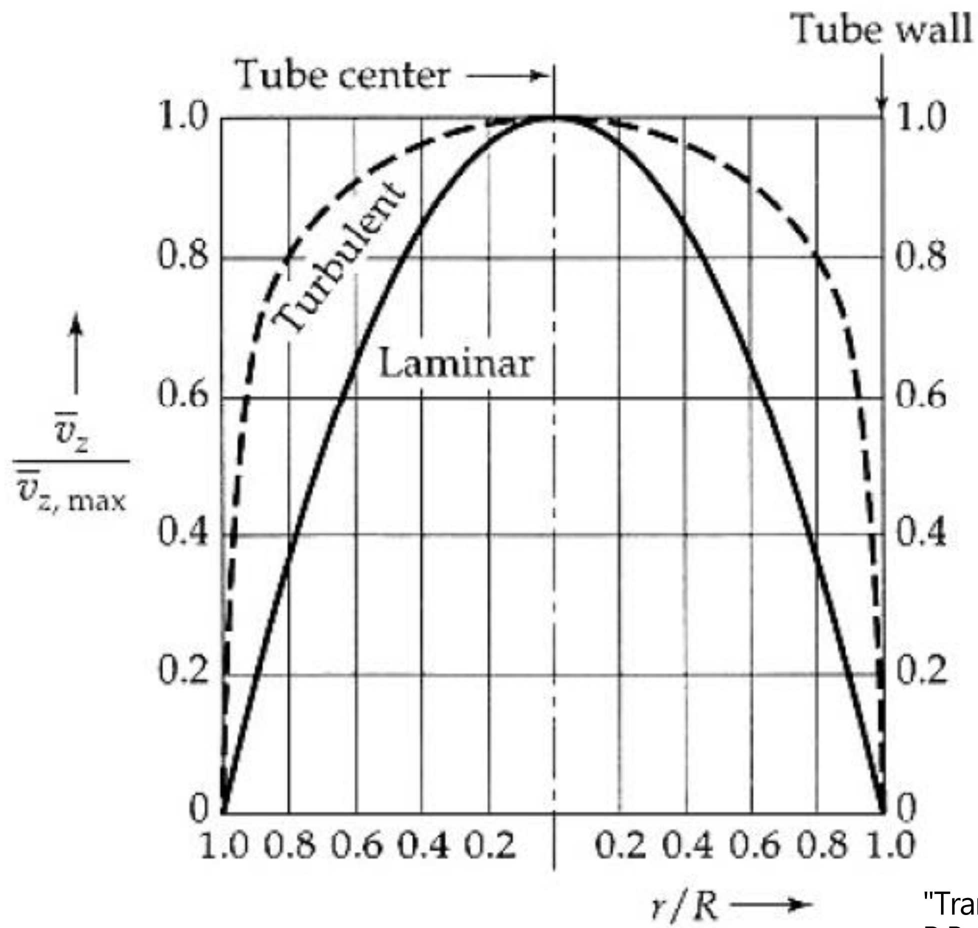
Chapter 5
Velocity Distributions in Turbulent Flow

- Comparisons of laminar and turbulent flows
- Time-smoothed equations of change for incompressible fluids
- The time-smoothed velocity profile near a wall
- Turbulent flow in ducts
- Turbulent flow in jets

Comparisons of laminar and turbulent flows

Laminar	Turbulent
Re < 2100	1e4<Re <1e5
$\frac{v_z}{v_{z,max}} = 1 - \left(\frac{r}{R}\right)^2$	$\frac{v_z}{v_{z,max}} = \left(1 - \frac{r}{R}\right)^{1/7}$
$\frac{\langle v_z \rangle}{v_{z,max}} = \frac{1}{2}$	$\frac{\langle v_z \rangle}{v_{z,max}} = \frac{4}{5}$

