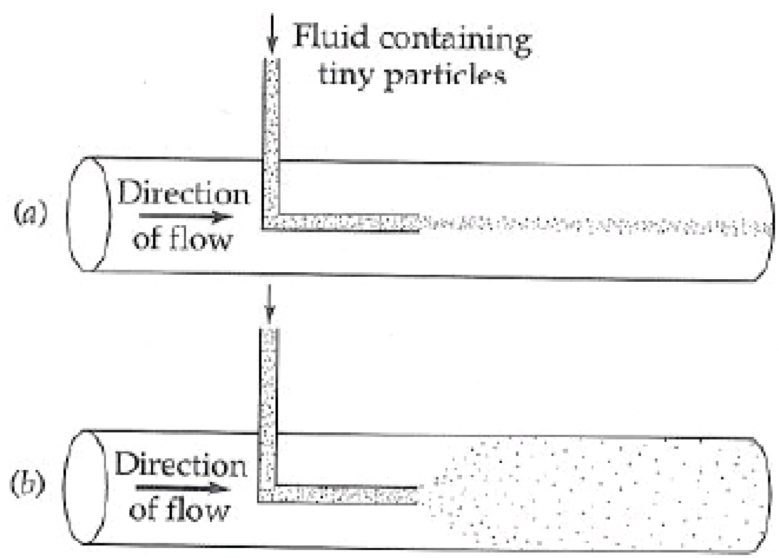
Chapter 2. Shell Momentum Balances and Velocity Distributions in Laminar Flow

- Shell momentum balances and boundary conditions
- Flow of a falling film
- Flow through a circular tube
- Flow through an annulus
- Flow of two adjacent immiscible fluids
- Creeping flow around a sphere

Shell momentum balances and velocity distributions in laminar flow

- Laminar and turbulent flow



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2.1 Shell momentum balances and Boundary conditions

$$\begin{bmatrix}\text{rate of momentum in}\\ \text{by convective transport}\end{bmatrix}-\begin{bmatrix}\text{rate of momentum out}\\ \text{by convective transport}\end{bmatrix}+$$

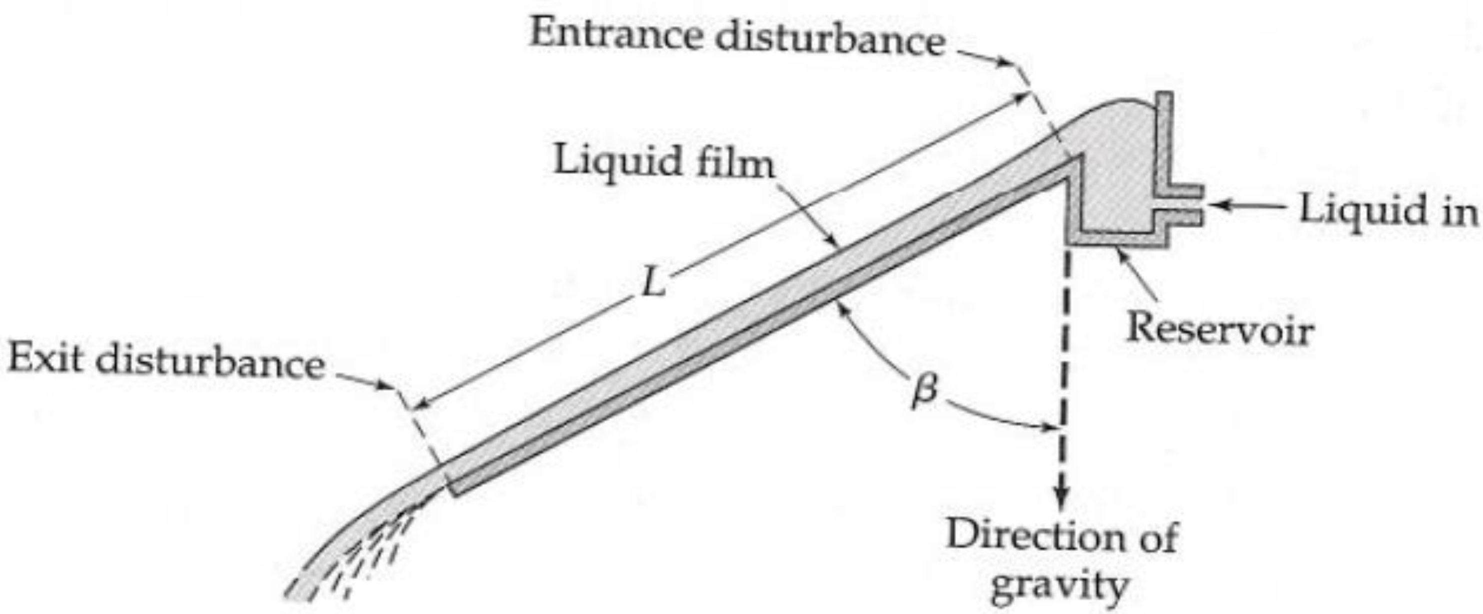
$$+\begin{bmatrix}\text{rate of momentum in}\\ \text{by molecular transport}\end{bmatrix}-\begin{bmatrix}\text{rate of momentum out}\\ \text{by molecular transport}\end{bmatrix}+$$

