{4}u^2(t)dt$$

  - HJB: $J_t^* + \min_u \mathcal{H}(x,u,t) = 0$ with $\mathcal{H} = \frac{1}{4}u^2 + J_x^*(x+u)$.
  - Find u^*: $\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow u^* = -2J_x^*(x,t)$ w/ $\frac{\partial^2 \mathcal{H}}{\partial x^2} = \frac{1}{2}$ (i.e., minimizer).
  - Need to solve PDE of J^*: $\frac{\partial J^*}{\partial t} - \left(\frac{\partial J^*}{\partial x}\right)^2 + \frac{\partial J^*}{\partial x}x = 0$ with **boundary condition** $J^*(x(T),T) = \frac{1}{4}x^2(T)$.
  - Assume **optimal cost** $J^*(x,t) = \frac{1}{2}K(t)x^2$ with $K(T) = \frac{1}{2}$ for BC.
  - HJB: $[\frac{1}{2}\dot{K} - K^2 + K]x^2 = 0 \Rightarrow K(t) = e^{T-t}/(e^{T-t} + e^{-T+t}) \to 1$.

Continuous-Time Finite-Horizon LQR

- Consider linear $\dot{x} = A(t)x(t) + B(t)u(t)$ with quadratic cost function:

$$J = \frac{1}{2}x_f^T H x_f + \int_{t_o}^{t_f} \frac{1}{2}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt$$

where $H, Q \succeq 0$ and $R \succ 0$, all symmetric; and t_f is fixed (not x_f).

- Hamiltonian: $\mathcal{H}(x, u, J_x^*, t) = \frac{1}{2}\left[x^T Q(t)x + u^T R(t)u\right] + J_x^* \cdot [A(t)x + B(t)u]$
- $\min_u \mathcal{H}$: $\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow u^*(t) = -R^{-1}(t)B^T(t)J_x^{*T}(x(t), t)$ w/ $\frac{\partial^2 \mathcal{H}}{\partial u^2} = R \succ 0$.
- HJB: $0 = J_t^* + \frac{1}{2}x^T Q x - \frac{1}{2}J_x^* B^T R^{-1} B J_x^* + J_x^* A x$ with $J^*(t_f) = \frac{1}{2}x_f^T H x_f$.
- Choose $J^*(x(t), t) = \frac{1}{2}x^T(t)P(t)x(t)$, with $P(t) \succeq 0$ symmetric.
- HJB: $0 = \frac{1}{2}x[\dot{P} + Q - PBR^{-1}B^T J_x^{*T} + PA + A^T P]x$.
- Optimal control: $u(t) = -R^{-1}(t)B^T(t)P(t)x(t)$ (**linear state feedback**)
- **DRE** (differential Riccati equation: offline solved/stored ahead of time):

$$-\dot{P} = PA + A^T P + Q - PBR^{-1}B^T P, \quad \text{from } P(t_f) = H$$

Continuous-Time Infinite-Horizon LQR

- Consider LTI system $\dot{x} = Ax + Bu$ with quadratic cost function:

$$J = \int_0^\infty \frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

Suppose (A, B) STLB, $Q = C^T C \succeq 0$ with (A, C) DETB, $R \succ 0$.

  - (A, B) not STLB or (A, C) not DETB $\Rightarrow x \to \infty$ (unstable).
  - (A, B) STLB and (A, C) DETB $\Rightarrow x \to 0$ (stable) $\Rightarrow H = 0$.
  - $P(t) \to P_\infty \succeq 0$, as $t \to 0$, which is also unique solution of **ARE** (algebraic Riccati equation):

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

  - ARE can possess many solutions, yet, only one solution if PSD.
  - $P_\infty \succ 0$ iff (A, B) STLB and (A, C) OBSV (i.e., more control needed).
  - **Intinite-Horizon LQR control**: $u(t) = -R^{-1}B^T P_\infty x$.
  - CL system with P_∞ still asymptotically stable.
  - Optimal cost: $J^*(x(0), 0) = \frac{1}{2}x^T(0)P_\infty x(0)$.

