Chapter 4. Velocity distribution with more than one independent variable

- Time-dependent flow of Newtonian fluids
- Solving flow problems using a stream function
- Flow of inviscid fluid by use of the velocity potential
- Flow near solid surface by boundary layer theory

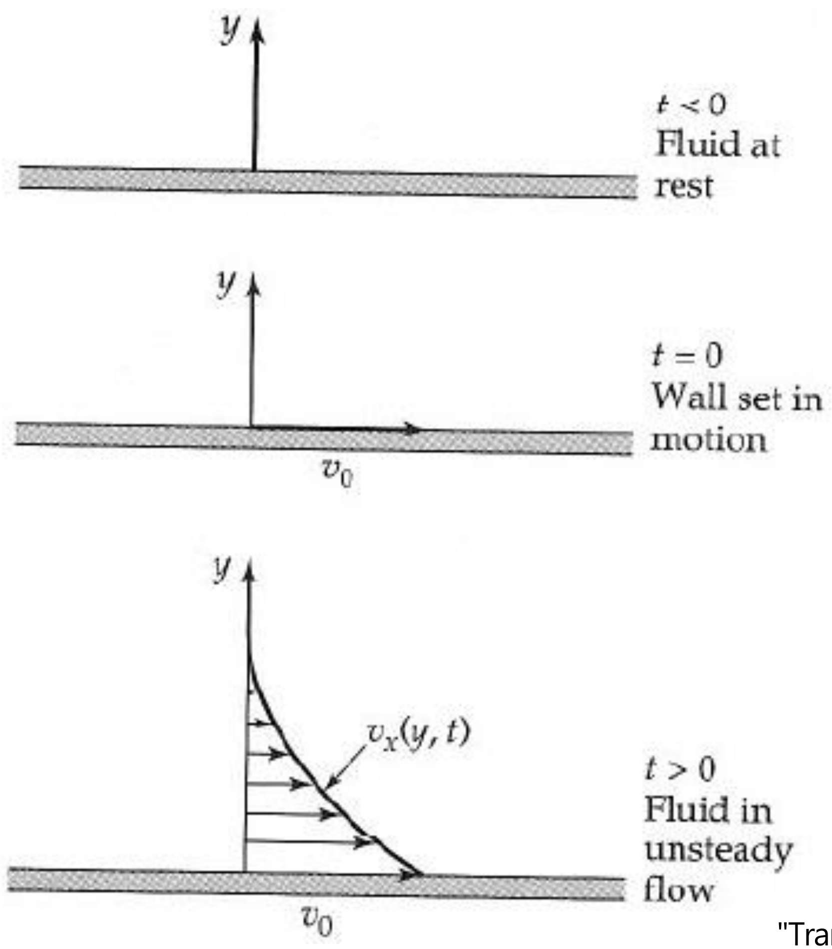
4.1. Time-dependent flow of Newtonian fluids

- Three methods to solve the differential equation.

  PDE is transform in one or more ODE

  - Combination of variables, semi-infinite regions
  - Separation of variables. Sturm-Liouville problems
  - Method of sinusoidal response.

Flow near a wall suddenly set in motion



- Semi-infinite body of liquid
- Constant density and viscosity

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Equations

- For the system

$$v_x = v_x(y,t); \;\; v_y = 0; \;\; v_z = 0$$

- Equation of motion

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

Equations

- Boundary conditions

$$
\begin{array}{llll}
\text{I.C.} & \text{at } t \leq 0 & v_x = 0 & \text{for all } y \\
\textbf{B}\text{.C.1} & \text{at } y = 0 & v_x = v_o & \text{for all } t > 0 \\
\textbf{B}\text{.C.2} & \text{at } y = \infty & v_x = 0 & \text{for all } t > 0
\end{array}
$$

- Dimensionless velocity

$$\phi = \frac{V_x}{V_o}$$

Equations

- New equation

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}$$

- Defining (from dimensional analysis)

$$\phi = \phi(\eta) \qquad \eta = \frac{y}{\sqrt{4\nu t}}$$

Equations

- Introducing derivatives

$$\frac{\partial^2 \phi}{\partial \eta^2} + 2\eta \frac{\partial \phi}{\partial \eta} = 0$$