$$+\left[\text{force of gravity acting on system}\right]=0$$

Boundary conditions

- At solid-fluid interfaces:
  - No-slip condition. $V_{solid} = V_{fluid}$
- At a liquid-liquid interfacial plane of constant x:
  - Continuous tangential velocity: v_y and v_z
  - Continuous stress tensor components: $\tau_{xx}, \tau_{xy}, \tau_{xz}$
- At a liquid-gas interfacial plane of constant x:
  - $\tau_{xy} = \tau_{xz} = 0$

2.2 Flow of a falling film



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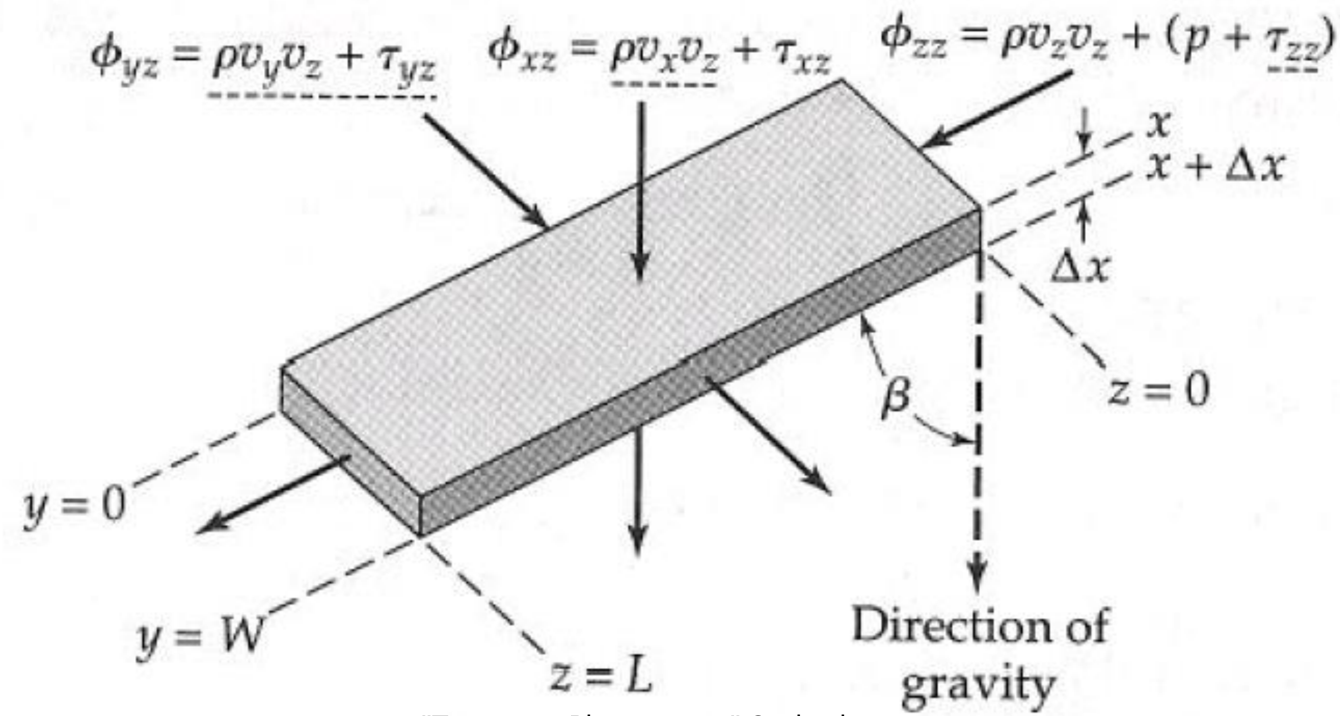
Assumptions