Velocity profiles



"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

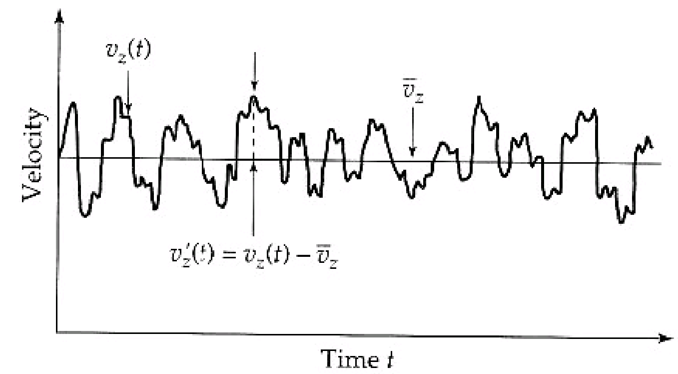
Time-smoothed equations of change for incompressible fluids

- The actual velocity in a point

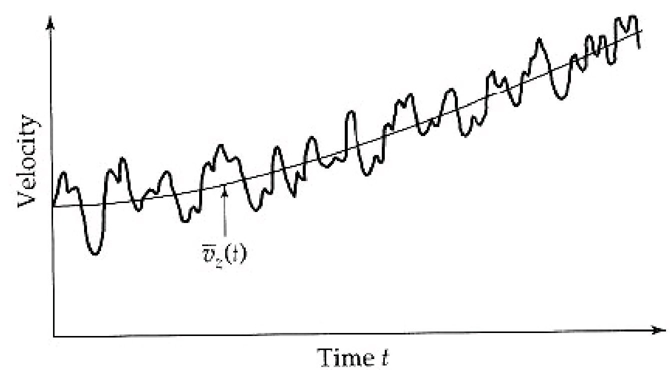
$$v_z = \overline{v}_z + v_z'$$

$$\overline{v}_z = \frac{1}{t_o}\int_{t-\frac{1}{2}t_o}^{t+\frac{1}{2}t_o} v_z(s)ds$$

- Local velocity as a function of time.
    - a) Time independent
    - b) Time dependent



(a)



(b)

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Some relationships

$$\overline{v_x'} = 0 \qquad \overline{v_y'} = 0 \qquad \overline{v_z'} = 0$$

$$\overline{\overline{v}_x v_x'} = 0 \qquad \overline{\overline{v}_y v_y'} = 0 \qquad \overline{\overline{v}_z v_z'} = 0$$

$$\overline{\frac{\partial}{\partial x} v_z} = \frac{\partial}{\partial x} \overline{v}_z \qquad \overline{\frac{\partial}{\partial t} v_z} = \frac{\partial}{\partial t} \overline{v}_z$$

Time smoothing of the equations of change

- Introducing

$$v_x = \overline{v}_x + v_x' \qquad v_y = \overline{v}_y + v_y'$$

$$v_z = \overline{v}_z + v_z' \qquad p = \overline{p} + p'$$

- into equations of continuity

$$\frac{\partial}{\partial x}(\overline{v}_x + v_x') + \frac{\partial}{\partial x}\left(\overline{v}_y + v_y'\right) + \frac{\partial}{\partial x}(\overline{v}_z + v_z') = 0$$

- Time-smoothed equation of continuity

$$\frac{\partial}{\partial x}(\overline{v}_x) + \frac{\partial}{\partial x}\left(\overline{v}_y\right) + \frac{\partial}{\partial x}(\overline{v}_z) = 0$$

Time smoothing of the equations of change

- into equation of motion (x-component)

$$\frac{\partial}{\partial t}(\overline{v}_x + v_x')$$
$$= -\frac{\partial}{\partial x}(\overline{p} + p')$$
$$-\left(\frac{\partial}{\partial x}\rho(\overline{v}_x + v_x')(\overline{v}_x + v_x') + \frac{\partial}{\partial y}\rho\left(\overline{v}_y + v_y'\right)(\overline{v}_x + v_x')\right.$$
$$\left.+\frac{\partial}{\partial z}\rho(\overline{v}_z + v_z')(\overline{v}_x + v_x')\right) + \mu\nabla^2(\overline{v}_x + v_x') + \rho g_x$$