- New boundary conditions

BC1 $\quad at\ \ \eta = 0 \quad \phi = 1$

BC2 + IC $\quad at\ \ \eta = \infty \quad \phi = 0$

Equations

- Integrating $$\frac{\partial \phi}{\partial \eta} = C_1 e^{-\eta^2}$$
- Integrating again $$\phi = C_1 \int_0^\eta e^{-x^2} dx + C_2$$
- With BCs

$$\phi = 1 - \frac{\int_0^\eta e^{-x^2} dx}{\int_0^\infty e^{-x^2} dx}$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-x^2} dx = 1 - erf(\eta)$$

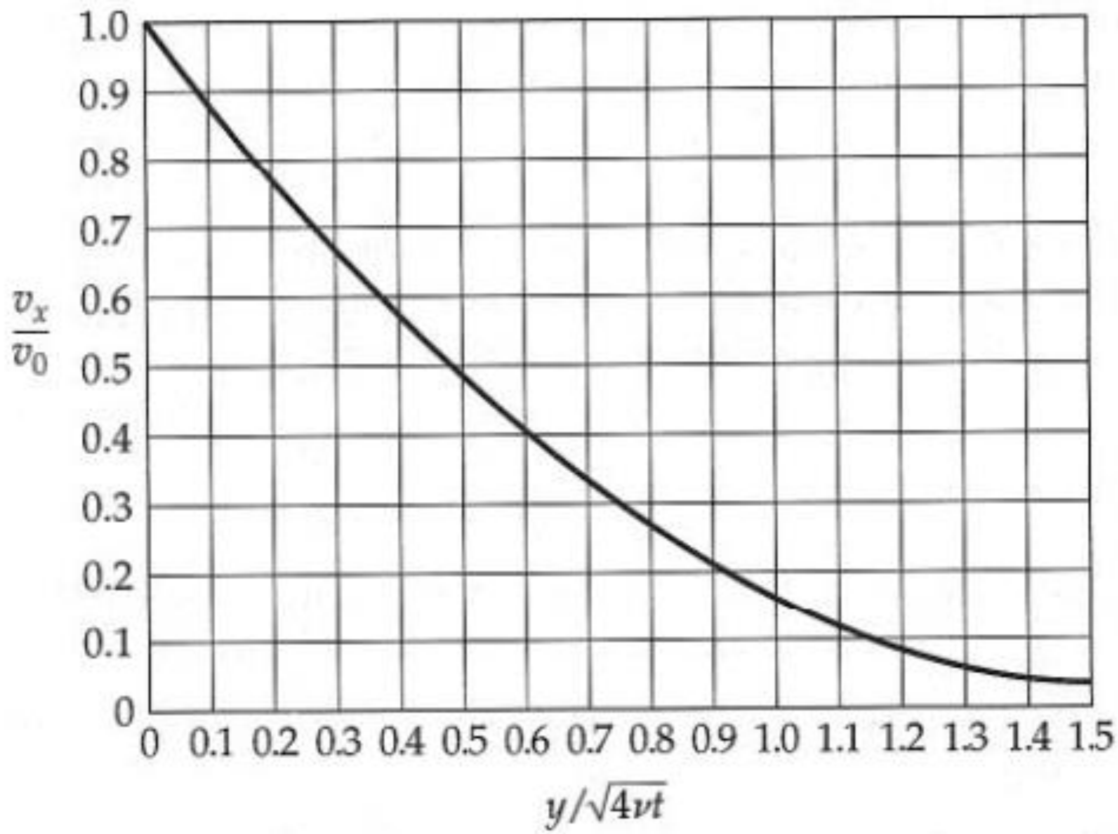
Velocity Profile

$$\frac{v_x(y,t)}{v_o} = 1 - erf\left(\frac{y}{\sqrt{4\nu t}}\right) = erfc\left(\frac{y}{\sqrt{4\nu t}}\right)$$

- Since erfc(2) ≈ 0 01
- We define the "Boundary layer thickness", δ as: δ = distance y for which v_x has dropped to 0.01 v_o

$$\delta = 4\sqrt{\nu t}$$

Results



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4.2. Resolving problems using a stream function(equation of change for the volticity)

- Taking the curl of the equation of motion: constant density and viscosity

$$\frac{\partial}{\partial t}(\nabla \times \underline{v}) - \left(\nabla \times \left(\underline{v} \times (\nabla \times \underline{v})\right)\right) = \nu \nabla^2 (\nabla \times \underline{v})$$