- Constant viscosity and density
- Laminar flow
- Far from the ends of the wall (no end effects)
- Velocities: $v_z = v_z(x)$
- Boundary conditions
  - $v_x = v_y = 0$ at $\mathrm{x} = 0,\ \tau_{xz} = 0$
    at $\mathrm{x} = \delta,\ v_z = 0$

The shell

where

$$\underline{\underline{\phi}} = \underline{\underline{\pi}} + \rho\underline{v}\underline{v} = p\underline{\underline{\delta}} + \underline{\underline{\tau}} + \rho\underline{v}\underline{v}$$



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Shell balance

- rate of z –momentum in across surface at z=0 $(W\Delta x)\phi_{zz}|_{z=0}$
- rate of z –momentum out in across surface at z=L, $(W\Delta x)\phi_{zz}|_{z=L}$

- rate of z –momentum in in across surface at x $(LW)\phi_{xz}|_{x}$
- rate of z –momentum out across surface at x+Δx, $(LW)\phi_{xz}|_{x+\Delta x}$

- gravity forces acting on fluid in the z - direction, $(LW\Delta x)\rho g\cos\beta$

Balance

- Introducing the z-momentum balance

$$LW\left(\phi_{xz}\Big|_{x} - \phi_{xz}\Big|_{x+\Delta x}\right) + W\Delta x\left(\phi_{zz}\Big|_{z=0} - \phi_{zz}\Big|_{z=L}\right)$$

$$+LW\Delta x(\rho g\cos\beta) = 0$$

- Dividing by $LW\Delta x$

$$\frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_{x}}{\Delta x} - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g\cos\beta$$

Differential equation

- Applying the limit $\Delta x \to 0$

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

- By integrating

$$\tau_{xz} = (\rho g \cos\beta) \cdot x + C_1$$

- Using BC1:

