Optimum Design

Fall 2016

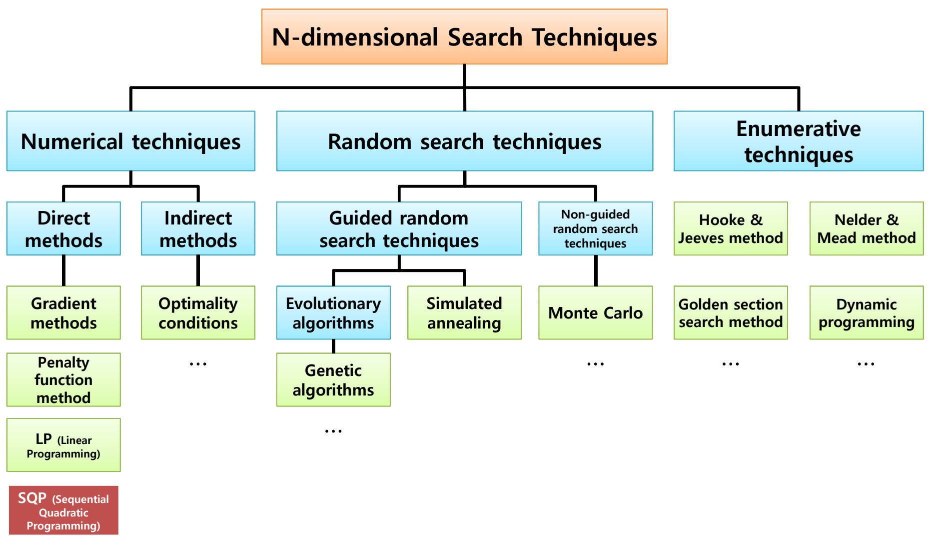
Myung-Il Roh

Department of Naval Architecture and Ocean Engineering
Seoul National University

Contents

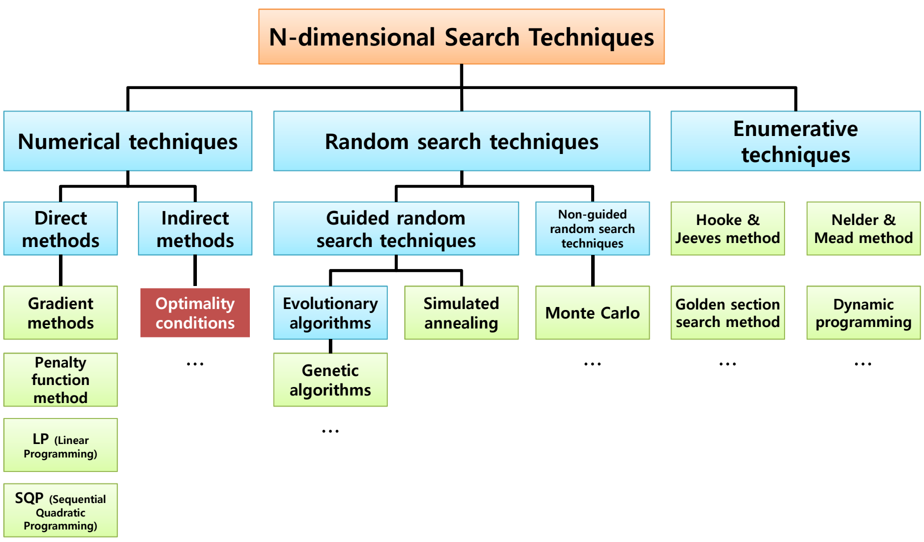
Ch. 6 Constrained Optimization Method: SQP (Sequential Quadratic Programming)

Classes of Search Techniques (1/4)



6.1 Optimality Condition Using Kuhn-Tucker Necessary Condition

Classes of Search Techniques (1/4)

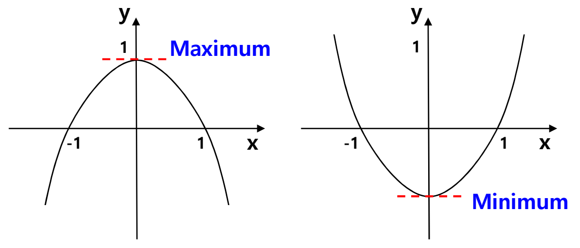


Optimal Solution Using Optimality Condition

Optimality Conditions for Function of Single Variable - The Maximum and Minimum of the Function

Review of the Course of High School

"수학의 정석" (Mathematics II) Review"6. Maximum, Minimum and Differentials"(p.104)



1) **Maximum value:** The increase of the value of the continuous function $f(x)$ is changed to the decrease of that at $x = x^*$.
2) **Minimum value:** The decrease of the value of the continuous function $f(x)$ is changed to the increase of that at $x = x^*$.

$$f'(x^*) = 0$$

(Necessary condition for $x = x^*$ to be a maximum or minimum)

Optimality Conditions for Function of Single Variable - First-Order Necessary and Sufficient Conditions (1/3)

- First-order necessary condition for the function of a single variable: $f'(x^*) = 0$

Proof) The Taylor series expansion of $f(x)$ at the point x^* is as follows.

$$f(x) = f(x^*) + \frac{df(x^*)}{dx}(x - x^*) + \frac{1}{2}\frac{d^2 f(x^*)}{dx^2}(x - x^*)^2 + \boxed{R}$$

Remainder
: If the difference between x and x^* is small, the value of the remainder is also very small.

Let $x - x^* = d$, the equation is as follows.

$$f(x) = f(x^*) + f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

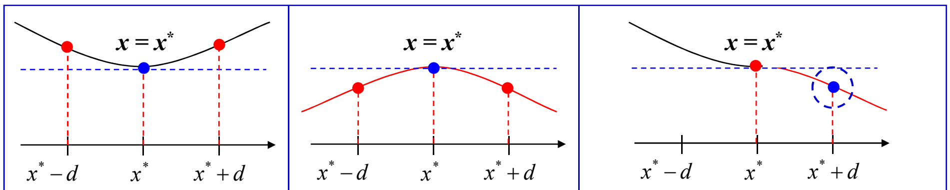
From this equation, the change in the function at x^*, i.e., $f(x) - f(x^*) = \Delta f(x)$ is given as

$$\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

Optimality Conditions for Function of Single Variable - First-Order Necessary and Sufficient Conditions (2/3)

$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

Δf must be positive, if x^* is a local minimum point.



$f(x^*)$ is minimum. | $f(x^*)$ is maximum. | $f(x^*)$ is neither minimum nor maximum (inflection point).

Since d $(= x - x^*)$ is small, the first-order term $f'(x^*)d$ dominates other terms.

And the sign of the term $f'(x^*)d$ is arbitrary.

Thus, the only way Δf can be positive regardless of the sign of d in the neighborhood of x^* is $f'(x^*) = 0$.

In the same way, Δf must be negative if x^* is a local maximum point. So, the only way Δf can be negative regardless of the sign of d in a neighborhood of x^* is $f'(x^*) = 0$.

Optimality Conditions for Function of Single Variable - First-Order Necessary and Sufficient Conditions (3/3)

$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

▪ Now, we need a sufficient condition to determine which of the stationary points are actually minimum for the function.

Since stationary points satisfy the necessary condition $f'(x^*) = 0$, the change in function $\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$ becomes as follows.

$$\Delta f(x) = \frac{1}{2}f''(x^*)d^2 + R$$

Since the second-order term dominates all other higher-order terms, the term can be positive for all $d \neq 0$, if

$$f''(x^*) > 0 \quad \text{(Sufficient condition)}$$

Summary

▪ First-order necessary condition

If x^* is a local minimum point, $f'(x^*) = 0$.

If $f'(x^*) = 0$, x^* is a stationary point (minimum, maximum, or inflection point).

▪ Sufficient condition

If x^* is a stationary point ($f'(x^*) = 0$) and $f''(x^*) > 0$, x^* is a local minimum point.

Optimality Conditions for Function of Several Variables

▪ Matrix form of the Taylor series expansion for the function of two variables

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

Element of the 2x2 Matrix

$$\left(\mathbf{x} = (x_1, x_2)^T,\ \mathbf{x}^* = (x_1^*, x_2^*)^T,\ \boxed{\mathbf{H} \in M_{2\times 2}} \right)$$

▪ Matrix form of the Taylor series expansion for the function of several variables

: It has the same form of the function of two variables.

$\mathbf{x}, \mathbf{x}^*, \nabla f$: n dimensional vector

$\mathbf{H} \in M_{n\times n}$

▪ By defining $\mathbf{x} - \mathbf{x}^* = \mathbf{d}$, the Taylor series expansion for the function of the several variables is as follows.

$$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \boxed{\nabla f(\mathbf{x}^*)^T \mathbf{d}} + \boxed{\frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d}} + R$$

$$\boxed{\nabla f(\mathbf{x}^*)^T = 0,\ \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0} \Rightarrow$$ Sufficient conditions for $\mathbf{x} = \mathbf{x}^*$ to be a local minimum

[Summary] Optimality Conditions for Function of Several Variables

- The Taylor series expansion of $f(\mathbf{x})$, which is the function of n variables, gives

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

- From this equation, the change in the function at $\mathbf{x}^*$, i.e., $\Delta f(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*)$, is given as

$$\Delta f = \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

- If we assume a local minimum is at $\mathbf{x}^*$, then Δf must be positive.

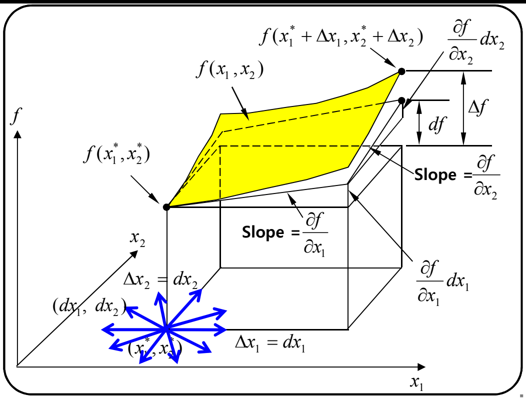
  1) the first-order necessary condition:
  If $\nabla f(\mathbf{x}^*) = 0$, i.e., $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0$, $(i = 1, 2, \cdots n)$, $\mathbf{x}^*$ is a stationary point (minimum, maximum, or inflection point).

  2) the sufficient condition:
  If $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$, then the stationary point $(\nabla f(\mathbf{x}^*)^T = 0 \Rightarrow \nabla f(\mathbf{x}^*) = 0)$ is a local minimum.
  To be $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$, $\mathbf{H}(\mathbf{x}^*)$ must be positive definite.

Lagrange Multiplier for Equality Constraints

Necessary Condition for a Stationary Point: Total Derivative

$df = 0$ ➡ grad $f = 0$



The change in the function $f(x_1, x_2)$ can be expressed using Taylor expansion as follows

$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2^2\right) + R$$

If $\Delta x_1 \to 0, \Delta x_2 \to 0$, the first-order term $\frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$ dominates other terms.

Therefore, Δf can be approximated as $\Delta f \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$.

The symbol "d" refers to the infinitesimal change. By definition of "d", we can write the change in function f as follows:

The change of the function in x_2 direction

$$df = \boxed{\frac{\partial f}{\partial x_1} dx_1} + \boxed{\frac{\partial f}{\partial x_2} dx_2}$$

The change of the function in x_1 direction

If $df = 0$, then x^* is a stationary point.

To be $df = 0$ regardless of the sign of dx_1 and dx_2, $\partial f / x_1$ and $\partial f / x_2$ must be zero.

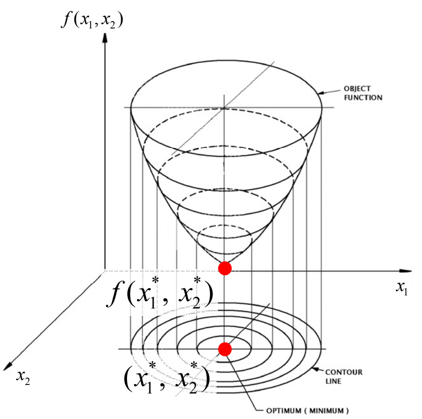
It means that the gradient of function f must be equal to zero.

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0 \Rightarrow \nabla f = 0$$

Definition of Stationary Point

Given: Minimize $f(x_1, x_2)$

Find: Stationary point (x_1^*, x_2^*)



The change in function (df) at the point (x_1^*, x_2^*) with the change in variables (dx_1, dx_2) is as follows.

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

The point at which the change in function (df) is zero is called stationary point.

It includes the minimum, maximum, and inflection (saddle) point.

Note: In the general engineering optimization problem, the optimum point (x) is more important than the optimum value (f).

[Example] Principal dimensions of a ship (L, B, D, C_B) to minimize the shipbuilding cost is more important than the shipbuilding cost itself.

Stationary Point for Unconstrained Optimization Problem

Given: $Minimize\ f(x_1, x_2, x_3)$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3$$

At the stationary point, the change in the function (df) is zero.

The gradient of the function at the stationary point must be zero, because the change in the function (df) can be only zero regardless of the sign of dx_1, dx_2, and dx_3.

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$$

$$\Rightarrow \nabla f = 0$$

Because 'Minimize f' is formulated as an equation (df=0), the number of equations is equal to the number of unknown variables. (Determinate problem)

Stationary Point for Constrained Optimization Problem (1/3)

Given: $Minimize\ f(x_1, x_2, x_3)$

$Subject\ to\ h(x_1, x_2, x_3) = 0$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

Method:

1. Express h (equality constraint) as an explicit function of x_1.
2. Substitute x_1 into f and find the stationary point by using $df = 0$.

In many problem, it may not be possible to express h (equality constraint) as an explicit function of x_1.

Example) It is difficult to express the following equality constraint as an explicit function.

$ex)\ \ h(x_1, x_2, x_3) = \tan x_1 + \cos x_2 + e^{x_3} = 0$

$\Rightarrow$ Is there any method to obtain the stationary point if the equality constraint can not be expressed as an explicit function?

Solution)

$df = 0$ at the stationary point.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \text{------ ①}$$

Since $h(x_1, x_2, x_3) = 0$, dh is also zero.

$$dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0 \quad \text{------ ②}$$

Since equation ① and ② are equal to zero, the following equation is always satisfied.

$df + \lambda \cdot dh = 0$, where λ is an undetermined coefficient 'Lagrange multiplier'.

Stationary Point for Constrained Optimization Problem (2/3)

Given: $Minimize\ f(x_1, x_2, x_3)$

$Subject\ to\ h(x_1, x_2, x_3) = 0$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

① $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$

② $dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0$

Because of the equality constraint h, dx_1, dx_2, and dx_3 are not independent.

$$df + \lambda \cdot dh = 0$$

λ: Undetermined Coefficient 'Lagrange multiplier'

This equation can be rearranged as follows.

$$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \lambda\left(\frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3\right) = 0$$

$$\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

Stationary Point for Constrained Optimization Problem (3/3)

Given: $Minimize\ f(x_1, x_2, x_3)$

$Subject\ to\ h(x_1, x_2, x_3) = 0$

Find: Stationary point (x_1^*, x_2^*, x_3^*)

① $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$

② $dh = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial x_3}dx_3 = 0$

Because of the equality constraint h, dx_1, dx_2, and dx_3 are not independent.

$$df + \lambda \cdot dh = 0$$

λ: Undetermined Coefficient 'Lagrange multiplier'

$$\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

If the dx_1, dx_2, and dx_3 were all independent of each other, all terms in the brackets should be zero to satisfy the equation. This however, is not the case because of the equality constraint h. Let's try to make the first term to be zero by determining a proper value of λ, so that the following equation is satisfied without considering the dx_1.

$$\left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right)dx_2 + \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right)dx_3 = 0$$

Since dx_2 and dx_3 are independent, the terms in the brackets must be zero to satisfy the equation.

$$\therefore \left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1}\right) = 0,\quad \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2}\right) = 0,\quad \left(\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3}\right) = 0$$

Therefore, the point (x_1, x_2, x_3, λ) that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1} + \lambda\frac{\partial h}{\partial x_1} = 0,\quad \frac{\partial f}{\partial x_2} + \lambda\frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} + \lambda\frac{\partial h}{\partial x_3} = 0,\quad h(x_1, x_2, x_3) = 0$$

4 Unknown variables: (x_1, x_2, x_3, λ)

4 Equations

There exists an unique solution.