- No other assumptions are needed
- It may be applied to different geometries

4.2. Resolving problems using a stream function

- For planar system, introduce the Stream Function, e.g.,

$$v_x = -\frac{\partial \psi}{\partial y} \qquad v_y = +\frac{\partial \psi}{\partial x}$$

4.3. Flow of inviscid fluids by use of the velocity potential

- Inviscid fluid = without viscosity
- Applicable for “low viscosity effect (fluid)”. Inadequate near the solid surfaces
- Assuming constant density and $\nabla \times \underline{v}$ (irrotational flow) potential flow is obtained
- Complete solution
  - Potential theory away from the solid surfaces
  - Boundary layer theory near the solid surfaces

Potential flow

- Equation of continuity

$$\left(\nabla \cdot \underline{v}\right) = 0$$

- Equation of motion (Euler equation, low viscosity limit)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \nabla\left(\frac{1}{2}v^2\right) - \left(\underline{v} \times \left(\nabla \times \underline{v}\right)\right)\right) = -\nabla P$$

Potential flow

- For 2-D, steady, irrotational flow

- Irrotational $$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0$$

- Continuity $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

- Motion

$$\frac{1}{2}\rho\left(v_x^2 + v_y^2\right) + P = 0$$

Stream function and velocity potential

- Stream function
$$v_x = -\frac{\partial \psi}{\partial y} \qquad v_y = +\frac{\partial \psi}{\partial x}$$
- Velocity potential
$$v_x = -\frac{\partial \phi}{\partial x} \qquad v_y = -\frac{\partial \phi}{\partial y}$$
- Cauchy-Rieman equations
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad and \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$
- Analytical function(complex potential)
$$w(z) = \phi(x, y) + i\psi(x, y)$$

Analytical functions

- Any analytical function w(z) may be the solution for some flow problem, and it yields a pair of function:
  - Velocity potential

$$\phi(x, y)$$

  - Stream function

$$\psi(x, y)$$

- Equipotential lines

$$\phi(x, y) = const$$

Stream lines

$$\psi(x, y) = const$$

Potential flow around a cylinder

- Complex potential

$$w(z) = -v_\infty R\left(\frac{z}{R} + \frac{R}{z}\right)$$

- Introducing $\qquad z = x + iy$

$$w(z) = -v_\infty x\left(1 + \frac{R^2}{x^2 + y^2}\right) - iv_\infty y\left(1 - \frac{R^2}{x^2 + y^2}\right)$$