$$\text{at } x = 0, \qquad \tau_{xz} = 0$$

$$\tau_{xz} = (\rho g \cos\beta) \cdot x$$

- Introducing the Newton's law of viscosity

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

Velocity distribution

- Differential equation

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x$$

- Integrating

$$v_z = -\left(\frac{\rho g \cos\beta}{2\mu}\right)x^2 + C_2$$

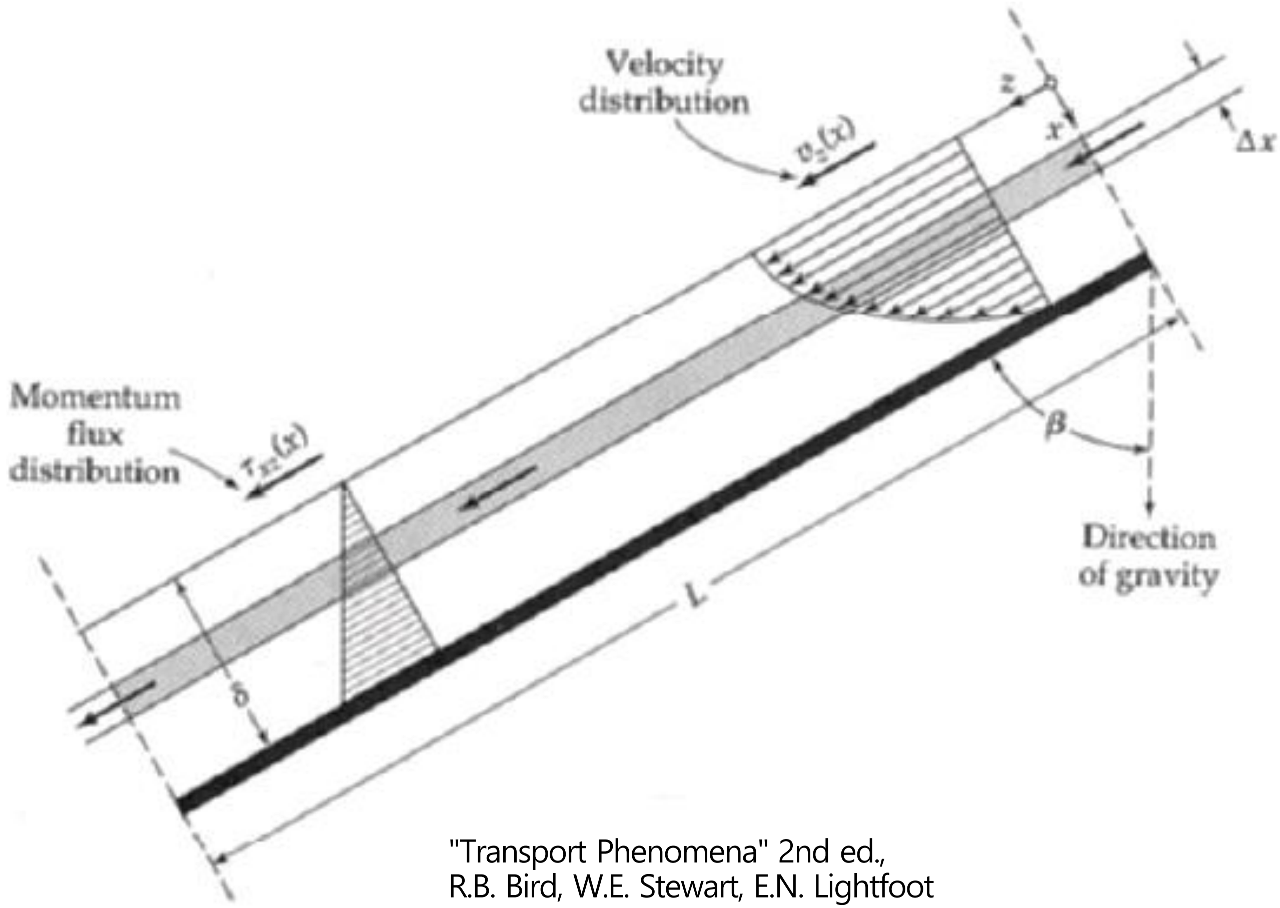
- BC2

$$\text{at } x = \delta, \qquad v_z = 0$$

- Velocity distribution

$$v_z = \left(\frac{\rho g \delta^2 \cos\beta}{2\mu}\right)\left[1 - \left(\frac{x}{\delta}\right)^2\right]$$

Results



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Other calculations

- The maximum velocity

$$v_z = \left(\frac{\rho g \delta^2 \cos\beta}{2\mu} \right)$$

- The average velocity

$$\langle v_z \rangle = \frac{\int_0^W \int_0^\delta v_z \, dx\, dy}{\int_0^W \int_0^\delta dx\, dy} = \frac{2}{3} v_{z,max}$$

Other calculations

- Mass rate of flow

$$w = \int_0^W \int_0^\delta \rho v_z \, dx\, dy = \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu}$$

- Force (z-direction) exerted by the fluid on the wall

$$F_z = \int_0^L \int_0^W \left(\tau_{xz}\big|_{x=\delta} \right) dy\, dz$$

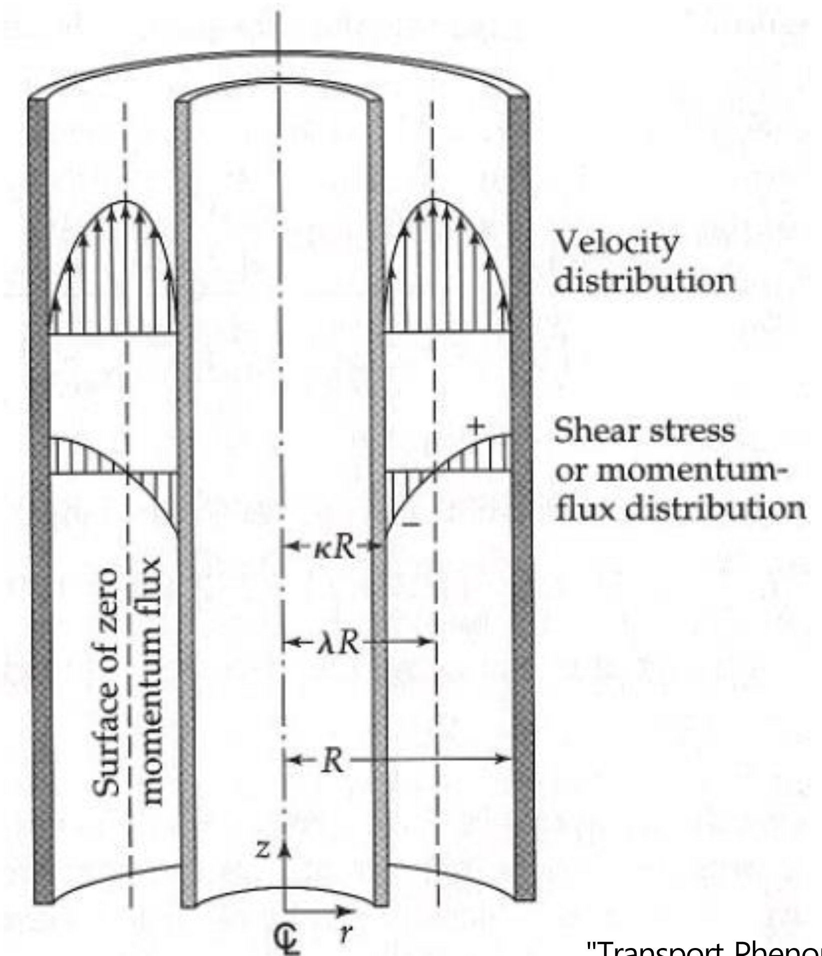
$$= \int_0^L \int_0^W \left(-\mu \frac{dv_z}{dx}\bigg|_{x=\delta} \right) dy\, dz$$