Lagrange Multiplier for Equality Constraints

The point (x_1, x_2, x_3, λ) that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1}+\lambda\frac{\partial h}{\partial x_1}=0,\quad \frac{\partial f}{\partial x_2}+\lambda\frac{\partial h}{\partial x_2}=0$$

$$\frac{\partial f}{\partial x_3}+\lambda\frac{\partial h}{\partial x_3}=0,\quad h(x_1,x_2,x_3)=0$$

It is convenient to write these equations in terms of a Lagrange function, L, defined as

$$L(x_1,x_2,x_3,\lambda)=f(x_1,x_2,x_3)+\lambda h(x_1,x_2,x_3)$$

$$\nabla L(x_1,x_2,x_3,\lambda)=0$$

Constrained optimization problem is transformed to an unconstrained optimization problem.

$$\frac{\partial L}{\partial x_1}=\frac{\partial f}{\partial x_1}+\lambda\frac{\partial h}{\partial x_1}=0\qquad \frac{\partial L}{\partial x_2}=\frac{\partial f}{\partial x_2}+\lambda\frac{\partial h}{\partial x_2}=0$$

$$\frac{\partial L}{\partial x_3}=\frac{\partial f}{\partial x_3}+\lambda\frac{\partial h}{\partial x_3}=0\qquad \frac{\partial L}{\partial \lambda}=h(x_1,x_2,x_3)=0$$

λ: Lagrange Multiplier
L: Lagrange Function

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (1/5)

Optimization Problem

Minimize $f(x_1,x_2,x_3)$ ···· ①
Subject to $h_1(x_1,x_2,x_3)=0$ ···· ②
$h_2(x_1,x_2,x_3)=0$ ···· ③

Number of variables: 3
Number of equation: 2

Necessary condition that minimize f is $df=0$.
$df=0$ corresponds to Eq. ①' as follows.

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\frac{\partial f}{\partial x_3}dx_3=0 \quad \cdots ①'$$

Subject to $h_1(x_1,x_2,x_3)=0$ ··· ②
$h_2(x_1,x_2,x_3)=0$ ··· ③

Number of variables: 3
Number of equations: 3

➡ We can solve this.

Because 'Minimize f' is formulated as an equation ($df=0$), the number of equations is equal to the number of unknown variables. (Determinate problem)

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (2/5)

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ …. ①

Subject to $h_1(x_1, x_2, x_3) = 0$ …. ②

$h_2(x_1, x_2, x_3) = 0$ …. ③

Number of variables: 3

Number of equation: 2

Necessary condition that minimize f is $df = 0$.
$df = 0$ corresponds to Eq. ①′ as follows.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

Subject to $h_1(x_1, x_2, x_3) = 0$ … ②

$h_2(x_1, x_2, x_3) = 0$ … ③

To find the relationships among dx_1, dx_2, dx_3, we modify the equation ② and ③ to the form of total derivative dh_1 and dh_2.

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \quad \cdots ②'$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \quad \cdots ③'$$

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (3/5)

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ …. ①

Subject to $h_1(x_1, x_2, x_3) = 0$ …. ②

$h_2(x_1, x_2, x_3) = 0$ …. ③

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \quad \cdots ①'$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \quad \cdots ②'$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \quad \cdots ③'$$

Are the equation ①', ②', and ③' differential equations with respect to f, h_1, and h_2?
Answer: If the problem were given as follows:

- Given: $\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0$, $\frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0$, $\frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0$,
- Find: Function f, h_1, h_2

Then the equation ①′, ②′, and ③′ would be differential equations.

However, the function f, h_1, and h_2 (equation ①, ②, ③) are given and differential quantities of dx_1, dx_2, and dx_3 are to find, the equation ①′, ②′, and ③′ are algebraic equations for the variables x_1, x_2, and x_3.

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (4/5)

Optimization Problem

Minimize $f(x_1, x_2, x_3)$ ···· ①

Subject to $h_1(x_1, x_2, x_3) = 0$ ···· ②

$h_2(x_1, x_2, x_3) = 0$ ···· ③

$$*df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = 0 \cdots ①'$$

$$dh_1 = \frac{\partial h_1}{\partial x_1}dx_1 + \frac{\partial h_1}{\partial x_2}dx_2 + \frac{\partial h_1}{\partial x_3}dx_3 = 0 \cdots ②'$$

$$dh_2 = \frac{\partial h_2}{\partial x_1}dx_1 + \frac{\partial h_2}{\partial x_2}dx_2 + \frac{\partial h_2}{\partial x_3}dx_3 = 0 \cdots ③'$$

* Since dx_1, dx_2, and dx_3 are not independent because of the equality constraints h_1 and h_2.

We multiply the equation ②' and ③' by λ_1 and λ_2, respectively and add it to the equation ①':

$$df + \lambda_1 dh_1 + \lambda_2 dh_2 = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial h_1}{\partial x_1} + \lambda_2\frac{\partial h_2}{\partial x_1}\right)}_{=0 \cdots ④}dx_1 + \underbrace{\left(\frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial h_1}{\partial x_2} + \lambda_2\frac{\partial h_2}{\partial x_2}\right)}_{=0 \cdots ⑤}dx_2 + \underbrace{\left(\frac{\partial f}{\partial x_3} + \lambda_1\frac{\partial h_1}{\partial x_3} + \lambda_2\frac{\partial h_2}{\partial x_3}\right)}_{=0 \cdots ⑥}dx_3 = 0$$

➡ Determine λ_1, λ_2 so that the first term in the brackets becomes zero*. (to eliminate dx_1)

➡ Determine λ_1, λ_2 so that the second term in the brackets becomes zero*. (to eliminate dx_2)

➡ Since dx_3 is an independent variable

5 variables: $(x_1, x_2, x_3, \lambda_1, \lambda_2)$

5 equations: 2,3,4,5,6

There exists a unique solution.

25

[Summary] Stationary Point for a Constrained Optimization Problem
- Solution for a Constrained Optimization problem by using the Lagrange Multiplier (5/5)

The point $(x_1, x_2, x_3, \lambda_1, \lambda_2)$ that satisfies the following equations is a stationary point.

$$\frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial h_1}{\partial x_1} + \lambda_2\frac{\partial h_2}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial h_1}{\partial x_2} + \lambda_2\frac{\partial h_2}{\partial x_2} = 0$$

$$\frac{\partial f}{\partial x_3} + \lambda_1\frac{\partial h_1}{\partial x_3} + \lambda_2\frac{\partial h_2}{\partial x_3} = 0, h_1(x_1, x_2, x_3) = 0, \quad h_2(x_1, x_2, x_3) = 0$$

It is convenient to write these equations in terms of a Lagrange function, L, defined as

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) + \lambda_1 h_1(x_1, x_2, x_3) + \lambda_2 h_2(x_1, x_2, x_3)$$

$$\nabla L(x_1, x_2, x_3, \lambda_1, \lambda_2) = 0$$

λ: Lagrange Multiplier

L: Lagrange Function

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial h_1}{\partial x_1} + \lambda_2\frac{\partial h_2}{\partial x_1} = 0 \;\cdots\cdots\; ④$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1\frac{\partial h_1}{\partial x_2} + \lambda_2\frac{\partial h_2}{\partial x_2} = 0 \;\cdots\; ⑤$$

$$\frac{\partial L}{\partial x_3} = \frac{\partial f}{\partial x_3} + \lambda_1\frac{\partial h_1}{\partial x_3} + \lambda_2\frac{\partial h_2}{\partial x_3} = 0 \;\cdots\; ⑥$$

$$\frac{\partial L}{\partial \lambda_1} = h_1(x_1, x_2, x_3) = 0 \;\cdots\; ②$$

$$\frac{\partial L}{\partial \lambda_2} = h_2(x_1, x_2, x_3) = 0 \;\cdots\; ③$$

The Lagrange function gives us a simple way of formulating the equations that have to be satisfied at a stationary point.

[Example] Lagrange Multiplier for Equality Constraints - Quadratic Programming Problem (1/2)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Original Problem

Minimize $f(x_1,x_2)=(x_1-1.5)^2+(x_2-1.5)^2$

Subject to $h(x_1,x_2)=x_1+x_2-2=0$

Lagrange Function

Minimize $L(x_1,x_2,\lambda)=f(x_1,x_2)+\lambda h(x_1,x_2)$

$$=(x_1-1.5)^2+(x_2-1.5)^2+\lambda(x_1+x_2-2)$$

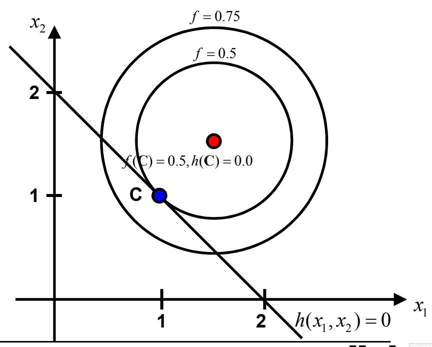
Necessary Condition: $\nabla L(x_1,x_2,\lambda)=0$

$$\frac{\partial L}{\partial x_1}=2(x_1-1.5)+\lambda=0$$

$$\frac{\partial L}{\partial x_2}=2(x_2-1.5)+\lambda=0$$

$$\frac{\partial L}{\partial \lambda}=x_1+x_2-2=0$$

$\Rightarrow x_1^*=x_2^*=1, \lambda^*=1$ (The point C is the stationary point.)



[Example] Lagrange Multiplier for Equality Constraints - Quadratic Programming Problem (2/2)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Original Problem

Minimize $f(\mathbf{x})=(x_1-1.5)^2+(x_2-1.5)^2$

Subject to $h(\mathbf{x})=x_1+x_2-2=0$

Lagrange Function

Minimize $L(\mathbf{x},\ v)=f(\mathbf{x})+vh(\mathbf{x})$

$$=(x_1-1.5)^2+(x_2-1.5)^2+v(x_1+x_2-2)$$

Necessary Condition: $\nabla L(\mathbf{x}^*,v^*)=0$

$$\nabla f(\mathbf{x}^*)+v^*\nabla h(\mathbf{x}^*)=0$$

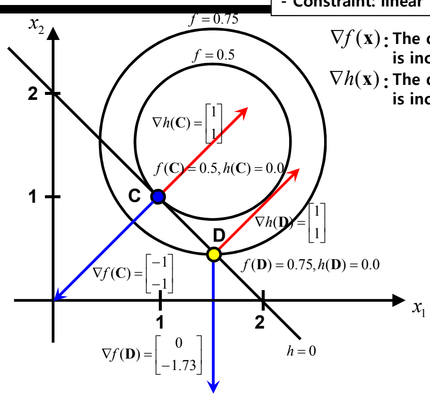
$$\therefore -\nabla f(\mathbf{x}^*)=v^*\nabla h(\mathbf{x}^*)$$

$$\nabla f(\mathbf{x})=\begin{bmatrix}2(x_1-1.5)\\2(x_2-1.5)\end{bmatrix},\ \nabla h(\mathbf{x})=\begin{bmatrix}1\\1\end{bmatrix}$$

$$-2(x_1^*-1.5)=v^*,\ -2(x_2^*-1.5)=v^*$$

$$x_1^*+x_2^*-2=0$$

$\Rightarrow x_1^*=x_2^*=1, v^*=1$ (The point C is the stationary point.)



$\nabla f(\mathbf{x})$: The direction where $f(\mathbf{x})$ is increased

$\nabla h(\mathbf{x})$: The direction where $h(\mathbf{x})$ is increased

At the candidate minimum C, the meaning of $-\nabla f(\mathbf{x}^*)=v^*\nabla h(\mathbf{x}^*)$ **is**

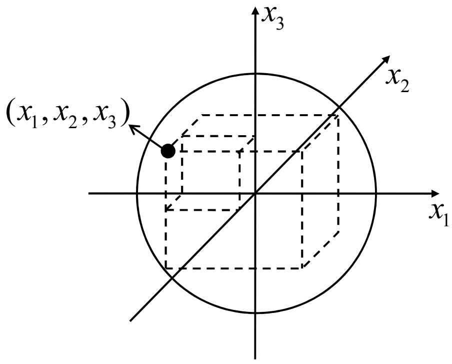
The gradient vector of the objective function and constraint are on the same line and proportional to each other, and the Lagrange multiplier v^* is the proportionality constant.

$$\nabla f(\mathbf{C})=\begin{bmatrix}-1\\-1\end{bmatrix},\ \nabla h(\mathbf{C})=\begin{bmatrix}1\\1\end{bmatrix},\ v^*=1$$

But the point D is not a candidate minimum, because the gradient vector of the objective function and constraint are not on the same line.

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (1/4)

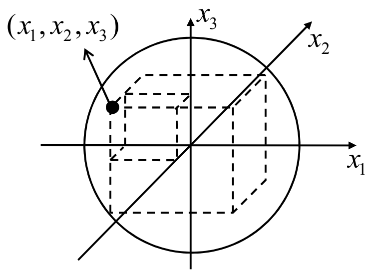
- ☑ **There is a sphere whose center is (0, 0, 0) and radius is c.**
- ☑ **Determine the maximum volume of the rectangular solid which is circumscribed* in the sphere.**

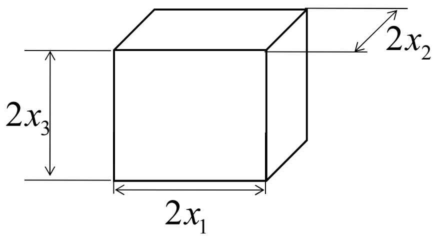


* To draw a geometric figure around another figure so that the two are in contact but do not intersect.

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (2/4)

- ☑ **Mathematical Modeling**





The volume of the rectangular sold f is

$$f(x_1, x_2, x_3) = 2x_1 \cdot 2x_2 \cdot 2x_3$$

Because the vertices of the rectangular solid are on the surface of the sphere,

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0$$

$$cf)\ equation\ for\ a\ sphere: x^2 + y^2 + z^2 = r^2$$

$$\begin{aligned} &Maximize\ f(x_1, x_2, x_3) = 2x_1 \cdot 2x_2 \cdot 2x_3 \\ &\qquad\qquad = 8x_1x_2x_3 \\ &Subject\ to \\ &\qquad h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0 \end{aligned}$$

$$\begin{aligned} &Minimize\ f(x_1, x_2, x_3) = -8x_1x_2x_3 \\ &Subject\ to \\ &\qquad h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - c^2 = 0 \end{aligned}$$

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (3/4)

☑ **Solution (1/2)**

$$\begin{aligned}&Minimize\ f(x_1,x_2,x_3) = -8x_1x_2x_3\\&Subject\ to\\&\qquad h(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2-c^2=0\end{aligned}$$

Lagrange function for this problem is as follow.

$$\begin{aligned}L(x_1,x_2,x_3,\lambda) &= f(x_1,x_2,x_3)+\lambda h(x_1,x_2,x_3)\\&= -8x_1x_2x_3+\lambda(x_1^2+x_2^2+x_3^2-c^2)\end{aligned}$$

$$\nabla L(x_1,x_2,x_3,\lambda)=0 \Rightarrow$$

$$\frac{\partial L}{\partial x_1} = -8x_2x_3+\lambda 2x_1=0 \qquad \frac{\partial L}{\partial x_2} = -8x_1x_3+\lambda 2x_2=0$$

$$\frac{\partial L}{\partial x_3} = -8x_1x_2+\lambda 2x_3=0 \qquad \frac{\partial L}{\partial \lambda} = x_1^2+x_2^2+x_3^2-c^2=0$$

[Example] Solving Nonlinear Constrained Optimization Problem by Using the Lagrange Multiplier (4/4)

☑ **Solution (2/2)**

$-8x_2x_3+\lambda 2x_1=0$ ········· ① **Equation ① × x_1** → $-8x_1x_2x_3+\lambda 2x_1^2=0$ → $x_1^2=\dfrac{4x_1x_2x_3}{\lambda}$

$-8x_1x_3+\lambda 2x_2=0$ ········· ② **Equation ② × x_2** → $-8x_1x_2x_3+\lambda 2x_2^2=0$ → $x_2^2=\dfrac{4x_1x_2x_3}{\lambda}$

$-8x_1x_2+\lambda 2x_3=0$ ········· ③ **Equation ③ × x_3** → $-8x_1x_2x_3+\lambda 2x_2^2=0$ → $x_3^2=\dfrac{4x_1x_2x_3}{\lambda}$

$x_1^2+x_2^2+x_3^2-c^2=0$ ········· ④

Substitute these into the equation ④

$$\frac{4x_1x_2x_3}{\lambda}+\frac{4x_1x_2x_3}{\lambda}+\frac{4x_1x_2x_3}{\lambda}-c^2=0$$

$$\frac{12x_1x_2x_3}{\lambda}=c^2$$

$$\frac{12x_1x_2x_3}{c^2}=\lambda \quad \text{·········} ⑤$$

Substitute the equation ⑤ into the equation ①

$$-8x_2x_3+\frac{12x_1x_2x_3}{c^2}2x_1=0$$

$$-8x_2x_3+\frac{24x_1^2x_2x_3}{c^2}=0$$

$$-8x_2x_3\left(1-\frac{3x_1^2}{c^2}\right)=0$$

If x_2 or x_3 are zero 0, the volume of the rectangular solid is zero and the solution is trivial. Therefore,

$$1-\frac{3x_1^2}{c^2}=0$$

$$\frac{3x_1^2}{c^2}=1$$

$$x_1^2=\frac{c^2}{3}$$

$$x_1=\pm\frac{c}{\sqrt{3}}$$

Because x_1 is a length, it must be positive.

x_2 and x_3 are found in the same way.

$$x_1=\frac{c}{\sqrt{3}},\ x_2=\frac{c}{\sqrt{3}},\ x_3=\frac{c}{\sqrt{3}}$$

So, the maximum volume is

$$8x_1x_2x_3=\frac{8c^3}{3\sqrt{3}}$$

[Summary] Constrained Optimization Method by Using the Lagrange Multiplier

☑ Constrained Optimization Problem

$$\textit{Minimize} \quad f(\mathbf{x}) = f(x_1, x_2, \ \dots \ , x_n)$$
$$\textit{Subject to} \quad h_i(\mathbf{x}) = 0, \quad i = 1, \ \dots \ , p$$

- Determination of the propeller principal dimensions by using the Lagrange multiplier
- Determination of the principal dimension of a ship by using the Lagrange multiplier

☑ Definition of the Lagrange function (L)

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x})$$
$$= f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x})$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix}$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign, i.e., they can be positive, negative, or zero.