Potential flow around a cylinder

- Stream function

$$\psi(x, y) = -v_{\infty} y \left(1 - \frac{R^2}{x^2 + y^2}\right)$$

- Velocity potential

$$\phi(x, y) = -v_{\infty} x \left(1 + \frac{R^2}{x^2 + y^2}\right)$$

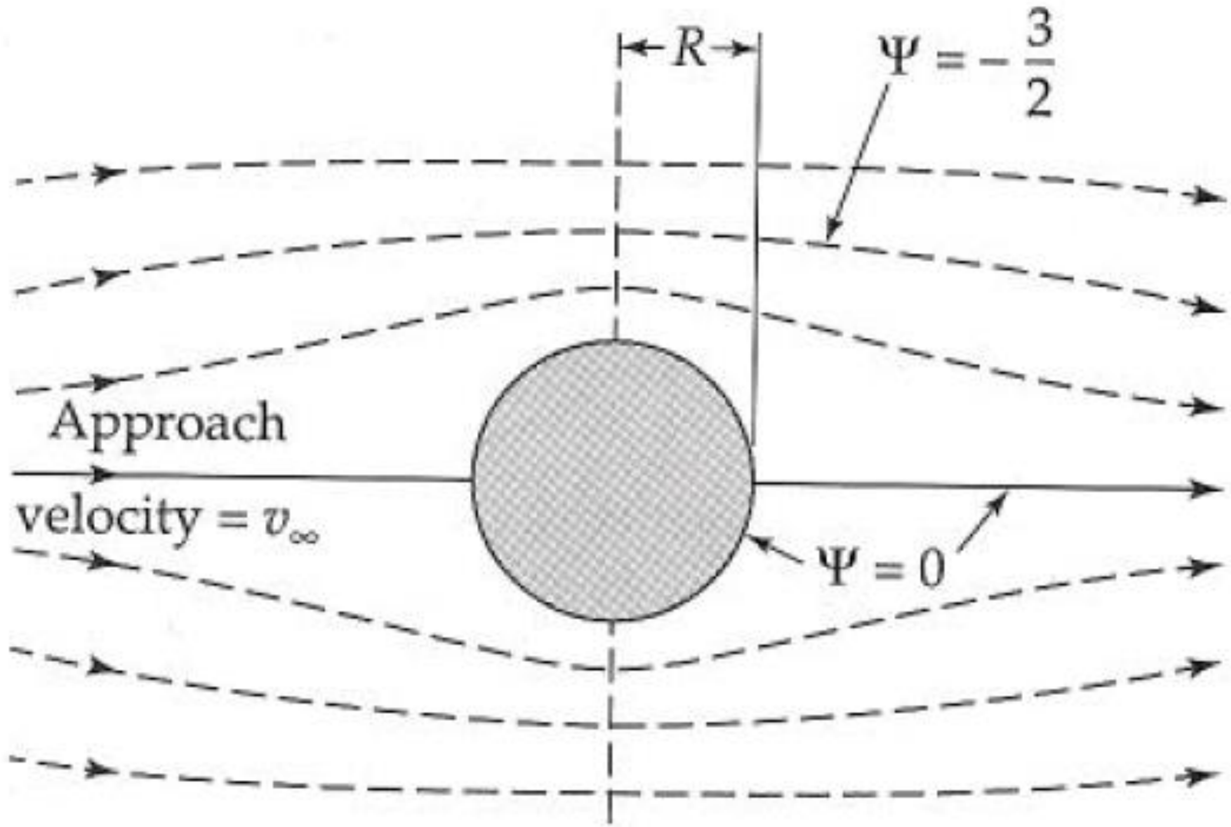
Streamlines

- Using dimensionless variables:

$$\Psi = \frac{\psi}{v_\infty} \qquad X = \frac{x}{R} \qquad Y = \frac{y}{R}$$

$$\Psi(X, Y) = -Y\left(1 - \frac{1}{X^2 + Y^2}\right)$$

The stream lines for the potential flow around a cylinder



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The velocity components

- Using the complex velocity

$$\frac{dw}{dz} = -v_x + iv_y = -v_\infty\left(1 - \frac{R^2}{z^2}\right) =$$

$$-v_\infty\left(1 - \frac{R^2}{r^2}e^{-2i\theta}\right)$$

$$\frac{dw}{dz} = -v_\infty\left(1 - \frac{R^2}{r^2}(\cos 2\theta - i\sin 2\theta)\right)$$