$$= \rho g \delta L W \cos\beta$$

More

- Reynolds number $$\mathbf{Re} = \frac{4\delta\langle \mathbf{v}_z \rangle \rho}{\mu}$$
- Laminar flow, negligible rippling $$\mathrm{Re} < 20$$
- Laminar flow, pronounced rippling $$20 < \mathrm{Re} < 1500$$
- Turbulent flow $$\mathrm{Re} > 1500$$

2.4 Flow through an annulus



- Steady-state axial flow between two coaxial cylinders
  - Incompressible fluid
  - Flowing upward
  - BC:

$$v_z(\kappa R) = 0$$
$$v_z(R) = 0$$

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Equations

- Momentum balance over a think cylindrical shell

$$\frac{d}{dr}\left(r\tau_{rz}\right)=\left(\frac{\left(p_0+\rho g\cdot 0\right)-\left(p_0+\rho g\cdot L\right)}{L}\right)=\left(\frac{P_0-P_L}{L}\right)$$

where $P=p+\rho g$

- Integrating

$$\tau_{rz}=\left(\frac{P_0-P_L}{2L}\right)\cdot r+\frac{C_1}{r}$$

  - C_1 cannot be determined immediately

Solving equations

- Substituting the Newton's Law and integrating

$$-\mu \cdot v_z = \left(\frac{P_0 - P_L}{4L} \right) \cdot r^2 + C_1 Ln(r) + C_2$$

- Constant are determined using BC's

$$v_z = \left(\frac{P_0 - P_L}{4\mu L} \right) \cdot R^2 \left[1 - \left(\frac{r}{R} \right)^2 - \frac{1-\kappa^2}{Ln(1/\kappa)} Ln\left(\frac{R}{r} \right) \right]$$

Other results

- Average velocity

$$v_z = \left(\frac{P_0 - P_L}{8\mu L} \right) \cdot R^2 \left[\frac{1-\kappa^{4}\,{}^{2}}{1-\kappa^{2}} - \frac{1-\kappa^{2}}{\mathrm{Ln}(1/\kappa)} \right]$$