Gain Selection

Consider LTI system $\dot{x} = Ax + Bu$ with quadratic cost function:

$$J = \int_0^\infty \tfrac{1}{2}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

- With (A, B) STLB, choose $Q = C^T C$ s.t., (A, C) DETB and $R \succ 0$.
- For LTV (or FH-LQR), offline solve DRE backward in time with BC $P(t_f) = H$ to obtain $P(t)$ and store $K(t) = -R^{-1}(t)B^T(t)P(t)$.
- For LTI (w/ IH-LQR), solve ARE to obtain $P_\infty \Rightarrow K = -R^{-1}B^T P_\infty$.
- Gain selection Q, R:

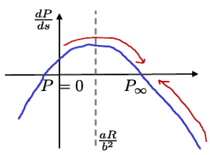
$$Q = \begin{bmatrix} \frac{1}{\tau_1 |x^1_{\max}|^2} & & \\ & \ddots & \\ & & \frac{1}{\tau_n |x^n_{\max}|^2} \end{bmatrix}, \quad R = \rho \begin{bmatrix} \frac{1}{|u^1_{\max}|^2} & & \\ & \ddots & \\ & & \frac{1}{|u^n_{\max}|^2} \end{bmatrix}$$

  where τ_i is time constant for x_i; $x^i_{\max}$ is constraint on x_i; $u^i_{\max}$ is constraint on u_i; and ρ is a weighting factor between regulation performance and control effort.
- Robustness of LQR: GM $= \infty$, PM $\geq 60°$.

LQR: Examples

Consider a scalar LTI system $\dot{x} = ax + bu$ $(b \neq 0)$ with quadratic cost function:

$$J = \int_0^\infty [Qx^2 + Ru^2]dt, \quad Q > 0, R > 0$$



- With $b \neq 0 \Rightarrow$ CTRB; $C = \sqrt{Q} \neq 0 \Rightarrow$ OBSV.
- Riccati equation: $-\dot{P} = 2aP + Q - \frac{b^2P^2}{R}$ with $P(t) > 0$.
- Optimal control: $u = K(t)x(t)$ or $u = K_\infty x(t)$

$$u(t) = -\frac{b}{R}P(t)x(t) \quad \text{(via DRE) or} \quad u(t) = -\frac{b}{R}P_\infty x(t) \quad \text{(via ARE)}$$

- $[-\mathtt{K}_\infty, \mathtt{P}_\infty, \lambda(\mathtt{A}_{\mathtt{cl}})] = \mathtt{lqr}(\mathtt{A}, \mathtt{B}, \mathtt{Q}, \mathtt{R}) \Rightarrow P_\infty = 20.9398, K_\infty = -0.5758$.
- P_∞ can be directly obtained from quadratic ARE (postive root):

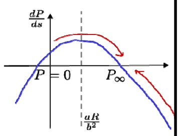
$$P_\infty = \frac{a + \sqrt{a^2 + b^2 Q/R}}{b^2/R}$$

- This P_∞ is in fact a stable equilibriutm of DRE backward in time, i.e., with $s = t_f - t$, $ds = -dt$, DRE becomes

$$\frac{dP}{ds} = 2aP + Q - \frac{b^2P^2}{R} \quad \Rightarrow \quad \frac{dP}{ds} = 0 \text{ with } P = P_\infty$$

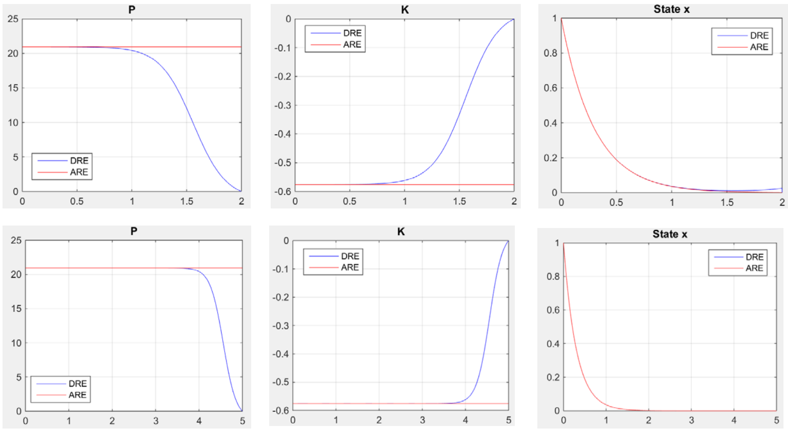
©Dongjun Lee

LQR: Examples



- $P_\infty = \frac{a+\sqrt{a^2+b^2Q/R}}{b^2/R}$ is GAS equilibrium of DRE for $P(t_f) \geq 0$:

