Chapter 11. The equations of change for non-isothermal system

- The energy equation
- Special form of the energy equation
- The Boussinesq equation of motion for forced and free convection
- Use of equation of change to solve steady state problems
- Dimensionless analysis of the equations of change for non-isothermal systems

11.1. The energy equation

$$\left\{\begin{array}{l}\text{rate of}\\ \text{increase of}\\ \text{kinetic and}\\ \text{internal}\\ \text{energy}\end{array}\right\} = \left\{\begin{array}{l}\text{net rate of kinetic}\\ \text{and internal}\\ \text{energy addition}\\ \text{by convective}\\ \text{transport}\end{array}\right\} + \left\{\begin{array}{l}\text{net rate of heat}\\ \text{addition by}\\ \text{molecular}\\ \text{transport}\\ \text{(conduction)}\end{array}\right\} +$$

$$\left\{\begin{array}{l}\text{rate of work}\\ \text{done on system}\\ \text{by molecular}\\ \text{mechanisms}\\ \text{(i.e., by stresses)}\end{array}\right\} + \left\{\begin{array}{l}\text{rate of work}\\ \text{done on system}\\ \text{by external}\\ \text{forces}\\ \text{(e.g., by gravity)}\end{array}\right\}$$

Equations

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right) = -\left(\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} + \frac{\partial e_z}{\partial z}\right) + \rho(v_x g_x + v_y g_y + v_z g_z)$$

- Vector form

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right) = -(\nabla \cdot \mathbf{e}) + \rho(\mathbf{v} \cdot \mathbf{g})$$

where

$$\mathbf{e} = \left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right)\mathbf{v} + p\mathbf{v} + [\boldsymbol{\tau} \cdot \mathbf{v}] + \mathbf{q}$$

Equations in vector form

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right) = -\left(\nabla \cdot \left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right)\mathbf{v}\right) - \left(\nabla \cdot \mathbf{q}\right)$$

rate of increase of energy per unit volume

rate of energy addition per unit volume by convective transport

rate of energy addition per unit volume by heat conduction

$$-\left(\nabla \cdot p\mathbf{v}\right) - \left(\nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}]\right) + \rho(\mathbf{v} \cdot \mathbf{g})$$

rate of work done on fluid per unit volume by pressure forces

rate of work done on fluid per unit volume by viscous forces

rate of work done on fluid per unit volume by external forces

Special form of the energy equation

- Subtract the mechanical energy equation in eq. 3.3-1 from the previous equation

$$\frac{\partial}{\partial t}\rho\hat{U} = -(\nabla\cdot\rho\hat{U}\mathbf{v}) - (\nabla\cdot\mathbf{q})$$

rate of increase in internal energy per unit volume	net rate of addition of internal energy by convective transport, per unit volume	rate of internal energy addition by heat conduction, per unit volume

$$-p(\nabla\cdot\mathbf{v}) \qquad -(\boldsymbol{\tau}:\nabla\mathbf{v})$$

reversible rate of internal energy increase per unit volume by compression	*irreversible* rate of internal energy increase per unit volume by viscous dissipation

3.3. The equation of mechanical energy

- Dot product between velocity and equation of motion:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\left(\nabla \cdot \frac{1}{2}\rho v^2 \underline{v}\right) - (\nabla \cdot p\underline{v}) - p(-\nabla \cdot \underline{v})$$

$$-\left(\nabla \cdot (\underline{\tau} \cdot \underline{v})\right) - (-\underline{\tau} : \nabla \underline{v}) + \rho\left(\underline{v} \cdot \underline{g}\right)$$

Special form of the energy equation

- Using the substantial derivative with continuity equation

$$\rho \frac{D\hat{U}}{Dt} = -(\nabla \cdot \mathbf{q}) - p(\nabla \cdot \mathbf{v}) - (\boldsymbol{\tau}:\nabla \mathbf{v})$$

  - In function of enthalpy with continuity equation and $\widehat{U} = \widehat{H} - p\widehat{V} = \widehat{H} - (\frac{p}{\rho})$

$$\rho \frac{D\hat{H}}{Dt} = -(\nabla \cdot \mathbf{q}) - (\boldsymbol{\tau}:\nabla \mathbf{v}) + \frac{Dp}{Dt}$$

- Equation of change for temperature (with eq. 9.8-7)

$$\rho \hat{C}_p \frac{DT}{Dt} = -(\nabla \cdot \mathbf{q}) - (\boldsymbol{\tau}:\nabla \mathbf{v}) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \frac{Dp}{Dt}$$