<Reason>

The solution does not change, even if the equality constraint is multiplied by the minus sign.

Comparison between Newton's Method and Method of Lagrange Multipliers

Newton' Method for Unconstrained Optimization Problem

Given: $\textit{Minimize} \ f(\mathbf{x})$

Find: Local minimum point

By defining $\mathbf{x} - \mathbf{x}^* = \mathbf{d}$, the Taylor series expansion for the function of multi variables is as follows.

$$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2} \mathbf{d}^T \mathbf{H}(\mathbf{x}^*) \mathbf{d} + R$$

Necessary condition for $\mathbf{x} = \mathbf{x}^*$ to be a candidate local minimum (stationary point) ⬅ $\nabla f(\mathbf{x}^*)^T = 0, \ \frac{1}{2} \mathbf{d}^T \mathbf{H}(\mathbf{x}^*) \mathbf{d} > 0$ ➡ Sufficient conditions for $\mathbf{x} = \mathbf{x}^*$ to be a local minimum

Method of Lagrange Multipliers for Constrained Optimization Problem

Given: $\textit{Minimize} \ f(\mathbf{x})$, $h(x_1, x_2, x_3) = 0$

Find: Local candidate minimum point

$$df + \lambda \cdot dh = 0$$

λ: Undetermined Coefficient 'Lagrange multiplier'

Define Lagrange function, $L = df + \lambda \cdot dh$

Necessary condition for $\mathbf{x} = \mathbf{x}^*$ to be a candidate local minimum ➡ $\nabla L = \mathbf{0}$ (stationary point)

Kuhn-Tucker Necessary Condition for Inequality Constraints

Quadratic Programming Problem with Inequality Constraint

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

We can transform an inequality constraint to an equality constraint by adding a new variable, called the slack variable.

Original Problem

Minimize $f(\mathbf{x}) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$

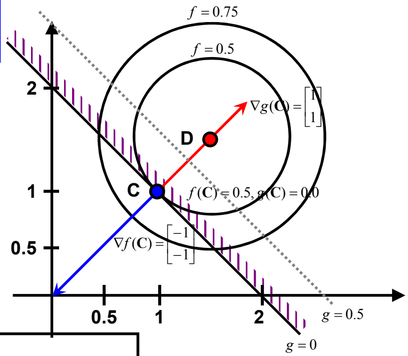
Subject to $g(\mathbf{x}) = x_1 + x_2 - 2 \le 0$

➡ $g(\mathbf{x}) + s^2 = x_1 + x_2 - 2 + \boxed{s^2} = 0$

Lagrange Function

Minimize $L(\mathbf{x},\ u,\ s) = f(\mathbf{x}) + u\left[g(\mathbf{x}) + s^2\right]$

$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + u(x_1 + x_2 - 2 + s^2)$$



Necessary Condition: $\nabla L(\mathbf{x}^*,\ u^*,\ s^*) = 0$

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + u = 0,\quad \frac{\partial L}{\partial x_2} = 2(x_2 - 1.5) + u = 0$$

$$\frac{\partial L}{\partial u} = x_1 + x_2 - 2 + s^2 = 0$$

Linear indeterminate equation

$$\frac{\partial L}{\partial s} = 2us = 0$$

Nonlinear indeterminate equation

$u \ge 0$

(1) If $s = 0$, (Inequality constraint is transformed to the equality constraint.)

$x_1^* = x_2^* = 1, u^* = 1$ ➡ Candidate minimum point (The point C)

(2) If $u = 0$, (Inequality constraint is not active.)

$x_1^* = x_2^* = 1.5, u^* = 0, s^2 = -1$ (The point D: the constraint is violated.)

- At first, we find the solution for the nonlinear indeterminate equation of $u = 0$ or $s = 0$
- And substitute u=0 or s=0 into the linear indeterminate equation.
- Then, solve the linear equation system.

Necessary Condition of Candidate Local Optimal Solution for the Inequality Constrained Problem (1/2)

[Ref] Lagrange function for the equality constrained problem

$$L(\mathbf{x},\mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x})$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.

Inequality constraint

$$g_i(\mathbf{x}) \le 0, \quad i = 1, \dots, m$$

To transform the inequality constraints to the equality constraints, the slack variables s_i^2 are introduced:

$$g_i(\mathbf{x}) + s_i^2 = 0, \quad i = 1, \dots, m$$

Lagrange function for the inequality constrained problem

Since the inequality constraint can be transformed to the equality constraint by introducing the slack variable, the Lagrange function is defined as

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2) = f(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2), \ \underline{u_i \ge 0}$$

u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.
s_i are the slack variables to transform the inequality constraints to the equality constraints.

Necessary Condition of Candidate Local Optimal Solution for the Inequality Constrained Problem (2/2)

Lagrange function for the inequality constrained problem

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2) = f(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

where, u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.
s_i are the slack variables to transform the inequality constraints to the equality constraints.

The necessary condition of the candidate local optimal solution for the inequality constrained problem

$$\nabla L(\mathbf{x}^*, \mathbf{u}^*, \mathbf{s}^*) = \mathbf{0}$$

⬇

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \dots, n$$

$$\frac{\partial L}{\partial u_i} \equiv g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \dots, \mathrm{m}$$

$$\frac{\partial L}{\partial s_i} \equiv u_i^* s_i^* = 0, \quad i = 1, \dots, m$$

$$u_i^* \ge 0, \quad i = 1, \dots, m$$

Kuhn-Tucker Necessary Condition for Inequality Constraints

Optimization Problem

Minimize $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$

Subject to $h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p$ **Equality constraints**

$g_i(\mathbf{x}) \le 0, \quad i = 1, \ldots, m$ **Inequality constraints**

Definition of the Lagrange function

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

$$= f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.
u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.
s_i are the slack variables to transform the inequality constraints to the equality constraints.

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial v_i} \equiv h_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, p$$

$$\frac{\partial L}{\partial u_i} \equiv g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \ldots, \mathrm{m}$$

$$\frac{\partial L}{\partial s_i} \equiv u_i^* s_i^* = 0, \quad i = 1, \ldots, m$$

$$u_i^* \ge 0, \quad i = 1, \ldots, m$$

If x* is the candidate local minimum point, the equations from the Kuhn-Tucker necessary condition have to be satisfied.
Therefore, K-T condition can be used to find the candidate local minimum point for the equality and inequality constrained problem.

The value of the objective function and gradient vectors have to be calculated a x*.

39

[Example] Nonlinear Constrained Optimization Problem #1 (1/2)

①

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g(\mathbf{x}) = x_1^2 + x_2^2 - 6 \le 0$

②

$$L(\mathbf{x}, u, s) = x_1^2 + x_2^2 - 3x_1x_2 + u(x_1^2 + x_2^2 - 6 + s^2)$$

③

$$\frac{\partial L}{\partial x_1} = 2x_1 - 3x_2 + 2ux_1 = 0 \quad \text{①}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 3x_1 + 2ux_2 = 0 \quad \text{②}$$

$$\frac{\partial L}{\partial u} = x_1^2 + x_2^2 - 6 + s^2 = 0, \; s^2 \ge 0, \; u \ge 0 \quad \text{③}$$

$$\frac{\partial L}{\partial s} = \boxed{2us = 0}$$ ➨ There are two cases.

CASE #1: $u = 0$ (The inequality constraint is considered as inactive at the solution point.)

$2x_1 - 3x_2 = 0$

$-3x_1 + 2x_2 = 0$

➨ **Point A:** $x_1^* = 0, x_2^* = 0, f(x_1^*, x_2^*) = 0$

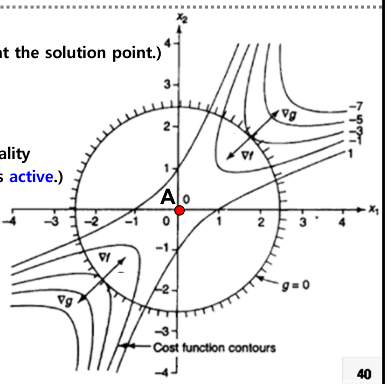
CASE #2: $s = 0$ (The solution point is on the boundary of the inequality constraint. The inequality constraint is considered as active.)

Rearrange the equation ①: $2x_1 - 3x_2 + 2ux_1 = 0, \quad u = -1 + \frac{3}{2}\frac{x_2}{x_1}$

Substitute u into the equation ②: $2x_2 - 3x_1 + 2(-1 + \frac{3}{2}\frac{x_2}{x_1})x_2 = 0$

$2x_2 - 3x_1 - 2x_2 + 3\frac{x_2^2}{x_1} = 0, \quad 3\frac{x_2^2}{x_1} = 3x_1, \quad x_2^2 = x_1^2$

Substitute x_2 into the equation ③: $2x_1^2 = 6, \quad x_1 = \pm\sqrt{3}$



40

[Example] Nonlinear Constrained Optimization Problem #1 (2/2)

CASE #1: $u = 0$ (The inequality constraint is considered as inactive at the solution point.)

$$2x_1 - 3x_2 = 0$$
$$-3x_1 + 2x_2 = 0$$
➡ **Point A:** $x_1^* = 0, x_2^* = 0, f(x_1^*, x_2^*) = 0$

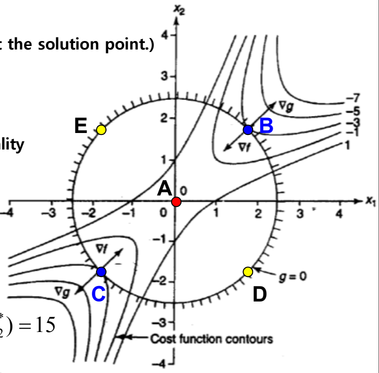
CASE #2: $s = 0$ (The solution point is on the boundary of the inequality constraint.)

$x_1 = x_2 = \sqrt{3},\ u = \frac{1}{2}$ ➡ **Point B:** $x_1^* = x_2^* = \sqrt{3}, f(x_1^*, x_2^*) = -3$

$x_1 = x_2 = -\sqrt{3},\ u = \frac{1}{2}$ ➡ **Point C:** $x_1^* = x_2^* = -\sqrt{3}, f(x_1^*, x_2^*) = -3$

$x_1 = -x_2 = \sqrt{3},\ u = -\frac{5}{2}$ ➡ **Point D:** $x_1^* = \sqrt{3}, x_2^* = -\sqrt{3}, f(x_1^*, x_2^*) = 15$

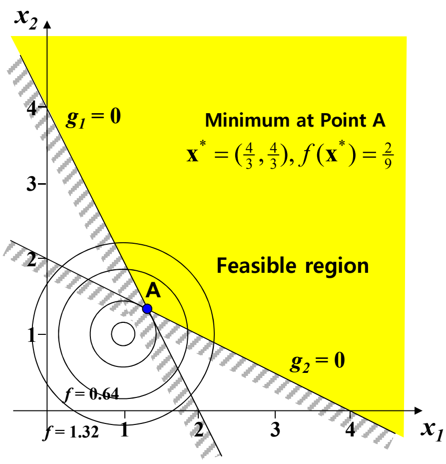
$x_1 = -x_2 = -\sqrt{3},\ u = -\frac{5}{2}$ ➡ **Point E:** $x_1^* = -\sqrt{3}, x_2^* = \sqrt{3}, f(x_1^*, x_2^*) = 15$



[Example] Nonlinear Constrained Optimization Problem #2
- Find the Optimal Solution for the Quadratic Programming Problem by using the Kuhn-Tucker Necessary Condition : xi are free in sign (1/3)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

$$\textit{Minimize}\quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to}\quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$



Lagrange function

$$L(\mathbf{x}, \mathbf{u}, \mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2)$$
$$+u_2(-x_1 - 2x_2 + 4 + s_2^2)$$

[Example] Nonlinear Constrained Optimization Problem #2
- Find the Optimal Solution for the Quadratic Programming Problem by using the Kuhn-Tucker Necessary Condition : xi are free in sign (2/3)

Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 + u_1(-2x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 - 2x_2 + 4 + s_2^2)$$

$$\Leftarrow \boxed{\begin{array}{l} f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \\ g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0 \\ g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0 \end{array}}$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 = 0$$

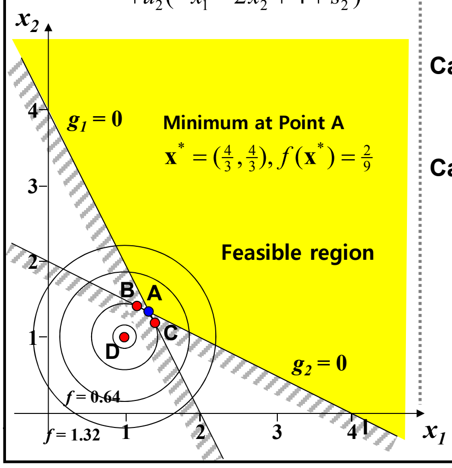
$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0 \qquad u_i \ge 0,\ i = 1,2$$

[Example] Nonlinear Constrained Optimization Problem #2
- Find the Optimal Solution for the Quadratic Programming Problem by using the Kuhn-Tucker Necessary Condition : xi are free in sign (3/3)

Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 + u_1(-2x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 - 2x_2 + 4 + s_2^2)$$



Case #1: s_1=s_2=0

$x_1 = x_2 = \frac{4}{3}, u_1 = u_2 = \frac{2}{9}$ **(Minimum at Point A)**

Case #2: u_1=s_2=0, (Point B)

$x_1 = \frac{6}{5}, x_2 = \frac{7}{5}, u_2 = \frac{2}{5}, s_1^2 = -\frac{1}{5}$

It has to be nonnegative (g_1).

Case #3: u_2=s_1=0, (Point C)

$x_1 = \frac{7}{5}, x_2 = \frac{6}{5}, u_1 = \frac{2}{5}, s_2^2 = -\frac{1}{5}$

It has to be nonnegative (g_2).

Case #4: u_1=u_2=0, (Point D)

$x_1 = x_2 = 1, s_1^2 = s_2^2 = -1$

It has to be nonnegative (g_1, g_2).

[Example] Nonlinear Constrained Optimization Problem #3
- Optimum Solution for the Case that x_i are "Nonnegative" (1/4)

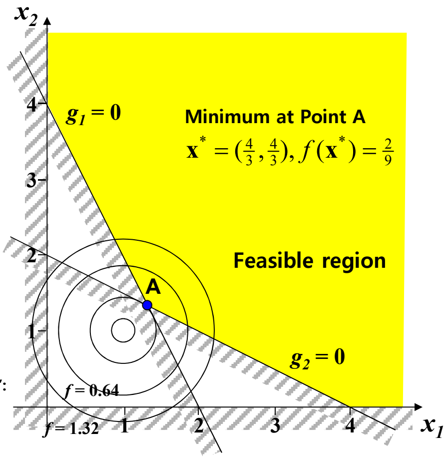
$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$
$$x_1 \ge 0, x_2 \ge 0$$

Minimum point: $\mathbf{x}^* = (\frac{4}{3}, \frac{4}{3}), f(\mathbf{x}^*) = \frac{2}{9}$

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 \le 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 \le 0$$
$$-x_1 \le 0, -x_2 \le 0$$

Inequality constraints whose form are "≤": Introduce the slack variable.