- Valid for laminar flow

$$\mathrm{Re} < 2000$$

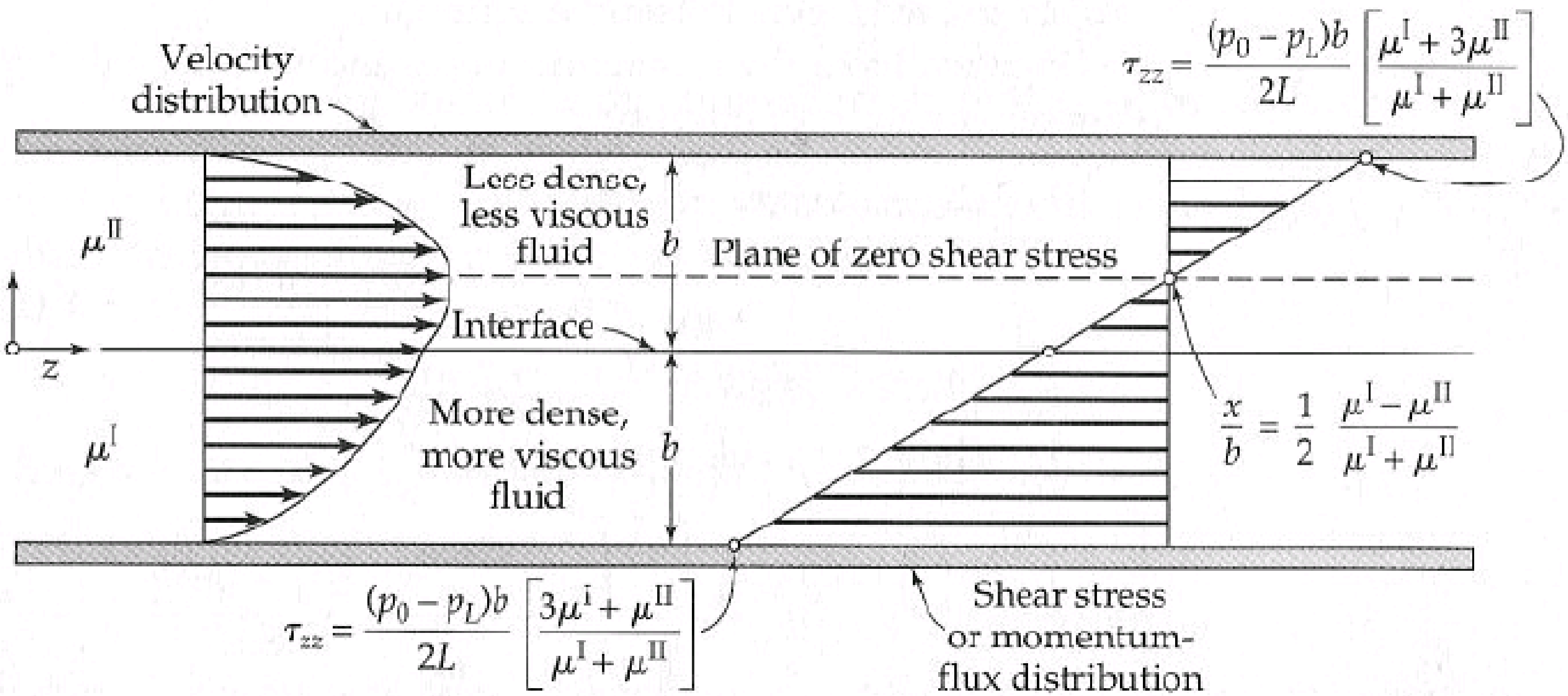
$$\mathrm{Re} = \frac{2R(1-\kappa)v_z\rho}{\mu}$$