Time smoothing of the equations of change

$$\frac{\partial}{\partial t}(\overline{v}_x)$$
$$= -\frac{\partial}{\partial x}(\overline{p}) - \left(\frac{\partial}{\partial x}\rho(\overline{v}_x)(\overline{v}_x) + \frac{\partial}{\partial y}\rho(\overline{v}_y)(\overline{v}_x) + \frac{\partial}{\partial z}\rho(\overline{v}_z)(\overline{v}_x)\right)$$
$$\boxed{-\left(\frac{\partial}{\partial x}\rho\left(\overline{v_x' v_x'}\right) + \frac{\partial}{\partial y}\rho\left(\overline{v_y' v_x'}\right) + \frac{\partial}{\partial z}\rho\left(\overline{v_z' v_x'}\right)\right)} + \mu\nabla^2(\overline{v}_x) + \rho g_x$$

Momentum transport associated
with the turbulent fluctuations

Remarks

- The equation of continuity is the same, except that:

$$v \ \ is \ \ replaced \ \ by \ \ \ \overline{v}$$

- In the equation of motion v and p are replaced by:

$$v \quad by \quad \overline{v} \qquad\qquad and \qquad\qquad p \quad by \quad \overline{p}$$

- In addition, it appears the boxed term associated with the turbulent fluctuations.

Turbulent momentum flux tensor

- Definition

$$\overline{\tau}_{xx}^{(t)} = \rho\overline{v_x' v_x'} \qquad \overline{\tau}_{xy}^{(t)} = \rho\overline{v_x' v_y'} \qquad \overline{\tau}_{xz}^{(t)} = \rho\overline{v_x' v_z'}$$

- Then

$$\left(\nabla \cdot \underline{\overline{v}}\right) = 0 \qquad and \qquad \left(\nabla \cdot \underline{v}'\right) = 0$$

$$\frac{\partial}{\partial t}\rho\underline{\overline{v}} = -\nabla\overline{p} - \left(\nabla \cdot \rho\underline{\overline{v v}}\right) - \left(\nabla \cdot \left(\underline{\overline{\tau}}^{(v)} + \underline{\overline{\tau}}^{(t)}\right)\right) + \rho\underline{g}$$

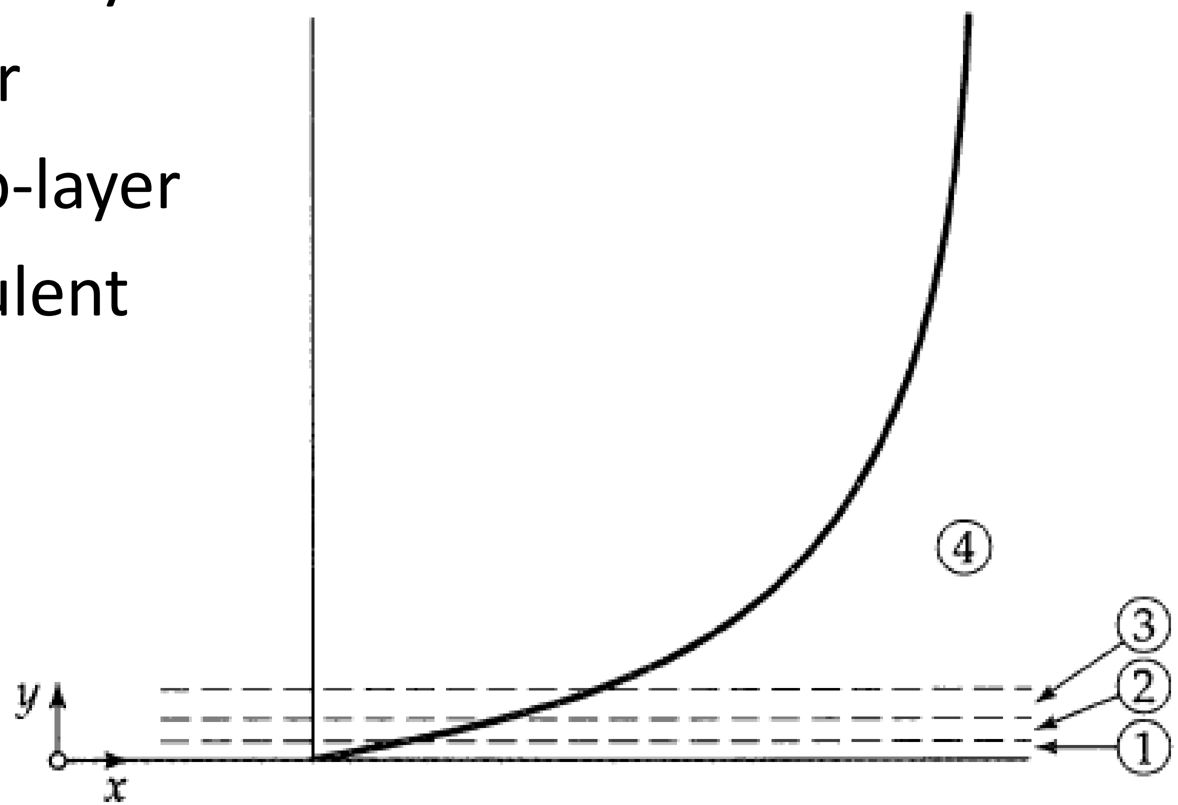
The time-smoothed velocity profile near a wall