Special cases

- Ideal gases $(\partial \ln \rho / \partial \ln T)_p = -1$

$$\rho \hat{C}_p \frac{DT}{Dt} = k\nabla^2 T + \frac{Dp}{Dt}$$

- For fluid flowing in a constant pressure system or a fluid with constant density

$$\rho \hat{C}_p \frac{DT}{Dt} = k\nabla^2 T$$

- For a stationary solid, v is zero

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k\nabla^2 T$$

The Boussinesq equation of motion for forced and free convection

- Boussinesq approach (simplified equation of state)

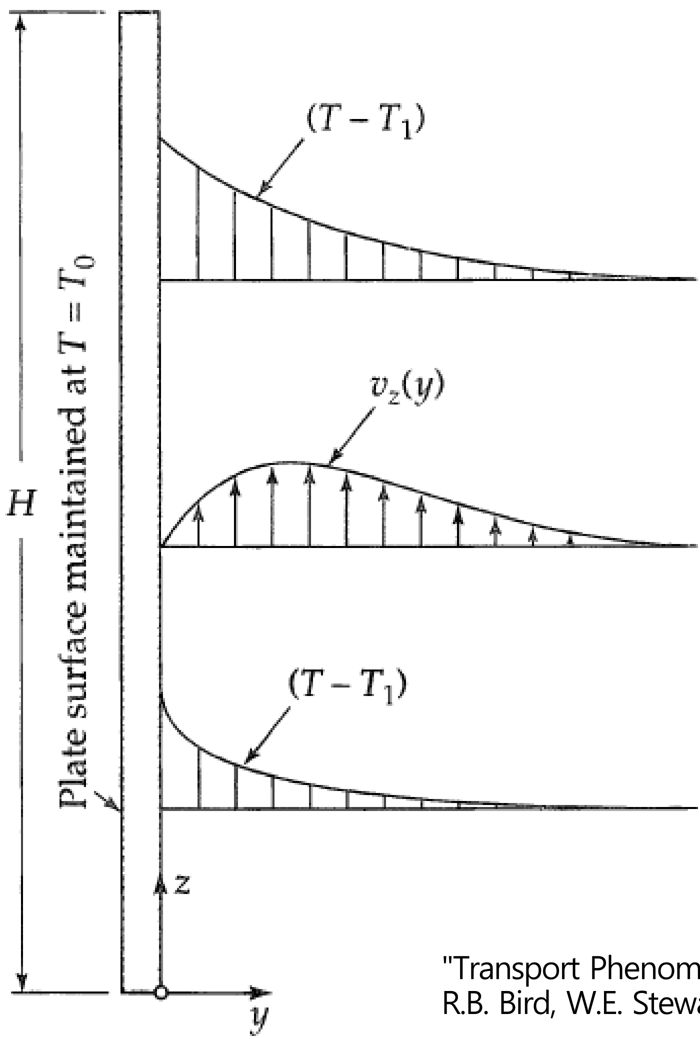
$$\rho(T) = \bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T})$$

$\bar{\beta}$ is $-(1/\rho)(\partial\rho/\partial T)_p$ evaluated at $T = \bar{T}$.

- Introducing only into ρg (not $\rho(D\mathbf{v}/Dt)$) in Eq B, Table 3.5-1. Boussinesq equation

$$\rho \frac{D\mathbf{v}}{Dt} = (-\nabla p + \bar{\rho}\mathbf{g}) - [\nabla \cdot \boldsymbol{\tau}] - \bar{\rho}\mathbf{g}\bar{\beta}(T - \bar{T})$$

Free Convection Heat transfer from a vertical plate



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- Ambient temperature, T_1
- Fluid rises because of the buoyant force
- Constant fluid physical properties, except for density

$$\rho = \rho(T)$$

Equations

- Equation of continuity

$$\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

- Equation of motion. Boussinesq approach

$$\bar{\rho}\left(v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)v_z = \mu\left(\frac{\partial^2}{\partial y^2} + \underline{\frac{\partial^2}{\partial z^2}}\right)v_z + \bar{\rho}g\bar{\beta}(T - T_1)$$

- Equation of energy

$$\bar{\rho}\hat{C}_p\left(v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(T - T_1) = k\left(\frac{\partial^2}{\partial y^2} + \underline{\frac{\partial^2}{\partial z^2}}\right)(T - T_1)$$

$\bar{\rho}$ and $\bar{\beta}$ are evaluated at the ambient temperature T_1.