The velocity components

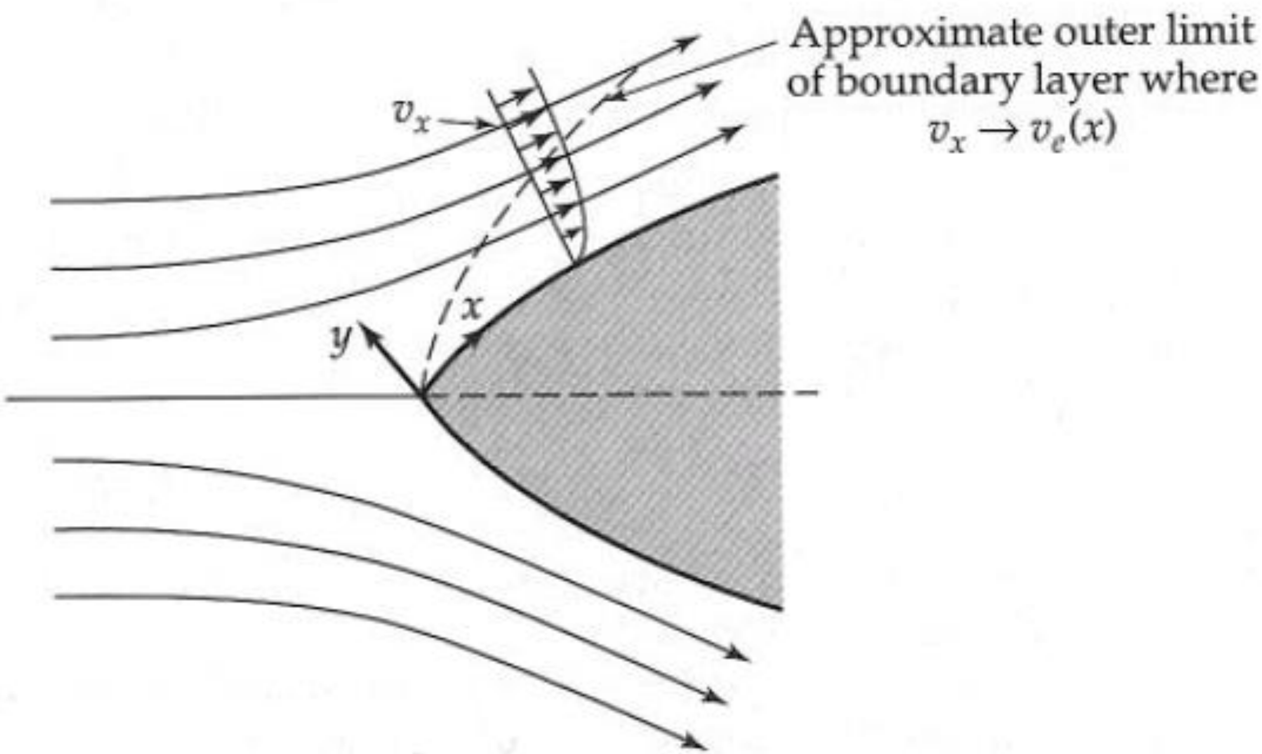
- x-component

$$v_x = -v_\infty \left(1 - \frac{R^2}{r^2}\cos 2\theta\right)$$

- y-component

$$v_y = -v_\infty \left(1 - \frac{R^2}{r^2}\sin 2\theta\right)$$

4.4. Flow near solid surfaces by boundary-layer theory



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Flow near solid surfaces by boundary-layer theory

- Equation of continuity and Navier-Stokes equations

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

Equations by dimensional analysis

- Considering orders of magnitude of the different terms, based in Prandtl boundary-layer approach

$$\boldsymbol{\delta_o} \ll \boldsymbol{l_o}$$

- Continuity $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

- Motion $$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

- Modified pressure is assumed to be known

Equations

- Boundary conditions
  - no-slip condition at the wall $(v_x = 0 \quad at\ y = 0)$
  - no mass transfer at the wall $\left(v_y = 0 \quad at\ y = 0\right)$

- Solving equation using continuity equation

$$v_x \frac{\partial v_x}{\partial x} - \left(\int_0^y \frac{\partial v_x}{\partial x} dy\right) \frac{\partial v_x}{\partial y} = v_e \frac{\partial v_e}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

Equations

- Von Karman momentum balance
  - Multiplying ρ and integrating with regard to y from 0 to ∞

$(v_x(x,y) \to v_e(x)\ \ as\ y \to \infty)$

$$\mu \frac{\partial v_x}{\partial y}\bigg|_{y=0} = \frac{d}{dx}\int_0^\infty \rho v_x(v_e - v_x)\, dy + \frac{dv_e}{dx}\int_0^\infty \rho(v_e - v_x)\, dy$$

Approximate boundary-layer solutions

- Approximate solutions starts assuming expressions for the velocity profile
- Example. Laminar flow along a flat plate (approximate solution). It uses