(1) Viscous sub-layer

(2) Buffer layer

(3) Inertial sub-layer

(4) Main turbulent stream



The time-smoothed velocity profile near a wall

- Viscous sub-layer, viscosity plays a key role
- Buffer layer, transition between viscous and inertial sub-layers
- Inertial sub-layer, viscosity plays a minor role
- Main turbulent stream, time-smoothed velocity distribution is nearly flat and viscosity is unimportant

The logarithmic velocity profiles in the inertial sub-layer

- Velocity gradient (does not depend on viscosity)

$$\frac{\partial \overline{v}_x}{\partial y} = \frac{1}{\kappa}\sqrt{\frac{\tau_o}{\rho}}\frac{1}{y}$$

where τ_o is the shear stress acting on the wall y is the distance from the wall

- Definition of the friction velocity v_*

$$v_* = \sqrt{\frac{\tau_o}{\rho}}$$

The logarithmic velocity profiles in the inertial sub-layer

- Integrating

$$\overline{v}_x = \frac{v_*}{\kappa} ln(y) + \lambda'$$

- λ' is a integration constant. Constant are determined by experiments

$$\frac{\overline{v}_x}{v_*} = 2.5ln\left(y\frac{v_*}{\nu}\right) + 5.5 \qquad for \quad y\frac{v_*}{\nu} > 30$$

Taylor-series development in the viscous sub-layer

- Taylor-series development

$$\overline{v}_x(y) = \overline{v}_x(0) + \left.\frac{\partial \overline{v}_x}{\partial y}\right|_{y=0} y + \frac{1}{2!}\left.\frac{\partial^2 \overline{v}_x}{\partial y^2}\right|_{y=0} y^2 + \frac{1}{3!}\left.\frac{\partial^3 \overline{v}_x}{\partial y^3}\right|_{y=0} y^3 + \cdots$$

- Shear stress for steady flow in a slit of thickness 2B

$$\overline{\tau}_{yx} = \overline{\tau}_{yx}^{(v)} + \overline{\tau}_{yx}^{(t)} = -\tau_0[1 - (y/B)]$$

$$+\mu \frac{\partial \overline{v}_x}{\partial y} - \rho \overline{v_x' v_y'} = \tau_0\left(1 - \frac{y}{B}\right)$$