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$
$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$

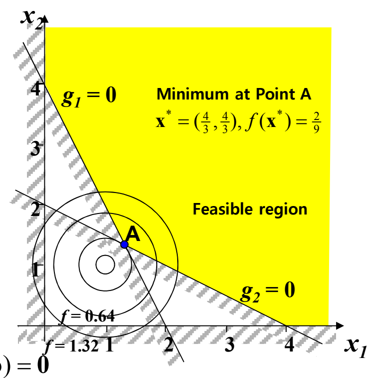


45

[Example] Nonlinear Constrained Optimization Problem #3
- Optimum Solution for the Case that x_i are "Nonnegative" (2/4)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = -2x_1 - x_2 + 4 + s_1^2 = 0$$
$$g_2(\mathbf{x}) = -x_1 - 2x_2 + 4 + s_2^2 = 0$$
$$-x_1 + \delta_1^2 = 0, -x_2 + \delta_2^2 = 0$$



Lagrange function

$$\begin{aligned} L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = & x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \\ & + u_1(-2x_1 - x_2 + 4 + s_1^2) \\ & + u_2(-x_1 - 2x_2 + 4 + s_2^2) \\ & + \zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2) \end{aligned}$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x}, \mathbf{u}, \mathbf{s}, \boldsymbol{\zeta}, \boldsymbol{\delta}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 - 2u_1 - u_2 - \zeta_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2x_2 - 2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0 \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_1^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0 \qquad \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0$$

$$\frac{\partial L}{\partial \zeta_1} = \delta_1^2 - x_1 = 0 \qquad \frac{\partial L}{\partial \zeta_2} = \delta_2^2 - x_2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0$$

46

[Example] Nonlinear Constrained Optimization Problem #3 - Optimum Solution for the Case that x_i are "Nonnegative" (3/4)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta})=\mathbf{0}$

$$\frac{\partial L}{\partial x_1}=2x_1-2-2u_1-u_2-\zeta_1=0 \qquad \frac{\partial L}{\partial x_2}=2x_2-2-u_1-2u_2-\zeta_2=0$$

$$\frac{\partial L}{\partial u_1}=-2x_1-x_2+4+s_1^2=0 \qquad \frac{\partial L}{\partial u_2}=-x_1-2x_2+4+s_2^2=0$$

$$\frac{\partial L}{\partial s_1}=2u_1s_1=0 \qquad \frac{\partial L}{\partial s_2}=2u_2s_2=0$$

$$\frac{\partial L}{\partial \zeta_1}=\delta_1^2-x_1=0\rightarrow\delta_1^2=x_1 \qquad \frac{\partial L}{\partial \zeta_2}=\delta_2^2-x_2=0\rightarrow\delta_2^2=x_2$$

Substitute   Substitute

$$\frac{\partial L}{\partial \delta_1}=2\zeta_1\delta_1=0\rightarrow 2\zeta_1\delta_1^2=0 \qquad \frac{\partial L}{\partial \delta_1}=2\zeta_2\delta_2=0\rightarrow 2\zeta_2\delta_2^2=0$$

Multiply δ_1 to the both sides.   Multiply δ_2 to the both sides.

$u_i,\zeta_i,\delta_i\geq 0,\ i=1,2$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta})=\mathbf{0}$

$$\frac{\partial L}{\partial x_1}=2x_1-2-2u_1-u_2-\zeta_1=0 \qquad \frac{\partial L}{\partial x_2}=2x_2-2-u_1-2u_2-\zeta_2=0$$

$$\frac{\partial L}{\partial u_1}=-2x_1-x_2+4+s_1^2=0 \qquad \frac{\partial L}{\partial u_2}=-x_1-2x_2+4+s_2^2=0$$

$$\frac{\partial L}{\partial s_1}=2u_1s_1=0 \qquad \frac{\partial L}{\partial s_2}=2u_2s_2=0$$

$$2\zeta_1x_1=0 \qquad 2\zeta_2x_2=0 \qquad u_i,\zeta_i\geq 0,\ i=1,2$$

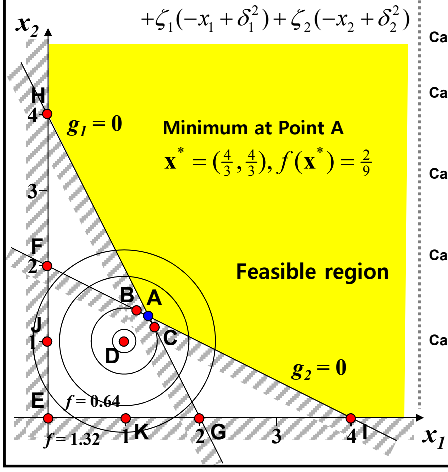
47

[Example] Nonlinear Constrained Optimization Problem #3 - Optimum Solution for the Case that x_i are "Nonnegative" (4/4)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Lagrangian function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta})=x_1^2+x_2^2-2x_1-2x_2+2$$
$$+u_1(-2x_1-x_2+4+s_1^2)$$
$$+u_2(-x_1-2x_2+4+s_2^2)$$
$$+\zeta_1(-x_1+\delta_1^2)+\zeta_2(-x_2+\delta_2^2)$$



Case #1: $s_1=s_2=\zeta_1=\zeta_2=0$, **(Point A)**

$x_1=x_2=\frac{4}{3},u_1=u_2=\frac{2}{9}$

Case #2: $u_1=s_2=\zeta_1=\zeta_2=0$, **(Point B)**

$x_1=\frac{6}{5},x_2=\frac{7}{5},u_2=\frac{2}{5},s_1^2=-\frac{1}{5}$ It has to be nonnegative.

Case #3: $u_2=s_1=\zeta_1=\zeta_2=0$, **(Point C)**

$x_1=\frac{7}{5},x_2=\frac{6}{5},u_1=\frac{2}{5},s_2^2=-\frac{1}{5}$ It has to be nonnegative.

Case #4: $u_1=u_2=\zeta_1=\zeta_2=0$, **(Point D)**

$x_1=x_2=1,s_1^2=s_2^2=-1$ It has to be nonnegative.

Case #5: $u_1=u_2=x_1=x_2=0$, **(Point E)**

$x_1=x_2=0,s_1^2=s_2^2=-4,$ It has to be nonnegative.

$\zeta_1=\zeta_2=-2$

Case #6: $u_1=s_2=x_1=x_2=0$, **(Point E)**

$x_1=x_2=0,s_1^2=-4,$ It has to be nonnegative.

$-x_1-2x_2+4+s_2^2\neq 0$ The constraint is violated.

Case #7: $u_2=s_1=x_1=x_2=0$, **(Point E)**

$x_1=x_2=0,s_2^2=-4,$ It has to be nonnegative.

$-2x_1-x_2+4+s_1^2\neq 0$ The constraint is violated.

Case #8: $s_1=s_2=x_1=x_2=0$, **(Point E)**

$x_1=x_2=0,-2x_1-x_2+4+s_1^2\neq 0,$

$-x_1-2x_2+4+s_2^2\neq 0$ The constraint is violated.

Case #9: $u_1=s_2=\zeta_2=x_1=0$, **(Point F)**

$x_1=0,x_2=2,u_2=1,$

$s_1^2=-2,\zeta_1=-3$ It has to be nonnegative.

Case #10: $u_2=s_1=\zeta_1=x_2=0$, **(Point G)**

$x_1=2,x_2=0,u_1=1,s_2^2=-2,$

$\zeta_2=-3$ It has to be nonnegative.

Case #11: $s_1=s_2=\zeta_1=x_2=0$, **(Point G)**

$x_1=2,x_2=0,$ The constraint is violated.

$-x_1-2x_2+4+s_2^2\neq 0$

Case #12: $u_2=s_1=\zeta_2=x_1=0$, **(Point H)**

$x_1=0,x_2=4,u_1=6,$

$s_2^2=4,\zeta_1=-14$ It has to be nonnegative.

Case #13: $s_1=s_2=\zeta_2=x_1=0$, **(Point H)**

$x_1=0,x_2=4,$ The constraint is violated.

$-x_1-2x_2+4+s_2^2\neq 0$

Case #14: $u_1=s_2=\zeta_1=x_2=0$, **(Point I)**

$x_1=4,x_2=0,u_2=6,$

$s_1^2=4,\zeta_2=-14$ It has to be nonnegative.

Case #15: $u_1=u_2=\zeta_2=x_1=0$, **(Point J)**

$x_1=0,x_2=1,s_1^2=-3,$

$s_2^2=-2,\zeta_1=-2$ It has to be nonnegative.

Case #16: $u_1=u_2=\zeta_1=x_2=0$, **(Point K)**

$x_1=1,x_2=0,s_1^2=-2,$

It has to be nonnegative. $s_2^2=-3,\zeta_2=-2$

48

6.2 Quadratic Programming (QP)

[Summary] Solution of QP Problem Using the Kuhn-Tucker Necessary Condition

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$

Subject to

$$-2x_1 - x_2 \le -4 \qquad\qquad -2x_1 - x_2 + 4 + s_1^2 = 0$$
$$-x_1 - 2x_2 \le -4 \quad \Rightarrow \quad -x_1 - 2x_2 + 4 + s_2^2 = 0$$
$$x_1 \ge 0, x_2 \ge 0 \qquad\qquad -x_1 + \delta_1^2 = 0,\ -x_2 + \delta_2^2 = 0$$

Lagrange function

$$L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$
$$+u_1(-2x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 - 2x_2 + 4 + s_2^2)$$
$$+\zeta_1(-x_1 + \delta_1^2) + \zeta_2(-x_2 + \delta_2^2) \quad \text{where, } u_i, \zeta_i \ge 0$$

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{u},\mathbf{s},\boldsymbol{\zeta},\boldsymbol{\delta}) = 0$

$$\frac{\partial L}{\partial x_1} = -2 + 2x_1 - 2u_1 - u_2 - \zeta_1 = 0, \qquad \frac{\partial L}{\partial x_2} = -2 + 2x_2 - u_1 - 2u_2 - \zeta_2 = 0$$

$$\frac{\partial L}{\partial u_1} = -2x_1 - x_2 + 4 + s_1^2 = 0, \qquad \frac{\partial L}{\partial u_2} = -x_1 - 2x_2 + 4 + s_2^2 = 0 \qquad \frac{\partial L}{\partial \zeta_1} = -x_1 + \delta_1^2 = 0$$

$$\frac{\partial L}{\partial s_1} = 2u_1 s_1 = 0,\ \frac{\partial L}{\partial s_2} = 2u_2 s_2 = 0 \qquad \frac{\partial L}{\partial \delta_1} = 2\zeta_1\delta_1 = 0,\ \frac{\partial L}{\partial \delta_2} = 2\zeta_2\delta_2 = 0 \qquad \frac{\partial L}{\partial \zeta_2} = -x_2 + \delta_2^2 = 0$$

where, $u_i, \zeta_i, \delta_i \ge 0$

Solution Procedure of Quadratic Programming (QP) Problem - Approximate the Original Problem as a QP Problem

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order(linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order(linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$
$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$
$d_i = \Delta x_i$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$: Quadratic objective function

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$: Linear equality constraints

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$: Linear inequality constraints

Solution Procedure of Quadratic Programming (QP) Problem - Construction of Lagrange Function

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$ ➡ $\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{b}_{(m\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 = \mathbf{0}$

Lagrange Function

$$L = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$$
$$+ \mathbf{u}^T{}_{(1\times m)}(\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)})$$
$$+ \mathbf{v}^T{}_{(1\times p)}(\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)})$$

Solution Procedure of Quadratic Programming (QP) Problem
- Applying the K-T Necessary Condition to the Lagrange Function

Lagrange Function

$$L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u}) = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$$
$$+\mathbf{u}^T{}_{(1\times m)}(\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)})$$
$$+\mathbf{v}^T{}_{(1\times p)}(\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)})$$

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u}) = \mathbf{0}$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{d}_{(n\times 1)}} = \mathbf{c}_{(n\times 1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times 1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{v}_{(p\times 1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} - \mathbf{e}_{(p\times 1)} = \mathbf{0}$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial \mathbf{u}_{(n\times 1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} + \mathbf{s}_{(m\times 1)}{}^2 - \mathbf{b}_{(m\times 1)} = 0$$

$$\frac{\partial L(\mathbf{d},\mathbf{v},\mathbf{s},\mathbf{u})}{\partial s_i} = u_i s_i = 0, \quad i = 0 \text{ to } m$$

Solution Procedure of Quadratic Programming (QP) Problem
- Method 1: Direct Solving the Eqs. from the K-T Conditions

Optimization problem

Minimize $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$

Subject to $h_i(\mathbf{x}) = 0, \quad i = 1,\ldots,p$ Equality constraint

$g_i(\mathbf{x}) \le 0, \quad i = 1,\ldots,m$ Inequality constraint

Definition of Lagrange function

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.
u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.
s_i are the slack variables to transform the inequality constraints to the equality constraints.

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial v_i} = h_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, p$$

$$\frac{\partial L}{\partial u_i} = g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \ldots, \mathrm{m}$$

Linear indeterminate equations

$$\frac{\partial L}{\partial s_i} = u_i^* s_i^* = 0, \quad i = 1, \ldots, m$$

Nonlinear indeterminate equations

$u_i^* \ge 0, \quad i = 1, \ldots, m$

Method 1:
- Find the solutions which satisfy the nonlinear indeterminate equations.
- Check whether the solutions satisfy the linear indeterminate equations and determine the solution of this problem.
- Human can find the solution of this problem easily by using this method.

Solution Procedure of Quadratic Programming (QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (1/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i \boxed{s_i} = 0, \quad i=0 \ to \ m \ \cdots\cdots \ ① \qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}_{(m\times1)}{}^2 - \mathbf{b}_{(m\times1)} = 0$$

Multiply s_i both side of the Equation ①

$$u_i s_i = 0 \ \Rightarrow \ u_i s_i^2 = 0$$

Although the Equation ① is multiplied by s_i, the solution ($u_i = 0 \ or \ s_i = 0$) is not changed.

Transform Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i \boxed{s_i^2} = 0, \quad i=0 \ to \ m \cdots\cdots \ ①' \qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}_{(m\times1)}{}^2 - \mathbf{b}_{(m\times1)} = 0$$

Solution Procedure of Quadratic Programming (QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (2/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i \boxed{s_i^2} = 0, \quad i=0 \ to \ m \cdots\cdots \ ①' \qquad \frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times1)} + \boxed{\mathbf{s}_{(m\times1)}{}^2} - \mathbf{b}_{(m\times1)} = 0$$

Replace s_i^2 with s_i' (where $s_i' \ge 0$)

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \quad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0, \ i=0 \ to \ m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(n\times1)}} = \mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}'_{(m\times1)} - \mathbf{b}_{(m\times1)} = 0$$

Linear indeterminate equations

Check whether the solutions obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables **d**, **s'**, **u**, **v**, this problem is a linear programming problem only having the equality constraints.

where, $u_i, s_i' \ \ge 0; \ i = 1 \ to \ m$

Solution Procedure of Quadratic Programming (QP) Problem
- Method 2: Formulate the Problem of the K-T Condition as a LP Problem (3/3)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \qquad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0,\ i = 0 \text{ to } m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(m\times1)}} = \mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}'_{(m\times1)} - \mathbf{b}_{(m\times1)} = 0$$

Linear indeterminate equations

Check whether the solutions obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables d, s', u, v, this problem is a linear programming problem only having the equality constraints.

where $u_i, s_i' \geq 0;\ i = 1 \text{ to } m$

Since the design variables $\mathbf{d}_{(n\times1)}$ are free in sign, we may decompose them as follows to use the Simplex method.

$$\mathbf{d}_{(n\times1)} = \mathbf{d}^+_{(n\times1)} - \mathbf{d}^-_{(n\times1)},\ (d_i^+ \geq 0,\ d_i^- \geq 0; i = 1 \text{ to } n)$$

Also, the Lagrange multipliers $\mathbf{v}_{(p\times1)}$ for the equality constraints are free in sign, we may decompose them as follows to use the Simplex method.

$$\mathbf{v}_{(p\times1)} = \mathbf{y}_{(p\times1)} - \mathbf{z}_{(p\times1)},\ (y_i \geq 0,\ z_i \geq 0;\ i = 1 \text{ to } p)$$

Solution Procedure of Quadratic Programming (QP) Problem
- Method 2: Simplex Method for Solving a QP Problem (1/2)

Kuhn-Tucker Necessary Condition: $\nabla L(\mathbf{d}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial \mathbf{d}_{(n\times1)}} = \mathbf{c}_{(n\times1)} + \mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times1)} + \mathbf{A}_{(n\times m)}\mathbf{u}_{(m\times1)} + \mathbf{N}_{(n\times p)}\mathbf{v}_{(p\times1)} = \mathbf{0}, \qquad \frac{\partial L}{\partial \mathbf{v}_{(p\times1)}} = \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times1)} - \mathbf{e}_{(p\times1)} = \mathbf{0}$$

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0,\ i = 0 \text{ to } m$$

Nonlinear indeterminate equations

$$\frac{\partial L}{\partial \mathbf{u}_{(m\times1)}} = \mathbf{A}^T_{(m\times n)}\mathbf{d}_{(n\times1)} + \mathbf{s}'_{(m\times1)} - \mathbf{b}_{(m\times1)} = 0$$

Linear indeterminate equations

Check whether the solutions obtained from the linear indeterminate equations satisfies the nonlinear indeterminate equations and determine the solution.