$$\frac{dP}{ds} = 2aP + Q - \frac{b^2P^2}{R} \quad \Rightarrow \quad \boxed{P(s) \to P_\infty \quad \text{if } P(t_f) \geq 0}$$



- Using P_∞, K_∞ instead of $P(t), K(t)$ adequate unless time-horizon is short and terminal condition is important (time-varying gain necessary).

LQR: Examples

Consider a scalar LTI system $\dot{x} = ax + bu$ $(b \neq 0)$ to minimuze

$$J = \int_0^\infty [Qx^2 + Ru^2]dt, \quad Q > 0, R > 0$$

- Infinite-horizon LQR: $u(t) = -\frac{b}{R}P_\infty x(t)$ with $P_\infty = \frac{a+\sqrt{a^2+b^2Q/R}}{b^2/R}$.
- Closed-loop dynamics with P_∞ and $K_\infty = -\frac{bP_\infty}{R} = -\frac{a+\sqrt{a^2+\frac{b^2Q}{R}}}{b}$:

$$\dot{x} = ax + bu = ax - \frac{b^2P_\infty}{R}x = -\sqrt{a^2 + \frac{b^2Q}{R}} \cdot x \quad \Rightarrow \quad \text{AS (ES)}$$

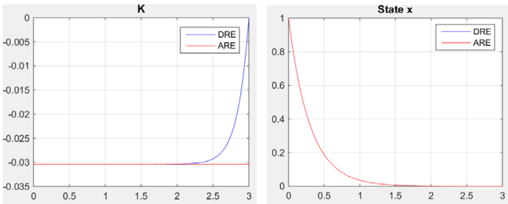
- **Cheap control:** $\frac{Q}{R} \to \infty$, $K_\infty = -\frac{Q}{R} \Rightarrow \boxed{\dot{x} = -b\frac{Q}{R}x} \Rightarrow x \to 0$ very quickly with large control cost.
- **Expensive control:** $\frac{Q}{R} \to 0$, $K_\infty = -\frac{1}{b}[a + |a|] \Rightarrow u = -\frac{1}{b}[a+|a|]x$
  - If $a > 0$ (i.e., OL-unstable) $\Rightarrow u = -\frac{2a}{b}x \Rightarrow$ CL-dynamics: $\boxed{\dot{x} = -ax}$ (i.e., symmetric OL/CL poles).
  - If $a < 0$ (i.e., OL-stable) $\Rightarrow \boxed{u = 0} \Rightarrow \dot{x} = ax$ with $x \to 0$ (i.e., avoid using expensive control at all).

LQR: Examples

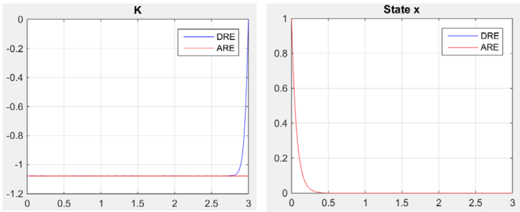
Consider a scalar LTI system $\dot{x} = ax + bu$ $(b \neq 0)$ to minimuze

$$J = \int_0^\infty [Qx^2 + Ru^2]dt, \quad Q > 0, R > 0$$

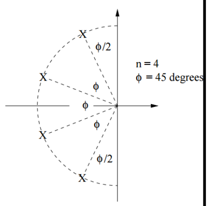
- Infinite-horizon LQR: $u(t) = -\frac{b}{R} P_\infty x(t)$ with $P_\infty = \frac{a + \sqrt{a^2 + b^2 Q/R}}{b^2/R}$.
- Expensive control
  $Q = 7, R = 400,\ A = -3.$