Equations

- Boundary conditions

$$\begin{array}{llll}
\text{B.C. 1:} & \text{at } y = 0, & v_y = v_z = 0 & \text{and} \quad T = T_0 \\
\text{B.C. 2:} & \text{as } y \to \pm\infty, & v_z \to 0 & \text{and} \quad T \to T_1 \\
\text{B.C. 3:} & \text{at } z = 0, & v_z = 0 &
\end{array}$$

Dimensionless variables

- Dimensionless temperature $$\Theta = \frac{T - T_1}{T_0 - T_1}$$
- Dimensionless vertical coordinate $$\zeta = \frac{z}{H}$$
- Dimensionless horizontal coordinate $$\eta = \left(\frac{B}{\mu \alpha H}\right)^{1/4} y$$
- Dimensionless vertical velocity $$\phi_z = \left(\frac{\mu}{\alpha B H}\right)^{1/2} v_z$$
- Dimensionless horizontal velocity $$\phi_y = \left(\frac{\mu H}{\alpha^3 B}\right)^{1/4} v_y$$

$$\alpha = k/\rho\hat{C}_p \text{ and } B = \bar{\rho} g \bar{\beta}(T_0 - T_1).$$

New Equations

- Equation of continuity

$$\frac{\partial \phi_y}{\partial \eta} + \frac{\partial \phi_z}{\partial \zeta} = 0$$

- Equation of motion. Boussinesq approach

$$\frac{1}{\mathrm{Pr}}\left(\phi_y \frac{\partial}{\partial \eta} + \phi_z \frac{\partial}{\partial \zeta}\right)\phi_z = \frac{\partial^2 \phi_z}{\partial \eta^2} + \Theta$$

- Equation of energy

$$\left(\phi_y \frac{\partial}{\partial \eta} + \phi_z \frac{\partial}{\partial \zeta}\right)\Theta = \frac{\partial^2 \Theta}{\partial \eta^2}$$

- Boundary conditions

B.C. 1:	at $\eta = 0$,	$\phi_y = \phi_z = 0$,	$\Theta = 1$
B.C. 2:	as $\eta \to \infty$,	$\phi_z \to 0$,	$\Theta \to 0$
B.C. 3:	at $\zeta = 0$,	$\phi_z = 0$	

Equations

- The average heat flux from one side of the plate may be written as

$$q_{\text{avg}} = \frac{1}{H}\int_0^H \left(-k\,\frac{\partial T}{\partial y}\right)\bigg|_{y=0} dz$$

- Using the dimensionless variables

$$q_{\text{avg}} = k(T_0 - T_1)\left(\frac{B}{\mu\alpha H}\right)^{1/4} \cdot \int_0^1 \left(-\frac{\partial\Theta}{\partial\eta}\right)\bigg|_{\eta=0} d\zeta$$

$$= k(T_0 - T_1)\left(\frac{B}{\mu\alpha H}\right)^{1/4} \cdot C$$

$$= C \cdot \frac{k}{H}(T_0 - T_1)\left(\left(\frac{\hat{C}_p\mu}{k}\right)\left(\frac{\bar{\rho}^2 g\bar{\beta}(T_0 - T_1)H^3}{\mu^2}\right)\right)^{1/4}$$

$$= C \cdot \frac{k}{H}(T_0 - T_1)(\text{GrPr})^{1/4}$$

- Average heat flux is proportional to

$$T^{\frac{5}{4}}$$

Ra = GrPr is referred to as the *Rayleigh number*.

Dimensional analysis of the equations of change for nonisothermal systems

- Continuity

$$(\nabla \cdot \mathbf{v}) = 0$$

- Equation of motion

$$\bar{\rho}\frac{D\mathbf{v}}{Dt} = -\nabla\mathcal{P} + \mu\nabla^2\mathbf{v} + \bar{\rho}\mathbf{g}\bar{\beta}(T - \bar{T})$$

- Equation of energy

$$\bar{\rho}\hat{C}_p\frac{DT}{Dt} = k\nabla^2 T + \mu\Phi_v$$

Introducing dimensionless numbers

$$\breve{x} = \frac{x}{l_0} \qquad \breve{y} = \frac{y}{l_0} \qquad \breve{z} = \frac{z}{l_0} \qquad \breve{t} = \frac{v_0 t}{l_0}$$

$$\breve{\mathbf{v}} = \frac{\mathbf{v}}{v_0} \qquad \breve{\mathcal{P}} = \frac{\mathcal{P} - \mathcal{P}_0}{\bar{\rho} v_0^2} \qquad \breve{T} = \frac{T - T_0}{T_1 - T_0}$$

$$\breve{\Phi}_v = \left(\frac{l_0}{v_0}\right)^2 \Phi_v \qquad \breve{\nabla} = l_0 \nabla \qquad \frac{D}{D\breve{t}} = \left(\frac{l_0}{v_0}\right) \frac{D}{Dt}$$