Coefficient

$\overline{v}_x(0) = 0$ by no slip condition

$$\left.\frac{\partial \overline{v}_x}{\partial y}\right|_{y=0} = \frac{\tau_0}{\mu}$$

$$\left.\frac{\partial^2 \overline{v}_x}{\partial y^2}\right|_{y=0} = \frac{\rho}{\mu}\left.\left(\overline{v_x' \frac{\partial v_y'}{\partial y} + v_y' \frac{\partial v_x'}{\partial y}}\right)\right|_{y=0} - \frac{\tau_0}{\mu B} = -\frac{\tau_0}{\mu B}$$

$$\left.\frac{\partial^3 \overline{v}_x}{\partial y^3}\right|_{y=0} = \frac{\rho}{\mu}\left.\left(\overline{v_x' \frac{\partial^2 v_y'}{\partial y^2} + 2\,\frac{\partial v_y'}{\partial y}\,\frac{\partial v_x'}{\partial y} + v_y' \frac{\partial^2 v_x'}{\partial y^2}}\right)\right|_{y=0}$$

$$= -\frac{\rho}{\mu}\left.\left(\overline{+2\left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_z'}{\partial z}\right)\frac{\partial v_x'}{\partial y}}\right)\right|_{y=0} = 0$$

Taylor-series development in the viscous sub-layer

- Assuming the next term and determining the coefficient experimentally

$$\frac{\bar{v}_x}{v_*} = \frac{yv_*}{\nu} - \frac{1}{2}\left(\frac{\nu}{v_* B}\right)\left(\frac{yv_*}{\nu}\right)^2 + C\left(\frac{yv_*}{\nu}\right)^4 + \cdots$$

$$\frac{\bar{v}_x}{v_*} = \frac{yv_*}{\nu}\left[1 - \frac{1}{2}\left(\frac{\nu}{v_* B}\right)\left(\frac{yv_*}{\nu}\right) - \frac{1}{4}\left(\frac{yv_*}{14.5\nu}\right)^3 + \cdots\right] \qquad 0 < \frac{yv_*}{\nu} < 5$$

Empirical expressions for the turbulent momentum flux

- The Eddy viscosity of Boussinesq. In analogy with Newton's law of viscosity

$$\bar{\tau}_{yx}^{(t)} = -\mu^{(t)} \frac{d\bar{v}_x}{dy}$$

- $\mu^{(t)}$ is the turbulent viscosity
(eddy viscosity: property of flow)

Empirical expressions for the turbulent momentum flux

- Expressions of eddy viscosity for some special cases:
  - For wall turbulence, valid only very near the wall

$$\mu^{(t)} = \mu\left(\frac{y v_*}{14.5\nu}\right)^3 \qquad 0 < \frac{y v_*}{\nu} < 5$$

  - For free turbulence

$$\mu^{(t)} = \rho \kappa_0 b(\bar{v}_{z,\max} - \bar{v}_{z,\min})$$

where b is width of mixing zone

κ_o is empirical parameter, dimensionless

Empirical expressions for the turbulent momentum flux

- The mixing length of Prandtl

$$\bar{\tau}_{yx}^{(t)} = -\rho l^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy}$$

- For
  - Wall turbulence (y=distance from the wall) $\quad l = \kappa_1 y$
  - Free turbulence (b=width of the mixing zone) $\quad l = \kappa_2 b$