- Cheap control
  $Q = 7, R = 4,\ A = -3.$



LQR: Examples



Consider $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ to minimuze

$$J = \int_0^\infty [x_1^2 + ru^2]dt$$

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0$ and $R = r > 0$.
- (A, B) CTRB and (A, C), $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ with $Q = C^T C$ OBSV.
- Solve ARE $0 = PA + A^T P + Q - PBR^{-1}B^T P$ for $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \Rightarrow$
  $P = \begin{bmatrix} \sqrt{2} r^{\frac{1}{4}} & r^{\frac{1}{2}} \\ r^{\frac{1}{2}} & \sqrt{2} r^{\frac{3}{4}} \end{bmatrix}$, which is PD with $p_{11} > 0$ and $p_{11} p_{22} - p_{12}^2 > 0$.
- Optimal gain $K = -\frac{1}{r} B^T P = -\begin{bmatrix} r^{-\frac{1}{2}} & \sqrt{2} r^{-\frac{1}{4}} \end{bmatrix}$.
- CL-dynamics: $A_{cl} = A + BK = \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{2} r^{-\frac{1}{4}} \end{bmatrix}$ with **symmetric eigenvalue pattern:**

$$\lambda(A_{cl}) = \left\{ r^{-\frac{1}{4}} \frac{-1 + j}{\sqrt{2}}, r^{-\frac{1}{4}} \frac{-1 - j}{\sqrt{2}} \right\}$$

Estimation Problem with Noises

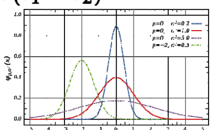
- Consider LTI system:

$$\dot{x} = Ax + Bu + Bw, \quad y = Cx + v$$

with $w \in \Re^n$ **process noise** (e.g., model uncertainty) and $v \in \Re^m$ **measurement noise** (e.g., sensor noise).

- Assume uncorrelated zero-mean Gaussian for $w \in \mathcal{N}(0, W)$ and $v \in \mathcal{N}(0, V)$, i.e., $E[w(t)] = 0$ and $E[v(t)] = 0$ and

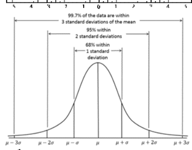
$$E[w(t_1)w^T(t_2)] = W(t)\delta(t_1 - t_2), \quad E[v(t_1)v^T(t_2)] = V(t)\delta(t_1 - t_2)$$
$$E[w(t_1)v^T(t_2)] = 0$$



- Luenberger observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

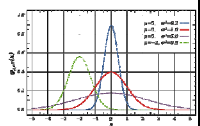
with estimation error dynamics: $\dot{e} = (A - LC)e + Bw + Lv$.



- Mean estimation: $E[\dot{e}] = (A - LC)E[e] \Rightarrow E[e] \to 0$.
- Large L results in faster $E[e] \to 0$, yet, larger amplification of v (i.e., wide scatter or large covariance $P_e(t) = E[e(t)e^T(t)]$ about x) $\Rightarrow$ **optimal** L?

Linear Quadratic Estimation (LQE)

- LTI system: $\dot{x} = Ax + Bu + Bw$, $y = Cx + v$, $w \in \mathcal{N}(0, W)$, $v \in \mathcal{N}(0, V)$.
- State estimator: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \Rightarrow e(t) \approx \mathcal{N}(0, P_e(t))$
- **Optimal estimation:** for CT-LTI system with Gaussian w, v, given $y(\tau)$, $\tau \in [0, t]$, design estimator to minimize $J(t) = \text{trace}[P_e(t)]$.



- Covariance evolution:

$$\dot{P}_e = [A - LC]P_e + P_e[A - LC]^T + BWB^T + LVL^T$$
$$= AP_e + P_eA^T + BWB^T + LVL^T - P_eC^TL^T - LCP_e$$

- ARE for LQR can be written as: with $K = -R^{-1}B^TP$,

$$-\dot{P} = A^TP + PA + Q + K^TRK + PBK + K^TB^TP$$

- **Kalman gain:** from similarity, $L^T = V^{-1}CP_e \Rightarrow L = P_e(t)C^T(t)V^{-1}(t)$.
- Covariance evolution with Kalman gain L:

$$\dot{P}_e = AP_e + P_eA^T + BWB^T - P_eC^TV^{-1}CP_e$$

- Duality of LQR and LQE: $(A, B) \approx (A^T, C^T)$, $(Q, R) \approx (BWB^T, V)$, $(K, P) \approx (-L^T, P_e)$.