Since these equations are linear in the variables d, s', u, v, this problem is a linear programming problem only having the equality constraints.

where $u_i, s_i' \geq 0;\ i = 1 \text{ to } m$

Because d and v are free in sign.

$$\mathbf{d}_{(n\times1)} = \mathbf{d}^+_{(n\times1)} - \mathbf{d}^-_{(n\times1)},\ (d_i^+ \geq 0,\ d_i^- \geq 0; i = 1 \text{ to } n)$$
$$\mathbf{v}_{(p\times1)} = \mathbf{y}_{(p\times1)} - \mathbf{z}_{(p\times1)},\ (y_i \geq 0,\ z_i \geq 0;\ i = 1 \text{ to } p)$$

Matrix From

$$\begin{bmatrix} \mathbf{H}_{(n\times n)} & -\mathbf{H}_{(n\times n)} & \mathbf{A}_{(n\times m)} & \mathbf{0}_{(n\times m)} & \mathbf{N}_{(n\times p)} & -\mathbf{N}_{(n\times p)} \\ \mathbf{A}^T{}_{(m\times n)} & -\mathbf{A}^T{}_{(m\times n)} & \mathbf{0}_{(m\times m)} & \mathbf{I}_{(m\times m)} & \mathbf{0}_{(m\times p)} & \mathbf{0}_{(m\times p)} \\ \mathbf{N}^T{}_{(p\times n)} & -\mathbf{N}^T{}_{(p\times n)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times p)} & \mathbf{0}_{(p\times p)} \end{bmatrix} \begin{bmatrix} \mathbf{d}^+_{(n\times1)} \\ \mathbf{d}^-_{(n\times1)} \\ \mathbf{u}_{(m\times1)} \\ \mathbf{s}'_{(m\times1)} \\ \mathbf{y}_{(p\times1)} \\ \mathbf{z}_{(p\times1)} \end{bmatrix} = \begin{bmatrix} -\mathbf{c}_{(n\times1)} \\ \mathbf{b}_{(m\times1)} \\ \mathbf{e}_{(p\times1)} \end{bmatrix}$$

Introduce the artificial variables, define the artificial objective function, and solve the linear programming problem by using the Simplex method.

Solution Procedure of Quadratic Programming (QP) Problem - Method 2: Simplex Method for Solving a QP Problem (2/2)

Matrix From

Introduce the artificial variables, define the artificial objective function, and solve the linear programming problem by using the Simplex method.

$$\begin{bmatrix} \mathbf{H}_{(n\times n)} & -\mathbf{H}_{(n\times n)} & \mathbf{A}_{(n\times m)} & \mathbf{0}_{(n\times m)} & \mathbf{N}_{(n\times p)} & -\mathbf{N}_{(n\times p)} \\ \mathbf{A}^T_{(m\times n)} & -\mathbf{A}^T_{(m\times n)} & \mathbf{0}_{(m\times m)} & \mathbf{I}_{(m\times m)} & \mathbf{0}_{(m\times p)} & \mathbf{0}_{(m\times p)} \\ \mathbf{N}^T_{(p\times n)} & -\mathbf{N}^T_{(p\times n)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times m)} & \mathbf{0}_{(p\times p)} & \mathbf{0}_{(p\times p)} \end{bmatrix} \begin{bmatrix} \mathbf{d}^+_{(n\times 1)} \\ \mathbf{d}^-_{(n\times 1)} \\ \mathbf{u}_{(m\times 1)} \\ \mathbf{s}'_{(m\times 1)} \\ \mathbf{y}_{(p\times 1)} \\ \mathbf{z}_{(p\times 1)} \end{bmatrix} + \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n+m+p} \end{bmatrix}}_{\text{Artificial variables}} = \begin{bmatrix} -\mathbf{c}_{(n\times 1)} \\ \mathbf{b}_{(m\times 1)} \\ \mathbf{e}_{(p\times 1)} \end{bmatrix}$$

How to define the artificial objective function

1. Define an one equation by sum of all the equations from the 1st row to $(n+m+p)$th row.
2. Define the sum of the all artificial variables ($Y_1 + Y_2 + \ldots + Y_{n+m+p}$) as an objective function (w).

- Determine an initial basic feasible solution (to satisfy the artificial objective function (w) to be zero) for the linear programming problem by using the Simplex method (Phase 1).
- Check whether the initial basic feasible solutions satisfy the following nonlinear indeterminate equations and determine it as a solution.

$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0, \;\; i = 0 \; to \; m$$

59

Solution Procedure of Quadratic Programming (QP) Problem - Summary of Method 2 of Simplex Method for Solving Quadratic Programming Problem

Kuhn-Tucker Necessary Condition(Matrix form)

$$\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}\mathbf{X}_{((2n+2m+2p)\times 1)} = \mathbf{D}_{((n+m+p)\times 1)}$$

Simplex Method for Solving Quadratic Programming Problem

1. The problem to solve the Kuhn-Tucker necessary condition is same as the problem having only the equality constraints (linear programming problem).

2. To solve the linear indeterminate equations, we introduce the artificial variables, define the artificial objective function, and determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{((n+m+p)\times(2n+2m+2p))}\mathbf{X}_{((2n+2m+2p)\times 1)} + \mathbf{Y}_{((n+m+p)\times 1)} = \mathbf{D}_{((n+m+p)\times 1)}$$

If any of the elements in D is (are) negative, the corresponding equation must be multiplied by -1 to have a nonnegative element on the right side.

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{n+m+p} Y_i = \sum_{i=1}^{n+m+p} D_i - \sum_{j=1}^{2(n+m+p)} \sum_{i=1}^{n+m+p} B_{ij} X_j = w_0 + \sum_{j=1}^{2(n+m+p)} C_j X_j$$

where $\underline{C_j = -\sum_{i=1}^{n+m+p} B_{ij}}$, $\underline{w_0 = \sum_{i=1}^{n+m+p} D_i}$ Initial value of the artificial objective function

Add the elements of the jth column of the matrix B and change its sign (Initial relative objective coefficient).

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following equation.

$u_i s_i' = 0; \; i = 1 \; to \; m$: This equation is used to check whether the solution satisfies this equation.

60

Solution Procedure of Quadratic Programming(QP) Problem - Comparison between Method 1 and Method 2

Optimization problem

Minimize $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$

Subject to $h_i(\mathbf{x}) = 0, \quad i = 1,\ldots,p$ Equality constraint

$g_i(\mathbf{x}) = 0, \quad i = 1,\ldots,m$ Inequality constraint

Definition of Lagrange function

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i (g_i(\mathbf{x}) + s_i^2)$$

v_i are the Lagrange multipliers for the equality constraints and are free in sign.
u_i are the Lagrange multiplier for the inequality constraints and have to be nonnegative.
s_i are the slack variables to transform the inequality constraints to the equality constraints.

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1, \ldots, n$$

$$\frac{\partial L}{\partial v_i} = h_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, p$$

$$\frac{\partial L}{\partial u_i} = g_i(\mathbf{x}^*) + s_i^{*2} = 0, \quad i = 1, \ldots, \mathrm{m}$$

Linear indeterminate equations

$$\frac{\partial L}{\partial s_i} = u_i^* s_i^* = 0, \quad i = 1, \ldots, m$$

Nonlinear indeterminate equations

$u_i^* \ge 0, \quad i = 1, \ldots, m$

Method 1:
- Find the solutions to satisfy the nonlinear indeterminate equations.
- Check whether the solutions satisfy the linear indeterminate equations and determine the solution of this problem.
- Human can find the solution of this problem easily by using this method.

Method 2:
- Find the solutions to satisfy the linear indeterminate equations by using the Simplex method.
- Check whether the solutions satisfy the nonlinear indeterminate equations and determine the solution of this problem.
- Since this method is more systematical, it is useful for the computational approach.

6.3 Sequential Linear Programming (SLP)

Sequential Linear Programming (SLP)

$\underbrace{\mathbf{x}^{(k+1)}}_{\text{Improved point}} = \underbrace{\mathbf{x}^{(k)}}_{\text{Starting point}} + \underbrace{\alpha_k}_{\text{Step size}} \underbrace{\mathbf{d}^{(k)}}_{\text{Search direction}}$

- Define **the linear programming (LP) problem** by linearizing the objective function and the constraints at the current design point.
- Compute the design change by solving the linear programming problem and obtain the improved design point.

$$\underbrace{\mathbf{x}^{(k+1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(k)}}_{\text{Current design point}} + \underbrace{\mathbf{d}^{(k)}}_{\text{Design change obtained by solving the LP problem.}}$$

- This method is to find the optimal solution by solving the linear programming problem **sequentially**.

Sequential Linear Programming (SLP) - [Example] Problem with Inequality Constraints (1/6)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$

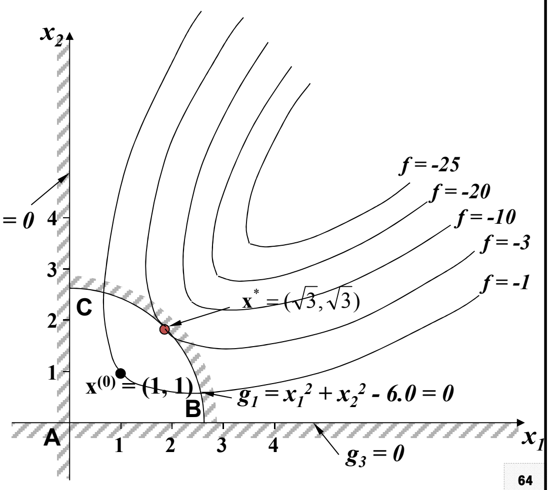
The starting design point: $\mathbf{x}^{(0)} = (1,1)$,

$\varepsilon_1 = \varepsilon_2 = 0.001$

Choose move limits such that a 15% design change is permissible.

The optimal solution:

$$\mathbf{x}^* = (\sqrt{3}, \sqrt{3}),\ f(\mathbf{x}^*) = -3$$



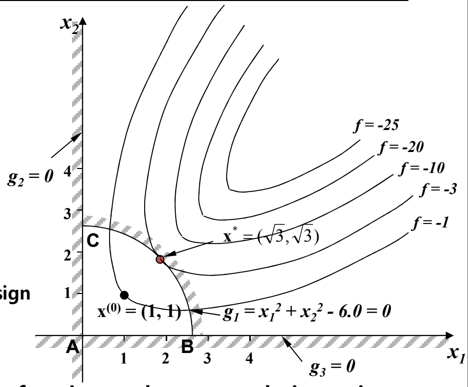
Sequential Linear Programming (SLP) - [Example] Problem with Inequality Constraints (2/6)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$



(1) Iteration 1 (k = 0)

(i) Step 1

From the given point (starting point), the current design point is as follows.

$\mathbf{x}^{(0)} = (1,1),\ \varepsilon_1 = \varepsilon_2 = 0.001$

(ii) Step 2: Evaluate the objective and constraint function at the current design point.

$f(1,1) = -1$

$g_1(1,1) = -\frac{2}{3} < 0$ ➡ **Constraint is satisfied.**

$g_2(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

$g_3(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

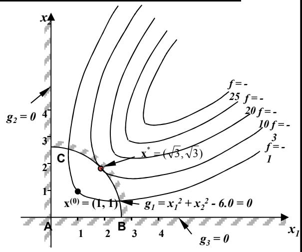
Sequential Linear Programming (SLP) - [Example] Problem with Inequality Constraints (3/6)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$



(1) Iteration 1 (k = 0) $\mathbf{x}^{(0)} = (1,1), \varepsilon_1 = \varepsilon_2 = 0.001$

(iii) Step 3: Define a LP problem (linearize the objective function).

Minimize $f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong f(\mathbf{x}^{(0)}) + \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}$

The first-order (linear) Taylor series expansion of the objective function

Minimize $f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}$

$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{d}^{(k)}$

$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla f^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$

$\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & x_2^{(k)} \end{bmatrix}^T$

$\mathbf{d}^{(k)} = \begin{bmatrix} d_1^{(k)} & d_2^{(k)} \end{bmatrix}^T = \begin{bmatrix} \Delta x_1^{(k)} & \Delta x_2^{(k)} \end{bmatrix}^T$

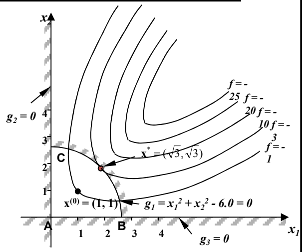
Minimize $f(\mathbf{x}^{(0)} + \mathbf{d}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \begin{bmatrix} 2x_1 - 3x_2 & 2x_2 - 3x_1 \end{bmatrix}_{\mathbf{x}^{(0)}} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix}$

$\bar{f}(\mathbf{d}^{(0)}) \cong (2x_1^{(0)} - 3x_2^{(0)})d_1^{(0)} + (2x_2^{(0)} - 3x_1^{(0)})d_2^{(0)} \leftarrow$ **Substitute** $\mathbf{x}^{(0)} = (1,1)$

$\boxed{\bar{f}(\mathbf{d}^{(0)}) \cong -d_1^{(0)} - d_2^{(0)}}$ ← **The linearized objective function**

Sequential Linear Programming (SLP)
- [Example] Problem with Inequality Constraints (4/6)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$



(1) Iteration 1 (k = 0) $\mathbf{x}^{(0)} = (1,1), \varepsilon_1 = \varepsilon_2 = 0.001$

(iii) Step 3: Define a LP problem (linearize the constraints).

$$\textit{Subject to} \quad g_j(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \Rightarrow \underline{g_j(\mathbf{x}^{(0)}) + \nabla g_j^{\,T}(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}} \le 0;\ j = 1 \textit{ to } m$$

The first-order (linear) Taylor series expansion of the constraints

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{d}^{(k)}$$

$$\nabla g_j^{\,T}(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le -g_j(\mathbf{x}^{(0)});\ j = 1 \textit{ to } m$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla g_j^T = \begin{bmatrix} \frac{\partial g_j}{\partial x_1} & \frac{\partial g_j}{\partial x_2} \end{bmatrix}, \nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} = \bar{g}_j(\Delta\mathbf{x}^{(0)}) = \bar{g}_j(\mathbf{d}^{(0)})$$

Subject to

$$\bar{g}_1(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} \frac{1}{3}x_1^{(0)} & \frac{1}{3}x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(\frac{1}{6}\left(x_1^{(0)}\right)^2 + \frac{1}{6}\left(x_2^{(0)}\right)^2 - 1.0\right)$$

$$\bar{g}_2(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} -x_1^{(0)} & 0 \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_1^{(0)}\right)$$

$$\bar{g}_3(\mathbf{d}^{(0)}) \Rightarrow \begin{bmatrix} 0 & -x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_2^{(0)}\right)$$

Substitute $\mathbf{x}^{(0)} = (1,1)$

$$\bar{g}_1(\mathbf{d}^{(0)}) = \tfrac{1}{3}d_1^{(0)} + \tfrac{1}{3}d_2^{(0)} \le \frac{2}{3}$$
$$\bar{g}_2(\mathbf{d}^{(0)}) = -d_1^{(0)} \le 1$$
$$\bar{g}_3(\mathbf{d}^{(0)}) = -d_2^{(0)} \le 1$$

The linearized constraints

$$g_1(1,1) = -\tfrac{2}{3}$$
$$g_2(1,1) = -1$$
$$g_3(1,1) = -1$$

67

Sequential Linear Programming (SLP)
- [Example] Problem with Inequality Constraints (5/6)

(iv) Step 4: Solve the LP problem for the design change ($\mathbf{d}^{(0)}$).