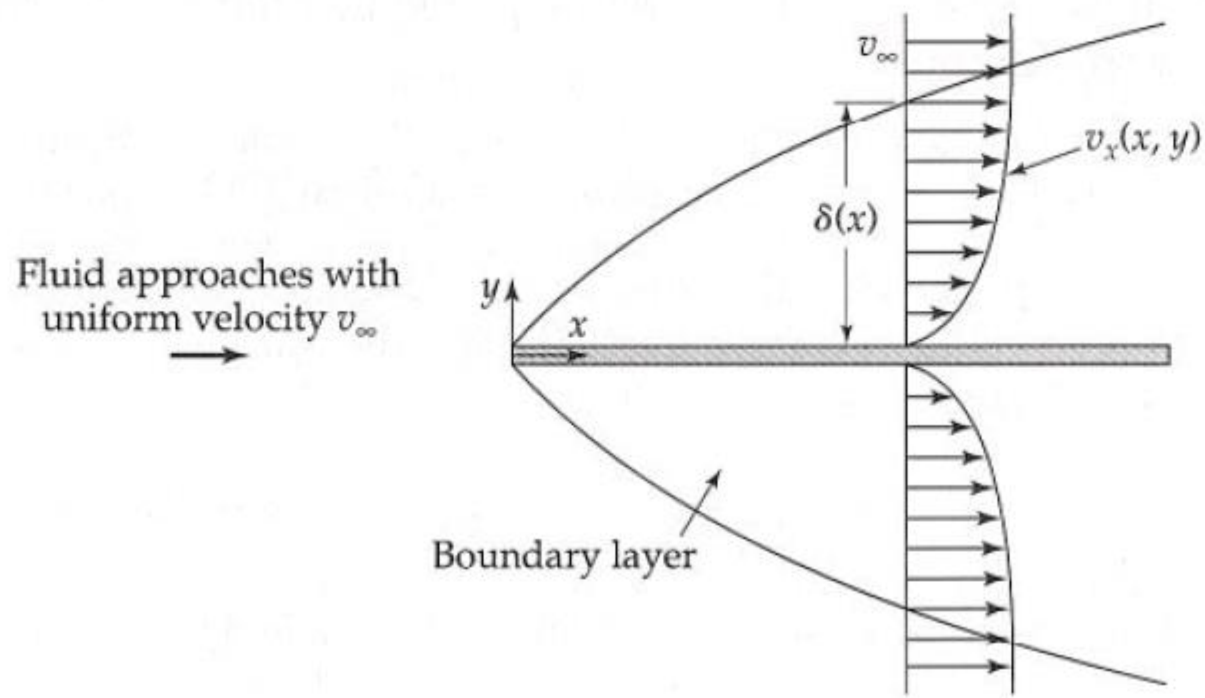
$$\frac{v_x}{v_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad for\ \ 0 \le y \le \delta(x)$$

Boundary-layer region

$$\frac{v_x}{v_\infty} = 1 \qquad for\ \ y \ge \delta(x)$$

Potential flow region

Approximate boundary-layer solutions



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Solutions

- By substitution of profile into von Karman integral balance give

$$\frac{3}{2}\frac{\mu v_\infty}{\delta} = \frac{d}{dx}\left(\frac{39}{280}\rho v_\infty^2 \delta\right)$$

- Boundary layer thickness

$$\delta(x) = \sqrt{\frac{280}{13}\frac{\nu x}{v_\infty}} = 4.64\sqrt{\frac{\nu x}{v_\infty}}$$

Solutions

- Velocity distribution

$$\frac{v_x}{v_\infty} = \frac{3}{2}\mathrm{y}\sqrt{\frac{13}{280}\frac{v_\infty}{\nu x}} - \frac{1}{2}\left(y\sqrt{\frac{13}{280}\frac{v_\infty}{\nu x}}\right)^3$$

- Drag force

$$F_x = 2\int_0^W \int_0^L \left(\mu\frac{\partial v_x}{\partial y}\right)_{y=0} dxdz = 1.293\sqrt{\rho\mu L W^2 {v_\infty}^3}$$

Laminar flow along a flat plate (exact solution)

- Solution uses Stream Functions. Table 4.2-1
  - Equations are solved by combination of variables

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\nu \frac{\partial^3 \psi}{\partial y^3}$$

- BC's

at $y = 0$, $\quad \frac{\partial \psi}{\partial x} = v_y = 0 \quad$ for $x \geq 0$

at $y = 0$, $\quad \frac{\partial \psi}{\partial y} = -v_x = 0 \quad$ for $x \geq 0$

as $y \rightarrow \infty$, $\quad \frac{\partial \psi}{\partial y} = -v_x \rightarrow -v_\infty \quad$ for $x \geq 0$

at $x = 0$, $\quad \frac{\partial \psi}{\partial y} = -v_x = -v_\infty \quad$ for $y > 0$

Solutions

- Variable (by dimensional analysis)

$$\frac{v_x}{v_\infty} = \Pi(\eta) \qquad where\ \ \eta = y\sqrt{\frac{1}{2}\frac{v_\infty}{\nu x}}$$

- Stream function that gives this velocity distribution

$$\psi(x,y) = -\sqrt{2v_\infty \nu x f(\eta)} \quad where\ \ f(\eta) = \int_0^\eta \Pi(x)' dx$$