Kalman-Bucy Filter

Kalman-Bucy (LQE) filter:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad L = P_e(t)C^T(t)V^{-1}(t)$$
$$\dot{P}_e = AP_e + P_eA^T + BWB^T - P_eC^TV^{-1}CP_e, \quad \hat{x}(0), P_e(0)$$

- Given $P_e(0)$, solve $P_e(t)$ forward in time (offline and store).
- $-K = \texttt{lqr}(A, B, Q, R)$, $L = \texttt{lqr}^T(A^T, C^T, BWB^T, V)$ $(A+BK/A-LC)$.
- $L = P_eC^TV^{-1} \uparrow \Leftarrow$ if state uncertainty $P_e \uparrow$ or sensing uncertainty $V \downarrow$.
- $\dot{P}_e$: 1) P_e propagates through state dynamics; 2) increase due to process noise BWB^T; 3) decrease due to sensing $P_eC^TV^{-1}CP_e$ (i.e., innovation).
- For LTI system, if $V \succ 0$, (A, C) DETB, (A, B) STZB, $P_e(t) \to P_e^\infty \succeq 0$, which is also PSD unique solution of ARE:

$$AP_e + P_eA^T + BWB^T - P_eC^TV^{-1}CP_e = 0$$

  with $P_e^\infty \succ 0$ iff (A, C) DETB and (A, B) CTRB.
- Filter with $L = P_e^\infty C^T(t)V^{-1}(t)$ still asymptotically stable.

Kalman Filtering w/ No Uncertainty

- Linear discrete plant dynamics with measurement (cf. EKF, UKF):

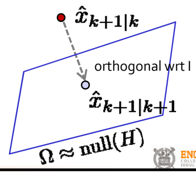
$$x_{k+1} = F_kx_k + G_ku_k, \quad y_k = H_kx_k$$

  where $x_k \in \Re^n$ is state, $u_k \in \Re^p$ input, and $y_k \in \Re^m$ measurement output.
- If F_k, G_k, u_k known (impractical: to be relaxed), state estimator:

$$\hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + G_ku_k$$

  where $\hat{x}_{k+1|k}$ is **prediction** of x_{k+1} given "best" estimate $\hat{x}_{k|k}$ of x_k propagated via dynamics over $[k, k+1]$ (can't do any better than this).
- Now, suppose measurement y_{k+1} given at $k+1$. Then, how to update $\hat{x}_{k+1|k}$ using this information?
- First of all, the estimate $\hat{x}_{k+1|k+1}$ of x_{k+1} should be consistent with this information $y_{k+1} = H_{k+1}x_{k+1}$, i.e.,

$$\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1}x\}$$



Kalman Filtering w/ No Uncertainty

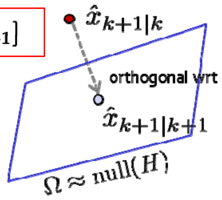
- Estimate $\hat{x}_{k+1|k+1}$ of x_{k+1} should consistent w/ $y_{k+1} = H_{k+1}x_{k+1}$, i.e.,

$$\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1}x\}$$

- Optimal estimate $\hat{x}_{k+1|k+1} \Rightarrow$ **correction** of $\hat{x}_{k+1|k}$ into its closest point on Ω with Euclidean norm.

- Using $\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k} = H_{k+1}^T\alpha$ and $y_{k+1} = H_{k+1}\hat{x}_{k+1|k+1}$,

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}[y_{k+1} - \hat{y}_{k+1}]$$



where $\hat{y}_{k+1} = H_{k+1}\hat{x}_{k+1|k}$ (best estimated output).