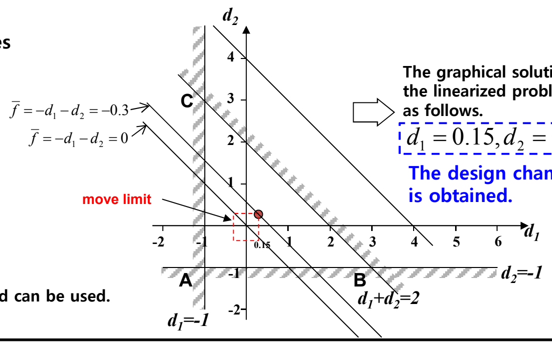
$$\textit{Minimize} \quad \bar{f} = -d_1 - d_2$$
$$\textit{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$
$$-d_1 \le 1$$
$$-d_2 \le 1$$
$$\boxed{\begin{matrix} -0.15 \le d_1 \le 0.15 \\ -0.15 \le d_2 \le 0.15 \end{matrix}}$$

Linearize the objective function and constraints.

$f(1,1) = -1, g_1(1,1) = -\frac{2}{3},$
$g_2(1,1) = -1, g_3(1,1) = -1$
$\nabla f = (-1,-1), \nabla g_1 = (\frac{1}{3}, \frac{1}{3}),$
$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$
$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_1 \le 0$$

Limits must be imposed on changes in design called move limit.



The graphical solution for the linearized problem is as follows.

$$d_1 = 0.15, d_2 = 0.15$$

The design change is obtained.

To solve the problem, the Simplex method can be used.

68

Sequential Linear Programming (SLP) - [Example] Problem with Inequality Constraints (6/6)

(v) Step 5: Check for convergence by using the obtained design change $\mathbf{d}^{(0)}$.

$$\mathbf{d}^{(0)} = (d_1, d_2) = (0.15, 0.15)$$

Since $\|\mathbf{d}^{(0)}\| = \sqrt{0.15^2 + 0.15^2} = 0.212 > \varepsilon_2 (= 0.001)$**, the criterion for convergence is not satisfied.**

(vi) Step 6: Update the design point as $\mathbf{x}^{(k+1)} = \mathbf{x} + \mathbf{d}^{(k)}$. Set $k = k+1$ and go to Step 2.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(1,1)} = \mathbf{x}^{(0)} + \mathbf{d}^{(0)} = (1,1) + (0.15, 0.15) = (1.15, 1.15)$$

$$k = k + 1 = 1$$

Summary of Sequential Linear Programming (SLP)

$$\textit{Minimize} \quad f(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong f(\mathbf{x}^{(k)}) + \nabla f^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)}$$

The first-order(linear) Taylor series expansion of the objective function

$$\textit{Subject to} \quad h_j(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong h_j(\mathbf{x}^{(k)}) + \nabla h_j^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} = 0; \; j = 1 \; to \; p$$

The first-order (linear) Taylor series expansion of the equality constraints

$$g_j(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong g_j(\mathbf{x}^{(k)}) + \nabla g_j^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} \le 0; \; j = 1 \; to \; m$$

The first-order (linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x} + \Delta\mathbf{x}) - f(\mathbf{x}),\; e_j = -h_j(\mathbf{x}),\; b_j = -g_j(\mathbf{x}),$
$c_i = \partial f(\mathbf{x}) / \partial x_i,\; n_{ij} = \partial h_j(\mathbf{x}) / \partial x_i,\; a_{ij} = \partial g_j(\mathbf{x}) / \partial x_i,$
$d_i = \Delta x_i$

$$\textit{Minimize} \quad \bar{f} = \sum_{i=1}^{n} c_i d_i$$

$$\textit{Subject to} \quad \sum_{i=1}^{n} n_{ij} d_i = e_j; \; j = 1 \; to \; p$$

$$\sum_{i=1}^{n} a_{ij} d_i \le b_j; \; j = 1 \; to \; m$$

where, $d_{il} \le d_i \le d_{iu} \; (\Delta x_{il}^{(k)} \le \Delta x_i^{(k)} \le \Delta x_{iu}^{(k)})$

<u>Matrix form</u>

$$\textit{Minimize} \quad \bar{f} = \mathbf{c}^T_{(1\times n)} \mathbf{d}_{(n\times 1)}$$: Linearized objective function

$$\textit{Subject to} \quad \mathbf{N}^T_{(p\times n)} \mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$$: Linearized equality constraint

$$\mathbf{A}^T_{(m\times n)} \mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$$: Linearized inequality constraint

➡ **Linear Programming Problem**

➡ **It can be solved by using the Simplex method.**

Summary of the SLP Method (1/2)

- Step 1: Estimate a starting design point as $\mathbf{x}^{(0)}$. Set $k = 0$. Specify two small numbers, ε_1, ε_2 (criterion for violating the constraints and convergence).
- Step 2: Evaluate objective and constraint function at current design point $\mathbf{x}^{(k)}$. Also evaluate the objective and constraint function gradients at the current design point.
- Step 3: Select the proper move limits $\Delta x_{il}^{(k)}$ and $\Delta x_{iu}^{(k)}$ as some fraction of the current design point. Define the linear programming problem.

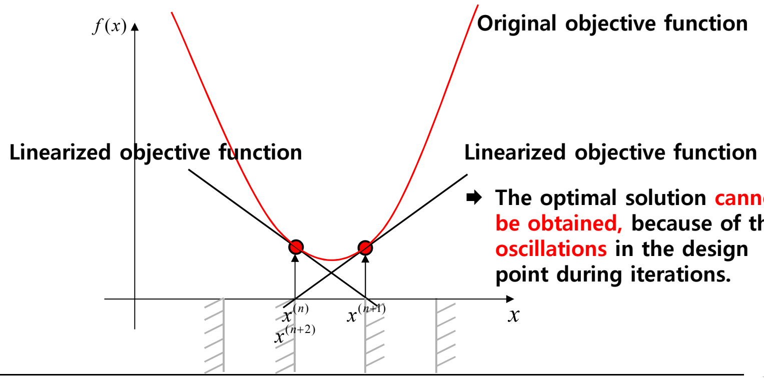
$$\Delta x_{il}^{(k)} \le \Delta x_i^{(k)} \le \Delta x_{iu}^{(k)}$$

Summary of the SLP Method (2/2)

- Step 4: Solve the linear programming problem for $\mathbf{d}^{(k)}$ by using the Simplex method.
- Step 5: Check for convergence. If, $g_i \le \varepsilon_1$ (i = 1 to m), $|h_i| \le \varepsilon_1$ (i = 1 to p), and $\|\mathbf{d}^{(k)}\| \le \varepsilon_2$, then stop and the current design point $\mathbf{x}^{(k)}$ is the optimal solution. Otherwise, continue.
- Step 6: Update the design point as $\mathbf{x}^{(k+1)} = \mathbf{x} + \Delta\mathbf{x}^{(k)}$, Set $k = k+1$ and go to Step 2.

Limitations of SLP Method

- ☑ The move limits of the design variables should be defined by the user.
- ☑ If the move limits are too small, it takes much time to find the optimal solution.
- ☑ If the move limits are too large, it can cause oscillations in the design point during iterations.
- ☑ Thus, the performance of the method depends heavily on selection of move limits.



➡ The optimal solution cannot be obtained, because of the oscillations in the design point during iterations.

6.4 Sequential Quadratic Programming (SQP)

Sequential Quadratic Programming (SQP)

- Define the quadratic programming (QP) problem by approximating quadratic form of the objective function and linear form of the constraints at the current design point.

- Compute the design change by solving the quadratic programming problem and obtain the improved design point.

$$\underbrace{\mathbf{x}^{(k+1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(k)}}_{\text{Current design point}} + \underbrace{\alpha_k}_{\text{Step size obtained by minimizing the penalty function.}} \overbrace{\underbrace{\mathbf{d}^{(k)}}_{\text{Search direction obtained by solving the QP problem.}}}^{\text{Design change}}$$

- This method is to find the optimal solution by solving the quadratic programming problem sequentially.

Formulation of the Quadratic Programming Problem to Determine the Search Direction

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order (linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order (linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$
$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$
$d_i = \Delta x_i$

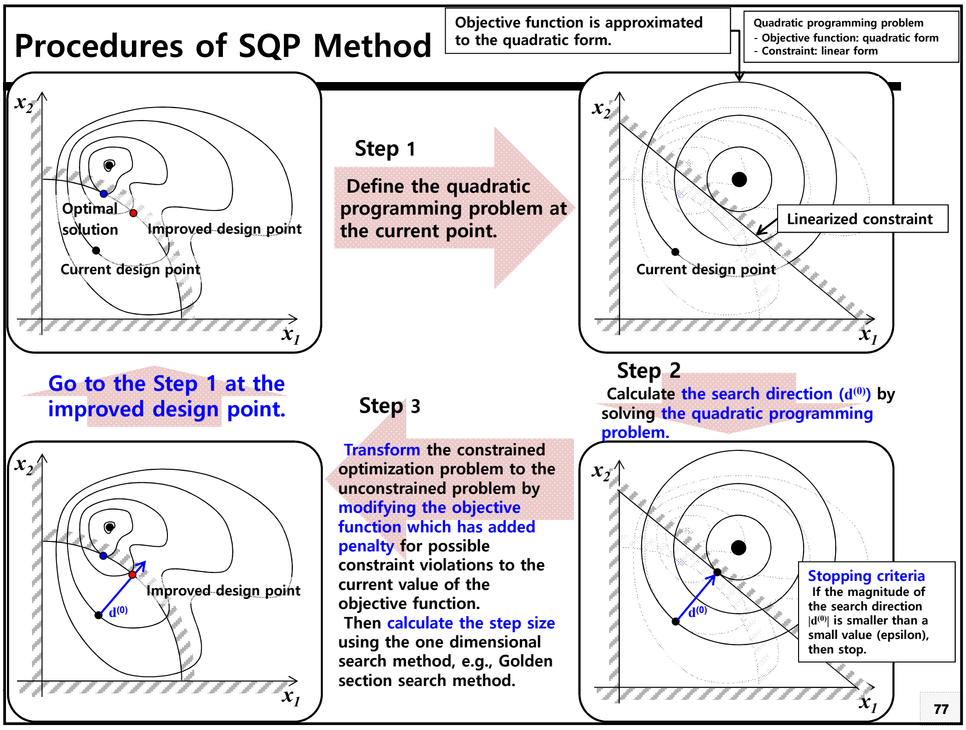
Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$: Quadratic objective function

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$: Linear equality constraints

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$: Linear inequality constraints

Procedures of SQP Method



Difference between Sequential Quadratic Programming (SQP) and CSD (Constrained Steepest Descent) Method

☑ Sequential Quadratic Programming (SQP)

- ① First, we define a quadratic programming problem for the objective function and constraints at the current design point, and find the search direction $\mathbf{d}^{(k)}$.
- ② We define the penalty function by adding a penalty for possible constraint violations to the current value of the objective function, and calculate the step size α_k to minimize the penalty function. For determination of the step size, one dimensional search method, e.g., Golden section search method can be used. And we determine the improved design point.
- ③ At the improved design point, we go to ①.
- The method is to find the optimal solution by solving the quadratic programming problem sequentially.

☑ CSD (Constrained Steepest Descent) method

- This method is a kind of the SQP method.
- When defining the quadratic programming problem, the Hessian matrix is assumed to be equal to the identity Matrix.
- This method uses the Pshenichny's penalty function.

Solution Procedure of SQP Using the Example - Determination of the Search Direction (1/5) [Iteration 1]

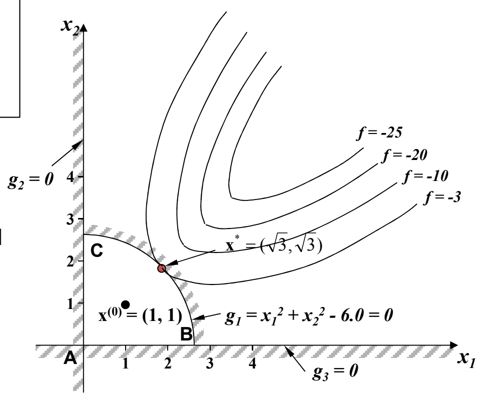
$$\textit{Minimize}\ \ f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to}\ \ g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$

Optimal solution: $\mathbf{x}^* = (\sqrt{3}, \sqrt{3}), f(\mathbf{x}^*) = -3$



Assume the starting point is $\mathbf{x}^{(0)} = (1,1)$.

(1) Iteration 1 (k = 0)

(i) Step 1: Evaluate the objective function and constraints at the current design point.

$f(1,1) = -1$

$g_1(1,1) = -\frac{2}{3} < 0$ ➡ **Constraint is satisfied.**

$g_2(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

$g_3(1,1) = -1 < 0$ ➡ **Constraint is satisfied.**

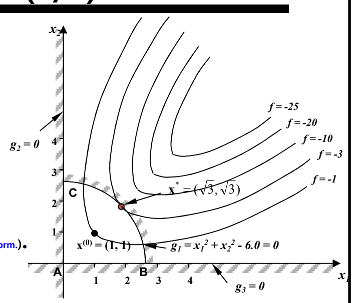
Solution Procedure of SQP Using the Example - Determination of the Search Direction (2/5)

$$\textit{Minimize}\ \ f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to}\ \ g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$



(1) Iteration 1 (k = 0) $\mathbf{x}^{(0)} = (1,1)$

(ii) Step 2: Define a QP problem (The objective function is approximated to the quadratic form.)

$$\textit{Minimize}\ \ f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong f(\mathbf{x}^{(0)}) + \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

$$\textit{Minimize}\ \ f(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \nabla f^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} + 0.5\Delta\mathbf{x}^{(0)T}\mathbf{H}\Delta\mathbf{x}^{(0)}$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla f^T = \left[\tfrac{\partial f}{\partial x_1} \ \ \tfrac{\partial f}{\partial x_2}\right], \mathbf{H} = \mathbf{I}$$

$$\textit{Minimize}\ \ f(\mathbf{x}^{(0)} + \mathbf{d}^{(0)}) - f(\mathbf{x}^{(0)}) \cong \left[2x_1 - 3x_2 \quad 2x_2 - 3x_1\right]_{\mathbf{x}^{(0)}} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\bar{f}(\mathbf{d}^{(0)}) \cong (2x_1^{(0)} - 3x_2^{(0)})d_1^{(0)} + (2x_2^{(0)} - 3x_1^{(0)})d_2^{(0)} + 0.5(d_1^{(0)2} + d_2^{(0)2})$$

$$\bar{f}(\mathbf{d}^{(0)}) \cong \underline{-d_1^{(0)} - d_2^{(0)}} + \underline{0.5(d_1^{(0)2} + d_2^{(0)2})}$$

Objective function is approximated to the first order term.

Objective function is approximated to the second order term.

Solution Procedure of SQP Using the Example - Determination of the Search Direction (3/5)

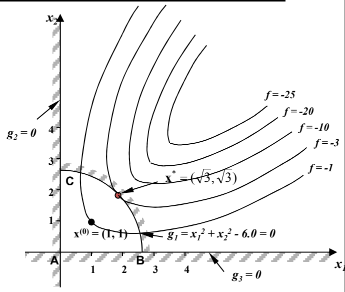
Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(1) Iteration 1 (k = 0) $\mathbf{x}^{(0)} = (1,1)$



Subject to $g_j(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \cong \underline{g_j(\mathbf{x}^{(0)}) + \nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)}} \le 0;\ j = 1\ to\ m$

The first-order (linear) Taylor series expansion of the constraints

$$\nabla g_j^T(\mathbf{x}^{(0)})\Delta\mathbf{x}^{(0)} \le -g_j(\mathbf{x}^{(0)});\ j = 1\ to\ m$$

$$\Delta\mathbf{x}^{(0)} = \mathbf{d}^{(0)}, \nabla g_j^T = \begin{bmatrix} \frac{\partial g_j}{\partial x_1} & \frac{\partial g_j}{\partial x_2} \end{bmatrix}$$

Subject to

$$\bar{g}_1(\mathbf{d}^{(0)}) : \begin{bmatrix} \frac{1}{3}x_1^{(0)} & \frac{1}{3}x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(\frac{1}{6}\left(x_1^{(0)}\right)^2 + \frac{1}{6}\left(x_2^{(0)}\right)^2 - 1.0 \right)$$

$$\bar{g}_2(\mathbf{d}^{(0)}) : \begin{bmatrix} -x_1^{(0)} & 0 \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_1^{(0)}\right)$$

$$\bar{g}_3(\mathbf{d}^{(0)}) : \begin{bmatrix} 0 & -x_2^{(0)} \end{bmatrix} \begin{bmatrix} d_1^{(0)} \\ d_2^{(0)} \end{bmatrix} \le -\left(-x_2^{(0)}\right)$$

Substitute $\mathbf{x}^{(0)} = (1,1)$

The linearized constraints

$$\boxed{\begin{aligned} &\tfrac{1}{3}d_1^{(0)} + \tfrac{1}{3}d_2^{(0)} \le \frac{2}{3} \\ &-d_1^{(0)} \le 1 \\ &-d_2^{(0)} \le 1 \end{aligned}}$$

Solution Procedure of SQP Using the Example - Determination of the Search Direction (4/5)

(iii) Step 3: Solve the QP problem to find the search direction ($\mathbf{d}^{(0)}$).