Turbulent flow in ducts

- Experimental measurements

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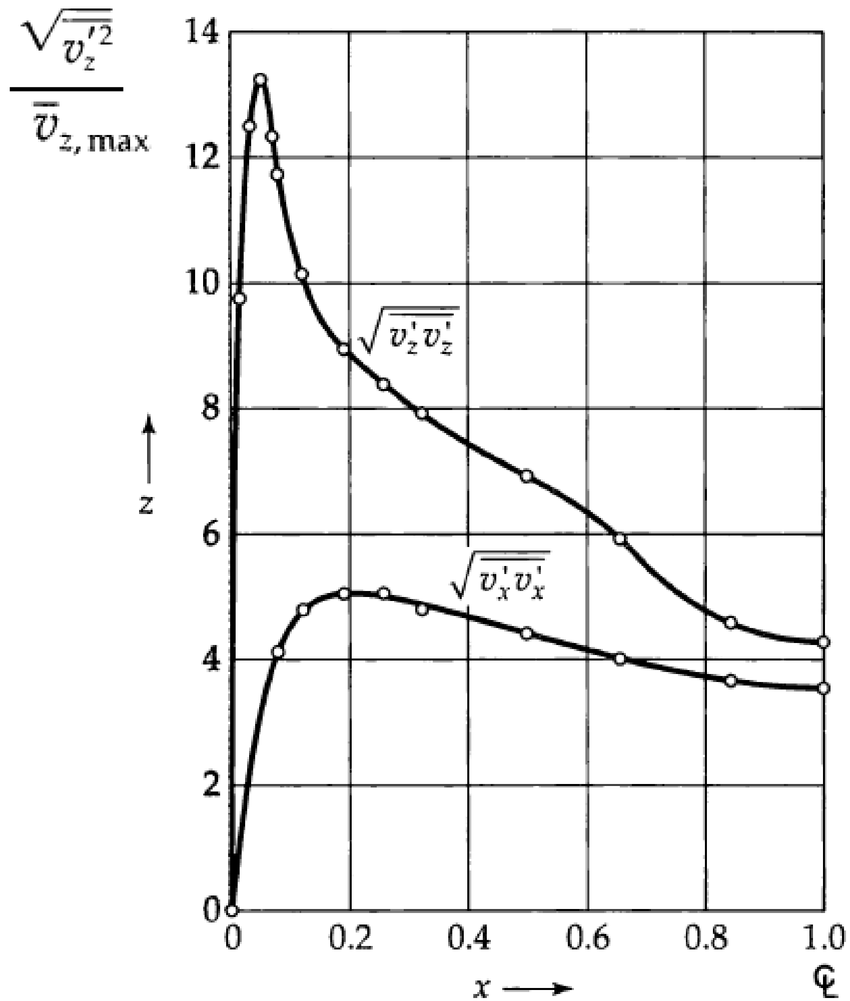
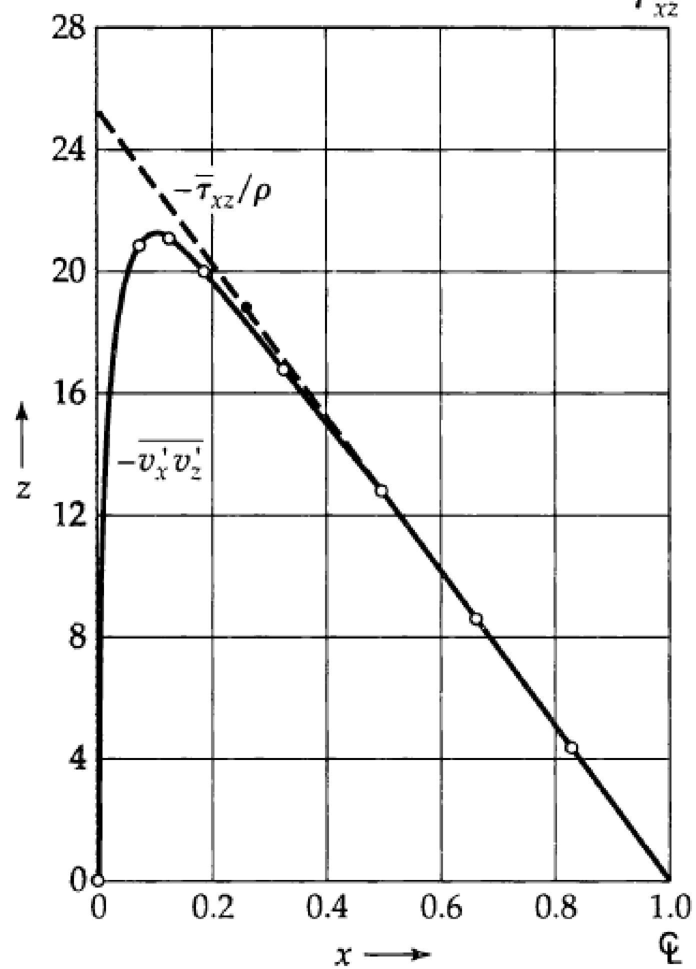
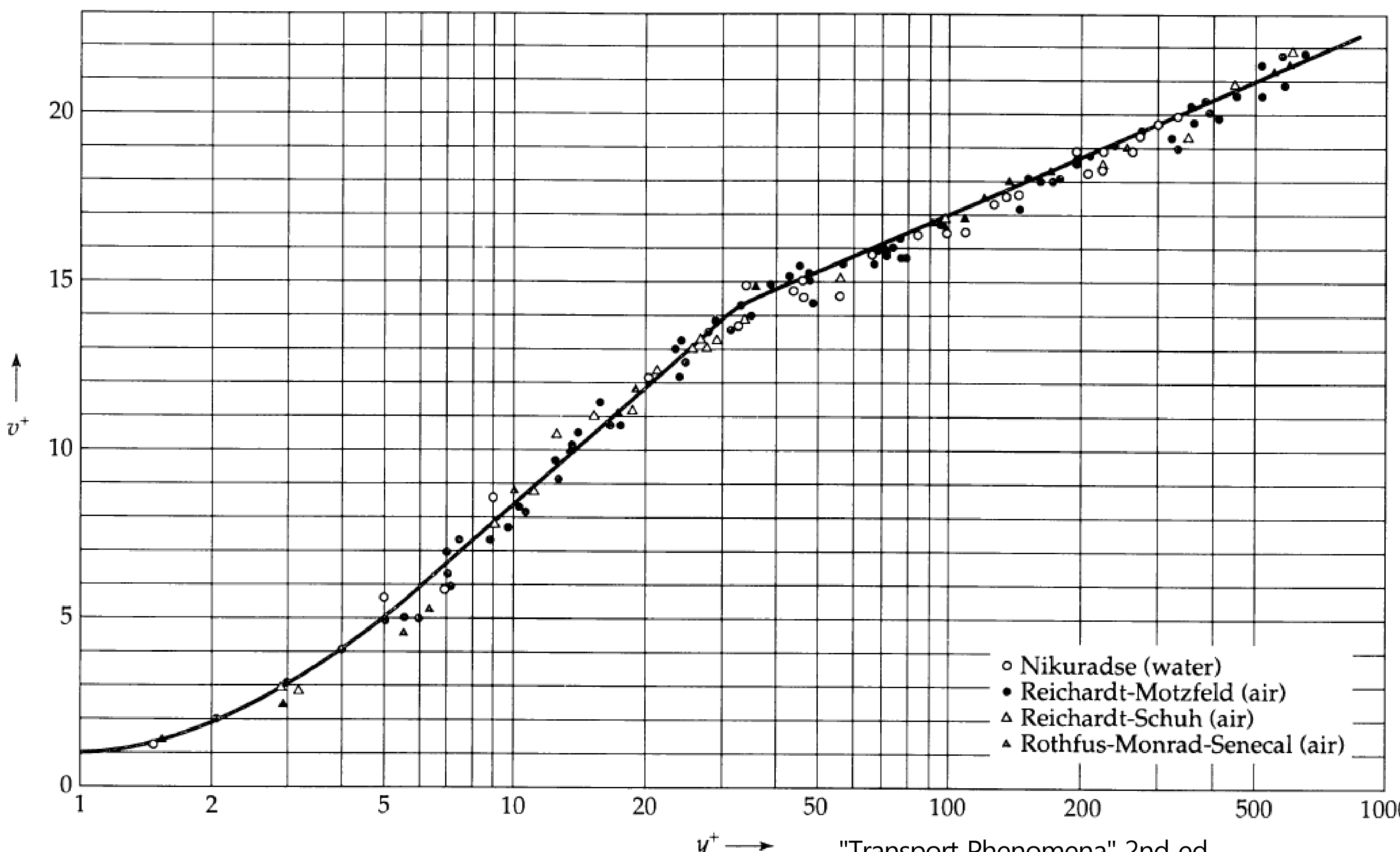
$$\bar{\tau}_{xz} = \bar{\tau}_{xz}^{(t)} + \bar{\tau}_{xz}^{(v)}$$

$$\bar{\tau}_{xz}^{(t)} = \rho \overline{v_x' v_z'}$$





Average velocity in a circular tube



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