- Kalman filtering w/ no uncertainty: with $\hat{x}_{0|0}$,
  1. Plant: $x_{k+1} = F_k x_k + G_k u_k$ w/ measurement $y_k = H_k x_k$
  2. **Prediction** (propagation): $\hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + G_k u_k$
  3. Measurement (output): $y_{k+1} = H_{k+1}x_{k+1}$
  4. Estimated measurement: $\hat{y}_{k+1} = H_{k+1}\hat{x}_{k+1|k}$
  5. **Correction** (update): $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}[y_{k+1} - \hat{y}_{k+1}]$

Kalman Filtering w/ Process Noise

- Plant dynamics with measurement and process noise v_k:

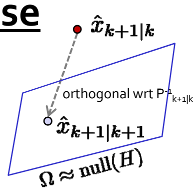
$$x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k$$

where $v_k \in \Re^n$ zero mean Gaussian w/ $E[v_k] = \bar{v}_k = 0$ and covariance $E[(v_k - \bar{v}_k)(v_k - \bar{v}_k)^T)] = V_k \in \Re^{n\times n}$ (e.g., uncertainty in actuation u_k, modeling F_k, G_k, unmodeled friction/slip, discretization).

- Now, x_k becomes RV $\Rightarrow$ need to estimate its mean and also covariance too, i.e., starting from $(\hat{x}_{0|0}, P_{o|o})$,
  - **Prediction** $(\hat{x}_{k+1|k}, P_{k+1|k})$: by propagating $(\hat{x}_{k|k}, P_{k|k})$ via plant dynamics with uncertainty V_k due to process noise.
  - **Correction** $(\hat{x}_{k+1|k+1}, P_{k+1|k+1})$: by using $r_{k+1} = y_{k+1} - \hat{y}_{k+1}$ with uncertainty S_{k+1} of r_{k+1} also taken into account.

- **Prediction:**
  - State (mean) prediction: $\hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + G_k u_k$
  - Uncertainty (covariance) propagation: $P_{k+1|k} = F_k P_{k|k} F_k^T + V_k$ where $P_{k+1|k} = E[(\hat{x}_{k+1|k} - x_{k+1})(\hat{x}_{k+1|k} - x_{k+1})^T]$, i.e., uncertainty from perfect estimate x_{k+1} (w/ v_k independent from $x_k, x_{k|k}$).

Kalman Filtering w/ Process Noise



- Plant dynamics with measurement and process noise v_k:

$$x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k$$

- **Prediction:**
  - State (mean) prediction: $\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k$
  - Uncertainty (covariance) propagation: $P_{k+1|k} = F_k P_{k|k} F_k^T + V_k$
- Now, given y_{k+1} is given. how to update $(\hat{x}_{k+1|k}, P_{k+1|k}) \rightarrow (\hat{x}_{k+1|k+1}, P_{k+1|k+1})$ while minimizing uncertainty?

$$\hat{x}_{k+1|k+1} = \arg\min_{x \in \Re^n} \frac{1}{\sqrt{(2\pi)^n |P_{k+1|k}|}} e^{-\frac{1}{2}(x-\hat{x}_{k+1|k})^T P^{-1}_{k+1|k}(x-\hat{x}_{k+1|k})}$$

subject to $\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1} x\} \Rightarrow$ equivalent to:

$$\min_x \frac{1}{2}(x - \hat{x}_{k+1|k})^T P^{-1}_{k+1|k}(x - \hat{x}_{k+1|k}), \quad \text{subj. to} \quad y_{k+1} = H_{k+1} x$$

- **Mahalanobis metric** $P^{-1}_{k+1|k}$: more weight and updating action for channels with smaller $P_{k+1|k}$ (i.e., high certainty).