Solutions

- Substitution into equation and new BCs

$$-ff'' = f'''$$

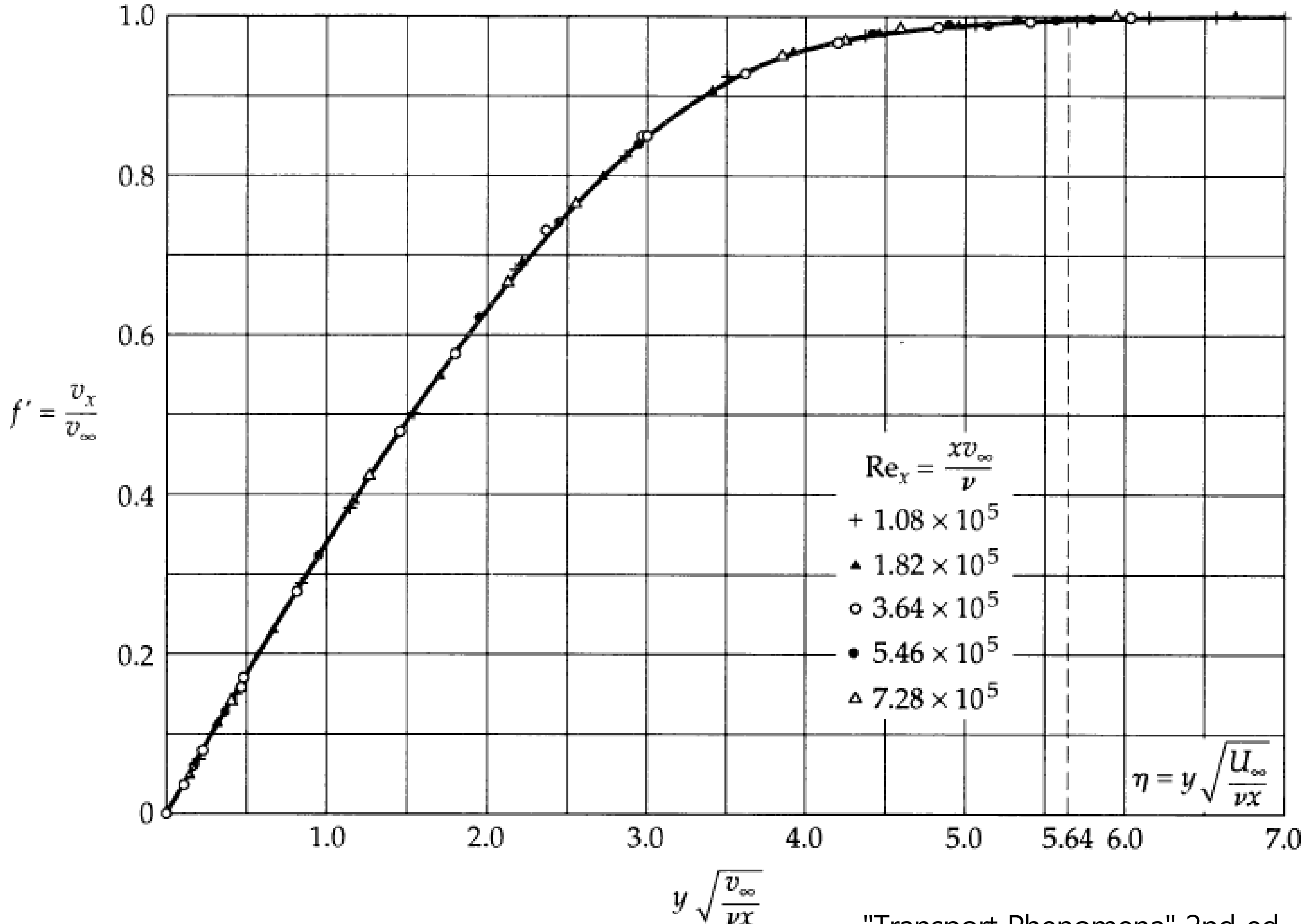
B.C. 1 and 2: at $\eta = 0$, $f = 0$ and $f' = 0$

B.C. 3 and 4: as $\eta \to \infty$, $f' \to 1$

- Drag force

$$F_x = 2\int_0^W \int_0^L \left(\mu \frac{\partial v_x}{\partial y}\right)_{y=0} dxdz = 1.328\sqrt{\rho\mu L W^2 {v_\infty}^3}$$

Predicted and observed velocity profiles for flow along a plate



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