- Forced convection

$$(\breve{\nabla} \cdot \breve{\mathbf{v}}) = 0$$

$$\frac{D\breve{\mathbf{v}}}{D\breve{t}} = -\breve{\nabla}\breve{\mathcal{P}} + \left[\!\left[\frac{\mu}{l_0 v_0 \bar{\rho}}\right]\!\right] \breve{\nabla}^2 \breve{\mathbf{v}} - \left[\!\left[\frac{g l_0 \bar{\beta}(T_1 - T_0)}{v_0^2}\right]\!\right] \left(\frac{\mathbf{g}}{g}\right) (\breve{T} - \breve{T})$$

$$\frac{D\breve{T}}{D\breve{t}} = \left[\!\left[\frac{k}{l_0 v_0 \bar{\rho} \hat{C}_p}\right]\!\right] \breve{\nabla}^2 \breve{T} + \left[\!\left[\frac{\mu v_0}{l_0 \bar{\rho} \hat{C}_p (T_1 - T_0)}\right]\!\right] \breve{\Phi}_v$$

Dimensionless groups

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Table 11.5-1 Dimensionless Groups in Equations 11.5-7, 8, and 9

Special cases →	Forced convection	Intermediate	Free convection (A)	Free convection (B)
Choice for v_0 →	v_0	v_0	ν / l_0	α / l_0
$\left[\frac{\mu}{l_0 v_0 \bar{\rho}}\right]$	$\frac{1}{\mathrm{Re}}$	$\frac{1}{\mathrm{Re}}$	1	Pr
$\left[\frac{g l_0 \bar{\beta}(T_1 - T_0)}{v_0^2}\right]$	Neglect	$\frac{\mathrm{Gr}}{\mathrm{Re}^2}$	Gr	GrPr^2
$\left[\frac{k}{l_0 v_0 \bar{\rho} \hat{C}_p}\right]$	$\frac{1}{\mathrm{RePr}}$	$\frac{1}{\mathrm{RePr}}$	$\frac{1}{\mathrm{Pr}}$	1
$\left[\frac{\mu v_0}{l_0 \bar{\rho} \hat{C}_p (T_1 - T_0)}\right]$	$\frac{\mathrm{Br}}{\mathrm{RePr}}$	$\frac{\mathrm{Br}}{\mathrm{RePr}}$	Neglect	Neglect

Dimensionless groups

ble 11.5-3 Physical Interpretation of Dimensionless Groups

$$\mathrm{Re} = \frac{\rho v_0^2/l_0}{\mu v_0/l_0^2} = \frac{\text{inertial force}}{\text{viscous force}}$$

$$\mathrm{Fr} = \frac{\rho v_0^2/l_0}{\rho g} = \frac{\text{inertial force}}{\text{gravity force}}$$

$$\frac{\mathrm{Gr}}{\mathrm{Re}^2} = \frac{\rho g \beta (T_1 - T_0)}{\rho v_0^2/l_0} = \frac{\text{buoyant force}}{\text{inertial force}}$$

$$\mathrm{P\acute{e}} = \mathrm{RePr} = \frac{\rho \hat{C}_p v_0 (T_1 - T_0)/l_0}{k(T_1 - T_0)/l_0^2} = \frac{\text{heat transport by convection}}{\text{heat transport by conduction}}$$

$$\mathrm{Br} = \frac{\mu (v_0/l_0)^2}{k(T_1 - T_0)/l_0^2} = \frac{\text{heat production by viscous dissipation}}{\text{heat transport by conduction}}$$

Table 11.5-2 Dimensionless Groups Used in Nonisothermal Systems

$\mathrm{Re} = [\![l_0 v_0 \rho/\mu]\!] = [\![l_0 v_0/\nu]\!]$	= *Reynolds number*
$\mathrm{Pr} = [\![\hat{C}_p \mu/k]\!] = [\![\nu/\alpha]\!]$	= *Prandtl number*
$\mathrm{Gr} = [\![g\beta(T_1 - T_0) l_0^3/\nu^2]\!]$	= *Grashof number*
$\mathrm{Br} = [\![\mu v_0^2/k(T_1 - T_0)]\!]$	= *Brinkman number*
$\mathrm{P\acute{e}} = \mathrm{RePr}$	= *Péclet number*
$\mathrm{Ra} = \mathrm{GrPr}$	= *Rayleigh number*
$\mathrm{Ec} = \mathrm{Br}/\mathrm{Pr}$	= *Eckert number*

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