Kalman Filtering w/ Process Noise

- The estimate $\hat{x}_{k+1|k+1}$ should again be consistent with $y_{k+1} = H_{k+1} x_{k+1}$:

$$\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1} x\}$$

- Optimal estimate $\hat{x}_{k+1|k+1}$: <u>correction</u> of $\hat{x}_{k+1|k}$ into its closest point on Ω with Mahalanobis norm $P^{-1}_{k+1|k}$.
- Using $\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k} = P_{k+1|k} H^T_{k+1} \alpha$ ($\perp$ null(H) w.r.t. $P^{-1}_{k+1|k}$), $y_{k+1} = H_{k+1} x_{k+1}$, and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$ (best estimated output):

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}]$$
$$K_{k+1} = P_{k+1|k} H^T_{k+1} S^{-1}_{k+1}, \quad S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1}$$

  - **Residual covariance** S_{k+1}: uncertainty in r_{k+1} (solely due to $\hat{y}_{k+1}$);
  - **Kalman gain** K_{k+1}: more update action for more uncertain state with more certain measurement information.
- Uncertainty update (reduction):

$$P_{k+1|k+1} = E\left[(\hat{x}_{k+1|k+1} - x_{k+1})(\hat{x}_{k+1|k+1} - x_{k+1})^T\right]$$
$$= P_{k+1|k} - P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} H_{k+1} P_{k+1|k}$$

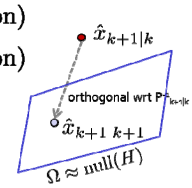
Kalman Filtering w/ Process Noise

- Plant dynamics with measurement and process noise v_k:

$$x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k$$

- **Prediction:**

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$



- **Measurement:** $y_{k+1} = H_{k+1} x_k$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$
- **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(state correction)}$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k} \qquad \text{(uncertainty reduction)}$$

  – Redisual variance: $S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T \; (= E[r_{k+1} r_{k+1}^T])$
  – Kalman gain: $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$
  – Uncertainty always reduced with certain measurement info y_k.
  – $H = I \Rightarrow P_{k+1|k+1} \to 0$, i.e., perfect estimation with $y_{k+1} = x_{k+1}$.

Kalman Filtering

- Plant dynamics with process noise v_k and measurement noise w_k:

$$x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k + w_k$$

$w_k \in \Re^n$ zero mean Gaussian w/ $E[w_k] = \bar{w}_k = 0$ and covariance $E[(w_k - \bar{w}_k)(w_k - \bar{w}_k)^T)] = W_k \in \Re^{n \times n}$.

- **Prediction** (same as before):

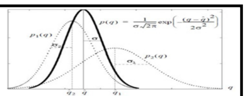
$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$

- **Measurement:** $y_{k+1} = H_{k+1} x_k + w_k$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$
  – Both y_{k+1} and $\hat{y}_{k+1}$ are now RVs with uncertainty, W_{k+1} and $\hat{W}_{k+1}$.
  – Given y_{k+1}, real measurement would likely distributed by $N(y_{k+1}, W_{k+1})$.
  – For $\hat{y}_{k+1}$, its covariance given by

$$\hat{W}_{k+1} = E\left[(\hat{y}_{k+1} - H_k x_{k+1})(\hat{y}_{k+1} - H_{k+1} x_{k+1})^T\right] = H_{k+1} P_{k+1|k} H_{k+1}^T$$

  – Given $N(y_{k+1}, W_{k+1})$ and $N(\hat{y}_{k+1}, \hat{W}_{k+1})$, **most like output** y_{k+1}^*?

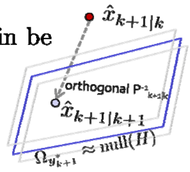
Kalman Filtering



- Merge measurement info $N(y_{k+1}, W_{k+1})$ and estimated measurement info $N(\hat{y}_{k+1}, \hat{W}_{k+1})$ using product of Gaussians (cf. Th.8.2.1) $\Rightarrow$ **most like measurement**: $y^*_{k+1} = \text{product}(y_{k+1}, \hat{y}_{k+1})$, which is still Gaussian with

$$y^*_{k+1} = \hat{y}_{k+1} + \hat{W}_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(mean)}$$
$$W^*_{k+1} = \hat{W}_{k+1} - \hat{W}_{k+1} S^{-1}_{k+1} \hat{W}_{k+1} \qquad \text{(covariance)}$$

where $S_{k+1} = \hat{W}_{k+1} + W_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1}$, i.e., combined uncertainty in $r_{k+1} = \hat{y}_{k+1} - y_{k+1}$ (residual variance).