Constrained Optimal Design Problem (Original problem)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

①

$f(1,1) = -1, g_1(1,1) = -\frac{2}{3},$
$g_2(1,1) = -1, g_3(1,1) = -1$
$\nabla f = (-1,-1), \nabla g_1 = (\frac{1}{3}, \frac{1}{3}),$
$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$

Quadratic Programming Problem

Minimize $\bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $\frac{1}{3}d_1 + \frac{1}{3}d_2 \le \frac{2}{3}$

$-d_1 \le 1$

$-d_2 \le 1$

②

Lagrange function

$$\begin{aligned} L = &(-d_1 - d_2) + 0.5(d_1^2 + d_2^2) \\ &+ u_1[\tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2] \\ &+ u_2(-d_1 - 1 + s_2^2) \\ &+ u_3(-d_2 - 1 + s_3^2) \end{aligned}$$

③

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\begin{aligned} &\tfrac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0 \\ &\tfrac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0 \\ &\tfrac{\partial L}{\partial u_1} = \tfrac{1}{3}\left(d_1 + d_2 - 2\right) + s_1^2 = 0 \\ &\tfrac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0 \\ &\tfrac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0 \\ &\tfrac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1,2,3 \end{aligned}$$

The search direction is

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0),$

$\mathbf{s}^{(0)} = (s_1, s_2, s_3) = (0, 1.414, 1.414),$

$$\boxed{\mathbf{d}^{(0)} = (d_1, d_2) = (1,1)}$$

* The search direction also can be determined using the Simplex method.

Solution Procedure of SQP Using the Example - Determination of the Search Direction (5/5)

$\boxed{\mathbf{d}^{(0)} = (d_1, d_2) = (1,1)}$ ← The search direction is determined.

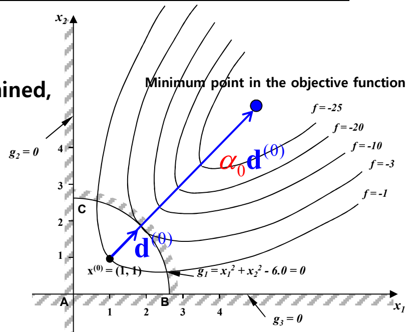
(iv) Step 4: After the search direction ($\mathbf{d}^{(0)}$) is determined, calculate the step size.

Calculate step size minimizing the value of the objective function along the search direction

$$\underbrace{\mathbf{x}^{(1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(0)}}_{\text{Current design point}} + \alpha_0 \underbrace{\mathbf{d}^{(0)}}_{\text{Search direction obtained from the QP problem}}$$

Find α_k : *Minimize* $f\left(\mathbf{x}^{(1)}\right) = f\left(\mathbf{x}^{(0)} + \alpha_0 \mathbf{d}^{(0)}\right) = f\left(\alpha_0\right)$

Find: α_0; Given: $\mathbf{x}^{(0)}$, $\mathbf{d}^{(0)}$



$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0),$
$\mathbf{d}^{(0)} = (d_1, d_2) = (1,1)$

The improved design point can be found along the search direction by minimizing the objective function. However, it may violate the original constraints.

Therefore, a penalty function should be constructed by adding the penalty for possible constraint violations to the current value of the objective function.

By property of the nature, the objective function is decreased when the constraints is violated.

Solution Procedure of SQP Using the Example - Definition of Penalty Function (Pshenichny's Descent Function) (1/2)

Penalty function (Pshenichny's descent function, $\Phi(\mathbf{x}^{(k)})$)

By adding a penalty for possible constraint violations to the current value of the objective function, the constrained optimization problem is transformed into the unconstrained optimization problem:

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

where,

k: iteration number how many times the QP problem is defined approximately

$f(\mathbf{x}^{(k)})$: current (k^{th} iteration) value of the objective function

$V(\mathbf{x}^{(k)})$ is either the maximum constraint violation among all the constraints or zero.

$V(\mathbf{x}^{(k)})$ is nonnegative. If all the constraints are satisfied, the value of the $V(\mathbf{x}^{(k)})$ is zero.

$$V(\mathbf{x}^{(k)}) = \max\{0; |h_1|, |h_2|, \cdots, |h_p|; g_1, g_2, \cdots, g_m\}$$

where,

h_p: value of the equality constraint function at the design point $\mathbf{x}^{(k)}$

g_p: value of the inequality constraint function at the design point $\mathbf{x}^{(k)}$

R_k is a positive number called the penalty parameter.

$$R_k = \max\left\{R_0, r_k\right\}$$

Initial value of R_k is specified by the user.

Summation of all the Lagrange multipliers

$$r_k = \sum_{i=1}^{p} \left|v_i^{(k)}\right| + \sum_{i=1}^{m} u_i^{(k)}$$

$v_i^{(k)}$: Lagrange multipliers for the equality constraints (free in sign)

$u_i^{(k)}$: Lagrange multiplier for the inequality constraints (nonnegative)

Solution Procedure of SQP Using the Example
- Definition of Penalty Function (Pshenichny's Descent Function) (2/2)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(v) Step 5: Calculate the penalty parameter R_k.
(In this example, the initial penalty parameter is assumed as R_0=10.)

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$ **and** $r_k = \sum_{i=1}^{p}\left|v_i^{(k)}\right| + \sum_{i=1}^{m}u_i^{(k)}$

$$r_0 = \sum_{i=1}^{m} u_i^{(0)} = 0$$

Since this problem does not have the equality constraints, we do not consider the v_i.

Therefore, $R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k)}),\ V(\mathbf{x}^{(k)}) = \max\{0, g_1(\mathbf{x}^{(k)}), g_2(\mathbf{x}^{(k)}), g_3(\mathbf{x}^{(k)})\},\ (k=0)$$

$g_1(\mathbf{x}^{(k)}) = \frac{1}{6}(x_1^{(k)})^2 + \frac{1}{6}(x_2^{(k)})^2 - 1.0$

$g_2(\mathbf{x}^{(k)}) = -x_1^{(k)}$

$g_3(\mathbf{x}^{(k)}) = -x_2^{(k)}$

Solution Procedure of SQP Using the Example
- Determination of the Step Size

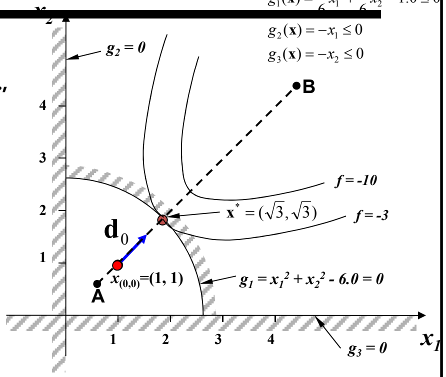
$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(vi) Step 6:
By using the one dimensional search method, e.g., Golden section search method, calculate the step size to minimize the penalty function along the search direction ($\mathbf{d}^{(0)}$), and determine the improved design point.



$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

$$= x_1^2 + x_2^2 - 3x_1x_2 + 10 \cdot V(\mathbf{x}^{(k)})$$

$$V(\mathbf{x}^{(k)}) = \max\{0, g_1(\mathbf{x}^{(k)}), g_2(\mathbf{x}^{(k)}), g_3(\mathbf{x}^{(k)})\},\ (k=0)$$

After the k-th search direction is found, one dimensional search for the step size is started.

$$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + \alpha_{(k,j)}\mathbf{d}^{(k)}$$

The iteration number k does not change during the one dimensional search for the step size.

$$\Phi\left(\mathbf{x}^{(k,j)}\right) = f\left(\mathbf{x}^{(k,j)}\right) + R_k \cdot V\left(\mathbf{x}^{(k,j)}\right),\ V(\mathbf{x}^{(k,j)}) = \max\{0, g_1(\mathbf{x}^{(k,j)}), g_2(\mathbf{x}^{(k,j)}), g_3(\mathbf{x}^{(k,j)})\}$$

The iteration number k does not change during the one dimensional search method.

After completing the one dimensional search, k is changed to k+1:

$\mathbf{x}^{(k,j)}$ **is changed to** $\mathbf{x}^{(k+1)}$.

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (1/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

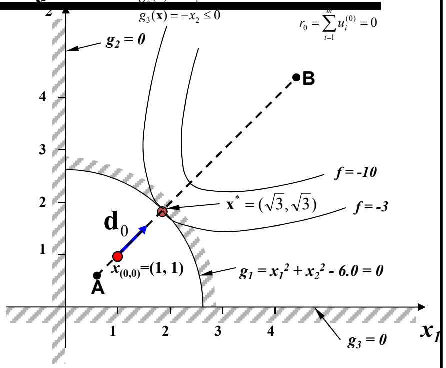
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i^{(0)} = 0$

(vi) Step 6:



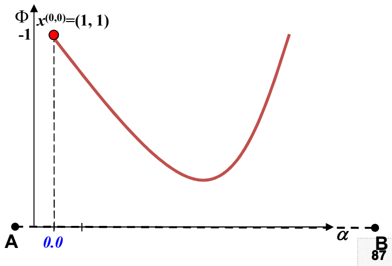
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0,\ j = 0$

When $\alpha_{(0,j)} = \alpha_{(0,0)} = 0.0$

$\mathbf{x}^{(0,j)} = \mathbf{x}^{(0,0)} = \mathbf{x}^{(0)} + \alpha_{(0,0)} \cdot d^{(0)} = (1,1) + 0 \cdot (1,1) = (1,1)$

$\Phi\left(\mathbf{x}^{(0,0)}\right) = f\left(\mathbf{x}^{(0,0)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,0)}\right) = -1 + 10 \times 0 = -1$

$where,\ V(\mathbf{x}^{(0,0)}) = \max\{0, g_1(\mathbf{x}^{(0,0)}), g_2(\mathbf{x}^{(0,0)}), g_3(\mathbf{x}^{(0,0)})\}$
$= \max\{0, -\frac{2}{3}, -1, -1\} = 0$



87

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (2/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

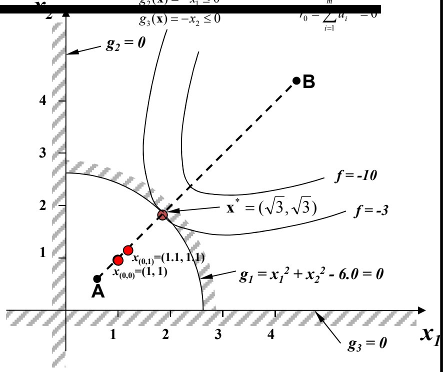
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i^{(0)} = 0$

(vi) Step 6:



Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0,\ j = 1$

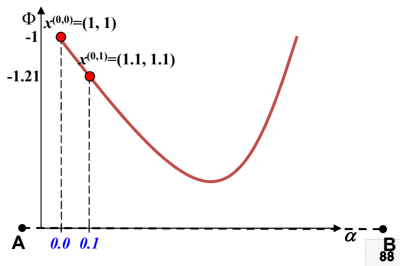
Assume $\alpha_{(0,1)} = 0.1$*,

$\mathbf{x}^{(0,1)} = \mathbf{x}^{(0)} + \alpha_{(0,1)} \cdot d^{(0)} = (1,1) + 0.1 \cdot (1,1) = (1.1, 1.1)$

$\Phi\left(\mathbf{x}^{(0,1)}\right) = f\left(\mathbf{x}^{(0,1)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,1)}\right) = -1.21 + 10 \times 0 = -1.21$

$where,\ V(\mathbf{x}^{(0,1)}) = \max\{0, g_1(\mathbf{x}^{(0,1)}), g_2(\mathbf{x}^{(0,1)}), g_3(\mathbf{x}^{(0,1)})\}$
$= \max\{0, -0.57, -1.1, -1.1\} = 0$

* The initial value of $\alpha_{(k,j)}$ (0.1) is given by the user. It can be given as another value, e.g., 0.5.



88

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (3/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

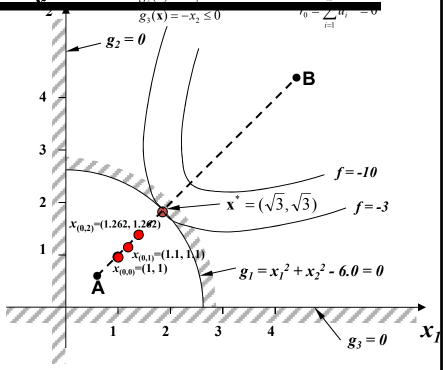
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i = 0$

(vi) Step 6:



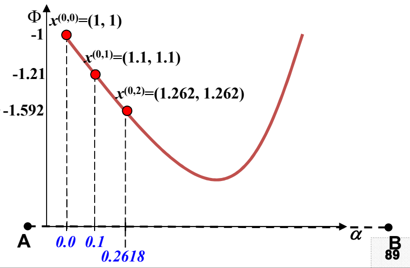
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \; j = 2$

When $\alpha_{(0,2)} = 0.1 + 1.618(0.1) = 0.2618$

$\mathbf{x}^{(0,2)} = \mathbf{x}^{(0)} + \alpha_{(0,2)} \cdot d^{(0)} = (1,1) + 0.262 \cdot (1,1) = (1.262, 1.262)$

$\Phi\left(\mathbf{x}^{(0,2)}\right) = f\left(\mathbf{x}^{(0,2)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,2)}\right) = -1.592 + 10 \times 0 = -1.592$

$where,\; V(\mathbf{x}^{(0,2)}) = \max\{0, g_1(\mathbf{x}^{(0,2)}), g_2(\mathbf{x}^{(0,2)}), g_3(\mathbf{x}^{(0,2)})\}$
$= \max\{0, -0.469, -1.262, -1.262\} = 0$



89

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (4/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

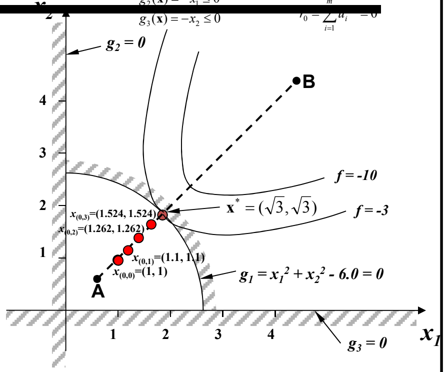
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i = 0$

(vi) Step 6:



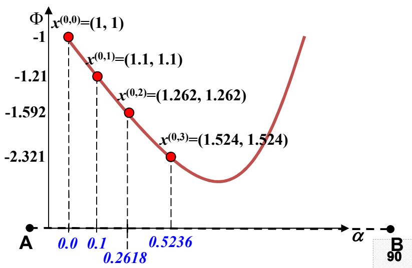
Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \; j = 3$

When $\alpha_{(0,3)} = 0.1 + 1.618(0.1) + 1.618^2(0.1) = 0.5236$

$\mathbf{x}^{(0,3)} = \mathbf{x}^{(0)} + \alpha_{(0,3)} \cdot d^{(0)} = (1,1) + 0.524 \cdot (1,1) = (1.524, 1.524)$

$\Phi\left(\mathbf{x}^{(0,3)}\right) = f\left(\mathbf{x}^{(0,3)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,3)}\right) = -2.321 + 10 \times 0 = -2.321$

$where,\; V(\mathbf{x}^{(0,3)}) = \max\{0, g_1(\mathbf{x}^{(0,3)}), g_2(\mathbf{x}^{(0,3)}), g_3(\mathbf{x}^{(0,3)})\}$
$= \max\{0, -0.226, -1.524, -1.524\} = 0$



90

Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (5/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