2.5 Flow of two adjacent immiscible fluids

No slip

At x=0,
$v_z^I = v_z^{II}$
$\tau_{xz}^I = \tau_{xz}^{II}$

No slip



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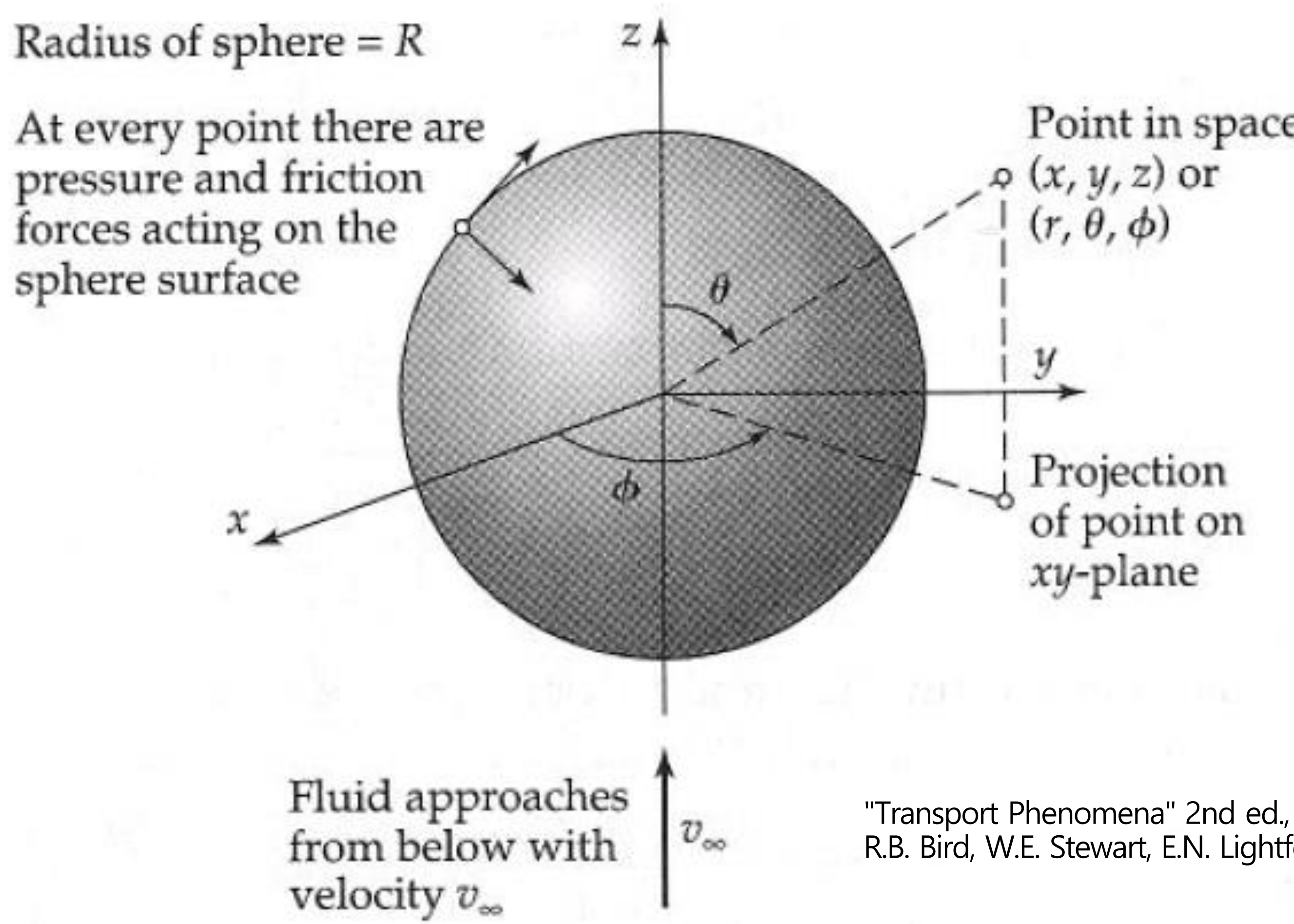
2.6 Creeping flow around a sphere



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Discussion using given results

$$v_r = v_\infty \left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3 \right] \cdot \mathrm{Cos}\theta$$

$$v_\theta = v_\infty \left[-1 + \frac{3}{4}\left(\frac{R}{r}\right) + \frac{1}{4}\left(\frac{R}{r}\right)^3 \right] \cdot \mathrm{Sin}\theta \qquad v_\phi = 0$$

$$p = p_0 - \rho g z - \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^2 \mathrm{Cos}\theta$$

Components of the stress tensor

- Using Table B.1

$$\tau_{rr} = -\tau_{\theta\theta} = \tau_{\phi\phi} = \frac{3\mu v_\infty}{R}\left[-\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right] \cdot \mathrm{Cos}\theta$$

$$\tau_{r\theta} = \tau_{\theta r} = \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^4 \mathrm{Sin}\theta$$

Normal force

$$F^{(n)} = \int_0^{2\pi}\int_0^{\pi}\left(-\left(p+\tau_{rr}\right)\Big|_{r=R} \mathrm{Cos}\theta\right)\cdot R^2 \mathrm{Sin}\theta \cdot d\theta d\phi$$

$$F^{(n)} = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R v_\infty$$

Bouyant force

Form drag

Tangential force

$$F^{(t)} = \int_0^{2\pi}\int_0^{\pi}\left(-\left(p+\tau_{r\theta}\right)\Big|_{r=R} \mathrm{Sin}\theta\right)\cdot R^2 \mathrm{Sin}\theta \cdot d\theta d\phi$$

$$F^{(t)} = 4\pi\mu R v_{\infty}$$

Friction drag

Total forces

$$F = F^{(n)} + F^{(t)} = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R v_\infty + 4\pi\mu R v_\infty$$

Bouyant force   Form drag   Friction drag

$$F = F_b + F_k = \frac{4}{3}\pi R^3 \rho g + 6\pi\mu R v_\infty$$

Bouyant force   Kinetic force

Stoke's law: $F_k = 6\pi\mu R v_\infty$