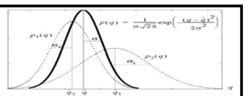
- With y^*_{k+1} as best measurement, estimate $\hat{x}_{k+1|k+1}$ should again be consistent w/ that information:

$$\hat{x}_{k+1|k+1} \in \Omega_{y^*_{k+1}} := \{x \in \Re^n \mid y^*_{k+1} = H_{k+1} x\}$$

- Optimal estimate $\hat{x}_{k+1|k+1}$: **correction** of $\hat{x}_{k+1|k}$ into its closest point on $\Omega_{y^*_{k+1}}$ with Mahalanobis norm $P^{-1}_{k+1|k}$:

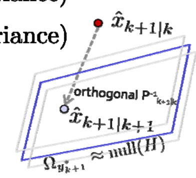
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} \hat{W}^{-1}_{k+1} \cdot [y^*_{k+1} - \hat{y}_{k+1}]$$
$$= \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}]$$

Kalman Filtering



- **Most like measurement**: $y^*_{k+1} = \text{product}(y_{k+1}, \hat{y}_{k+1})$ with

$$y^*_{k+1} = \hat{y}_{k+1} + \hat{W}_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(mean)}$$
$$W^*_{k+1} = \hat{W}_{k+1} - \hat{W}_{k+1} S^{-1}_{k+1} \hat{W}_{k+1} \qquad \text{(covariance)}$$
$$S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1} \qquad \text{(residual variance)}$$



- Optimal estimate $\hat{x}_{k+1|k+1}$:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}]$$

which is in the same form as before (yet, different residual variance $S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1}$ instead of $S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1}$).

- Uncertainty update (reduction):

$$P_{k+1|k+1} = E\left[(\hat{x}_{k+1|k+1} - x_{k+1})(\hat{x}_{k+1|k+1} - x_{k+1})^T\right]$$
$$= E\left[\left((I - P_{k+1|k} H^T_{k+1} S^{-1}_{k+1})(\hat{x}_{k+1|k} - x_{k+1})\right)(\ldots)^T\right]$$
$$= P_{k+1|k} - P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} H_{k+1} P_{k+1|k}$$

which is again in the same form as before.

Kalman Filtering

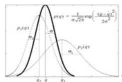
- Plant dynamics with process noise v_k and measurement noise w_k:

$$x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k + w_k$$

- **Prediction:**

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$

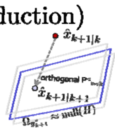
- **Measurement:** $y_{k+1} = H_{k+1} x_k + w_{k+1}$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$



- **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(state correction)}$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k} \qquad \text{(uncertainty reduction)}$$

  - Redisual variance: $S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T + W_{k+1}$
  - Kalman gain: $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$ direction with state uncertainty X weighted gain of measurement



  - K_{k+1} automatically and optimally adjusting, incorporating measurement uncertainty and state estimate uncertainty.
  - With $H_k = I$: 1) If $W_{k+1} = 0 \Rightarrow K_{k+1} = I$ and $P_{k+1|k+1} = 0$; 2) If $W_{k+1} = \infty \Rightarrow K_{k+1} = 0$ and $P_{k+1|k+1} = P_{k|k}$.

Linear Quadratic Gaussian (LQG)

- Consider CT-LTI system:

$$\dot{x} = Ax + Bu + w, \quad y = Cx + v$$

where $w \in \mathcal{N}(0, W(t))$, $v \in \mathcal{N}(0, V(t))$ are zero-mean Gaussian. We want to design optimal control to minimize quadratic cost function:

$$J = \lim_{T \to 0} E\left[\frac{1}{T}\int_0^T [x^T(t) Q x(t) + u^T(t) R u(t)] dt\right]$$

- **LQG solution:** $u = -K_{\text{LQR}} \cdot \hat{x}$ with the Kamlan estimation $\hat{x}$.
  - Combination of optimal control (LQR) and optimal estimator (KF or LQE).
  - CL-system stability guaranteed from the separation principle.
  - Optimal over all causal, linear and even nonlinear controllers.
  - Robustness not guaranteed → RS/RP, loop shaping, LTR, etc.