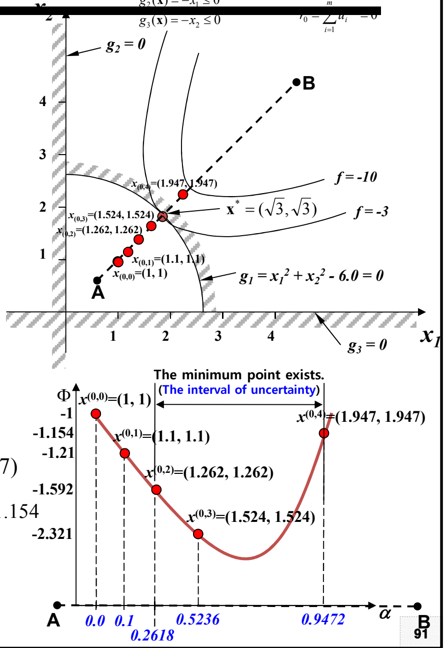
$g_3(\mathbf{x}) = -x_2 \le 0$

$R_0 = \max\{R_0, r_0\} = \max\{10, 0\} = 10$

$\mathbf{u}^{(0)} = (u_1, u_2, u_3) = (0,0,0)$

$r_0 = \sum_{i=1}^{m} u_i = 0$

(vi) Step 6:



Search direction: $\mathbf{d}_0 = (1,1), \quad k = 0, \ j = 4$

When $\alpha_{(0,4)} = 0.1 + 1.618(0.1) + 1.618^2(0.1) + 1.618^3(0.1) = 0.9472$

$$\mathbf{x}^{(0,4)} = \mathbf{x}^{(0)} + \alpha_{(0,4)} \cdot d^{(0)} = (1,1) + 0.947\cdot(1,1) = (1.947, 1.947)$$

$$\Phi\left(\mathbf{x}^{(0,4)}\right) = f\left(\mathbf{x}^{(0,4)}\right) + R_0 \cdot V\left(\mathbf{x}^{(0,4)}\right) = -3.792 + 10\times 0.2638 = -1.154$$

$$where,\ V(\mathbf{x}^{(0,4)}) = \max\{0, g_1(\mathbf{x}^{(0,4)}), g_2(\mathbf{x}^{(0,4)}), g_3(\mathbf{x}^{(0,4)})\} = \max\{0, 0.2638, -1.947, -1.947\} = 0.2638$$

91

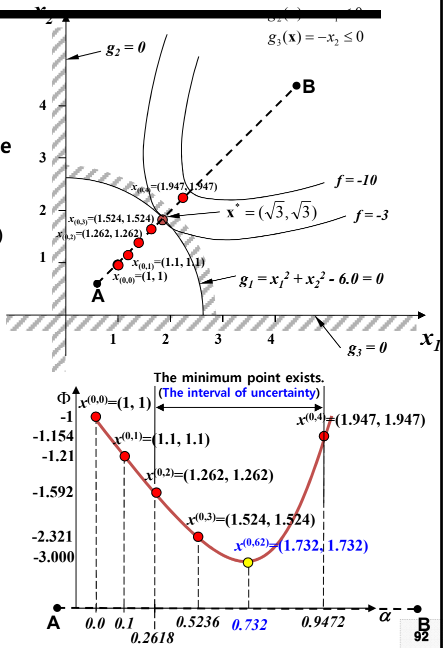
Solution Procedure of SQP Using the Example - Determination of the Step Size Using the Golden Section Search Method (6/6)

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

(vi) Step 6:



The value of the $\alpha_0 = 0.732$ is found at which the penalty function is minimized in the interval between $\mathbf{x}^{(0,2)}$ and $\mathbf{x}^{(0,4)}$.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha_0 \cdot \mathbf{d}^{(0)} = (1,1) + 0.732\cdot(1,1) = (1.732, 1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f\left(1.732, 1.732\right) = -3$$

92

Solution Procedure of SQP Using the Example
- Determination of the Search Direction (1/3) [Iteration 2]

(2) Iteration 2 (k = 1)

(i) Step 1: Calculate maximum constraint violation of all the constraints.

From the previous stage,

$$\mathbf{x}^{(1)} = (1.732, 1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f\left(1.732, 1.732\right) = -2.999824$$

$$g_1\left(\mathbf{x}^{(1)}\right) = g_1\left(1.732, 1.732\right) = -5.866\times10^{-5}$$ ➡ **Constraint is satisfied.**

$$g_2\left(\mathbf{x}^{(1)}\right) = -1.732$$ ➡ **Constraint is satisfied.**

$$g_3\left(\mathbf{x}^{(1)}\right) = -1.732$$ ➡ **Constraint is satisfied.**

$$V_1 = V\left(\mathbf{x}^{(1)}\right) = \max\{0; -5.866\times10^{-5}, -1.732, -1.732\} = 0$$

And,

$$\nabla f(\mathbf{x}^{(1)}) = (2x_1 - 3x_2, 2x_2 - 3x_1) = (-1.732, -1.732)$$

$$\nabla g_1(\mathbf{x}^{(1)}) = (\tfrac{1}{3}x_1, \tfrac{1}{3}x_2) = (0.577, 0.577), \nabla g_2 = (-1, 0), \nabla g_3 = (0, -1)$$

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

Solution Procedure of SQP Using the Example
- Determination of the Search Direction (2/3)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

(ii&iii) Step 2&3: Define and solve the QP problem to determine the search direction ($\mathbf{d}^{(1)}$).

Constrained Optimal Design Problem (Original problem)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$

①

Quadratic Programming Problem

Minimize $\bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $0.577d_1 + 0.577d_2 \le 5.866\times10^{-5}$

$-d_1 \le 1.732$

$-d_2 \le 1.732$

where,

$d_1 = x_1 - 1.732,$

$d_2 = x_2 - 1.732$

$f(1.732, 1.732) = -3, \nabla f = (-1.732, -1.732)$
$g_1(1.732, 1.732) = -5.866\times10^{-5}, \nabla g_1 = (0.577, 0.577)$
$g_2(1.732, 1.732) = -1.732, \nabla g_2 = (-1, 0)$
$g_3(1.732, 1.732) = -1.732, \nabla g_3 = (0, -1)$

②

Lagrange function

$$L = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$$
$$+ u_1[0.577(d_1 + d_2) - 5.866\times10^{-5} + s_1^2]$$
$$+ u_2(-d_1 - 1.732 + s_2^2)$$
$$+ u_3(-d_2 - 1.732 + s_3^2)$$

③

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = 0.577\left(d_1 + d_2\right) - 5.866\times10^{-6} + s_1^2 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2^2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3^2 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1, 2, 3$$

The search direction is

$\mathbf{d}^{(1)} = (d_1, d_2)$
$= (5.081\times10^{-5}, 5.081\times10^{-5})$

$\mathbf{u}^{(1)} = (u_1, u_2, u_3)$
$= (3, 0, 0)$

$\mathbf{s}^{(1)} = (s_1, s_2, s_3)$
$= (0,\ 1.316,\ 1.316)$

* The search direction also can be determined using the Simplex method.

Solution Procedure of SQP Using the Example - Determination of the Search Direction (3/3)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

(iv) Step 4: Check for the following stopping criteria.

$$\mathbf{d}^{(1)} = (d_1, d_2) = (5.081\times10^{-5},\ \ 5.081\times10^{-5})$$

$$\left\|\mathbf{d}^{(1)}\right\| = \sqrt{\left(5.081\times10^{-5}\right)^2 + \left(5.081\times10^{-5}\right)^2} = 7.186\times10^{-5} < \varepsilon_2 (= 0.001)$$

The stopping criteria is satisfied.

(v) Step 5: Stop the iteration.

The candidate minimum solution: $\mathbf{x}^* = (\sqrt{3}, \sqrt{3}), f(\mathbf{x}^*) = -3$

where the Lagrange multiplier are:

$$\mathbf{u}^* = (3, 0, 0), \mathbf{s}^* = (0,\ 1.316,\ 1.316)$$

Summary of Sequential Quadratic Programming (SQP)

Optimization Problem

Minimize $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$

Subject to $h_i(\mathbf{x}) = 0,\ \ i = 1, \ldots, p$ **Equality constraints**

$g_i(\mathbf{x}) = 0,\ \ i = 1, \ldots, m$ **Inequality constraints**

Pshenichny's descent function: the penalty function is constructed by adding a penalty for possible constraint violations to the current value of the objective function

$$\Phi(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}) + R_k \cdot V(\mathbf{x}^{(k)})$$

(k is the iteration number how many times the QP problem is defined.)

$V(\mathbf{x}^{(k)})$ is either the maximum constraint violation of all the constraints or zero.

$V(\mathbf{x}^{(k)})$ is nonnegative. If all the constraints are satisfied, the value of the $V(\mathbf{x}^{(k)})$ is zero.

$$V(\mathbf{x}^{(k)}) = \max\{0; |h_1|, |h_2|, \cdots, |h_p|; g_1, g_2, \cdots, g_m\}$$

R_k is a positive number called the penalty parameter (initially specified by the user).

$$R_k = \max\left\{R_0,\ \underline{r_k}\left(= \sum_{i=1}^{p}\left|v_i^{(k)}\right| + \sum_{i=1}^{m} u_i^{(k)}\right)\right\}$$

: Summation of all the Lagrange multipliers

The improved design point is determined as follows:

$$\underbrace{\mathbf{x}^{(k+1)}}_{\text{Improved design point}} = \underbrace{\mathbf{x}^{(k)}}_{\text{Current design point}} + \alpha_k \cdot \mathbf{d}^{(k)}$$

Search direction obtained from the QP problem ($\mathbf{d}^{(k)}$)

Step size calculated by one dimensional search method (α_k)
(ex. Golden section search method)

6.5 Determine the Search Direction of the Quadratic Programming Problem by Using the Simplex Method

Formulation of the Quadratic Programming Problem (1/5)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Solve the QP problem to determine the search direction ($\mathbf{d}^{(0)}$).

Constrained Optimal Design Problem (Original problem)

$$\textit{Minimize} \quad f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$\textit{Subject to} \quad g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$$

$$g_2(\mathbf{x}) = -x_1 \le 0$$

$$g_3(\mathbf{x}) = -x_2 \le 0$$

①

$f(1,1) = -1, g_1(1,1) = -\frac{2}{3}$,
$g_2(1,1) = -1, g_3(1,1) = -1$
$\nabla f = (-1,-1), \nabla g_1 = (\frac{1}{3}, \frac{1}{3})$,
$\nabla g_2 = (-1,0), \nabla g_3 = (0,-1)$

Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$

$$-d_1 \le 1$$

$$-d_2 \le 1$$

where
$d_1 = x_1 - 1,\ d_2 = x_2 - 1$

②

Lagrange function

$$L = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$
$$+ u_1[\tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2]$$
$$+ u_2(-d_1 - 1 + s_2^2)$$
$$+ u_3(-d_2 - 1 + s_3^2)$$

③

Kuhn-Tucker necessary condition

$$\tfrac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$

$$\tfrac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$

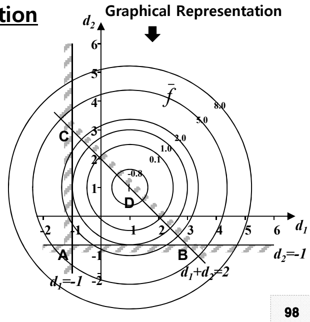
$$\tfrac{\partial L}{\partial u_1} = \tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2 = 0$$

$$\tfrac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0$$

$$\tfrac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0$$

$$\tfrac{\partial L}{\partial s_i} = u_i s_i = 0,\ u_i \ge 0,\ i = 1,2,3$$

Graphical Representation



Formulation of the Quadratic Programming Problem (2/5)

Quadratic Programming Problem

Minimize $\bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $\frac{1}{3}d_1 + \frac{1}{3}d_2 \le \frac{2}{3}$

$-d_1 \le 1$

$-d_2 \le 1$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}(d_1 + d_2 - 2) + s_1^2 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2^2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3^2 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u_i \ge 0,\ i = 1,2,3 \Rightarrow u_i s_i^2 = 0,\ u_i \ge 0,\ i = 1,2,3$$

Multiply the both side of equations by s_i

Kuhn-Tucker necessary condition

Replace s_i^2 with s_i': $s_i^2 = s_i' \ge 0$

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}(d_1 + d_2 - 2) + s_1' = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2' = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3' = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i' = 0$$
$$u_i, s_i' \ge 0,\ i = 1,2,3$$

Kuhn-Tucker necessary condition

Represent s_i' to s_i for the convenience

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}(d_1 + d_2 - 2) + s_1 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0$$
$$u_i, s_i \ge 0,\ i = 1,2,3$$

99

Formulation of the Quadratic Programming Problem (3/5)

Quadratic Programming Problem

Minimize $\bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $\frac{1}{3}d_1 + \frac{1}{3}d_2 \le \frac{2}{3}$

$-d_1 \le 1$

$-d_2 \le 1$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1 + d_1 + \tfrac{1}{3}u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1 + d_2 + \tfrac{1}{3}u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = \tfrac{1}{3}(d_1 + d_2 - 2) + s_1 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1 + s_2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1 + s_3 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0$$
$$u_i, s_i \ge 0,\ i = 1,2,3$$

How can we express the Kuhn-Tucker necessary condition in matrix form ($\mathbf{d}$, $\mathbf{c}$, $\mathbf{H}$, $\mathbf{A}$, $\mathbf{b}$)?

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times2)}\mathbf{d}_{(2\times1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times2)}\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}$

Subject to $\mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)} \le \mathbf{b}_{(3\times1)}$

Assume that $\mathbf{H}_{(2\times2)}$ is equal to $\mathbf{I}_{(2\times2)}$.

where, $\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$

$\mathbf{A}_{(2\times3)} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{1}{3} & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$

100

Formulation of the Quadratic Programming Problem (4/5)

$$\mathbf{d}_{(2\times1)}=\begin{bmatrix}d_1\\d_2\end{bmatrix},\mathbf{c}_{(2\times1)}=\begin{bmatrix}-1\\-1\end{bmatrix},\mathbf{H}_{(2\times2)}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\mathbf{A}_{(2\times3)}=\begin{bmatrix}\frac{1}{3}&-1&0\\\frac{1}{3}&0&-1\end{bmatrix},\mathbf{b}_{(3\times1)}=\begin{bmatrix}\frac{2}{3}\\1\\1\end{bmatrix}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1}=-1+d_1+\tfrac{1}{3}u_1-u_2=0$$
$$\frac{\partial L}{\partial d_2}=-1+d_2+\tfrac{1}{3}u_1-u_3=0$$
$$\frac{\partial L}{\partial u_1}=\tfrac{1}{3}\left(d_1+d_2-2\right)+s_1=0$$
$$\frac{\partial L}{\partial u_2}=-d_1-1+s_2=0$$
$$\frac{\partial L}{\partial u_3}=-d_2-1+s_3=0$$
$$\frac{\partial L}{\partial s_i}=u_is_i=0$$
$$u_i,s_i\ge0,\ i=1,2,3$$

$$\begin{bmatrix}-1+d_1+\frac{1}{3}u_1-u_2\\-1+d_2+\frac{1}{3}u_1-u_3\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}+\begin{bmatrix}\frac{1}{3}&-1&0\\\frac{1}{3}&0&-1\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$
$$=\mathbf{c}_{(2\times1)}+\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}+\mathbf{A}_{(2\times3)}\mathbf{u}_{(3\times1)}=\mathbf{0}$$

$$\begin{bmatrix}\frac{1}{3}(d_1+d_2-2)+s_1\\-d_1-1+s_2\\-d_2-1+s_3\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\-1&0\\0&-1\end{bmatrix}\begin{bmatrix}d_1\\d_2\end{bmatrix}+\begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}-\begin{bmatrix}\frac{2}{3}\\1\\1\end{bmatrix}$$
$$=\mathbf{A}^T_{(3\times2)}\mathbf{d}_{(2\times1)}+\mathbf{s}_{(3\times1)}-\mathbf{b}_{(3\times1)}=\mathbf{0}$$

$$\downarrow$$

Matrix form

Formulation of the Quadratic Programming Problem (5/5)

In order to use the Simplex method, we decompose into two variables, because the design variables $\mathbf{d}_{(2\times1)}$ are free in sign: $\mathbf{d}_{(2\times1)}=\mathbf{d}^+_{(2\times1)}-\mathbf{d}^-_{(2\times1)}$

Kuhn-Tucker necessary condition in Matrix form: $\nabla L(\mathbf{d}^+,\mathbf{d}^-,\mathbf{u},\mathbf{s})=\mathbf{0}$

$$\underbrace{\begin{bmatrix}\mathbf{H}_{(2\times2)}&-\mathbf{H}_{(2\times2)}&\mathbf{A}_{(2\times3)}&\mathbf{0}_{(2\times3)}\\\mathbf{A}^T_{(3\times2)}&-\mathbf{A}^T_{(3\times2)}&\mathbf{0}_{(3\times3)}&\mathbf{I}_{(3\times3)}\end{bmatrix}}_{=\mathbf{B}_{(5\times10)}}\underbrace{\begin{bmatrix}\mathbf{d}^+_{(2\times1)}\\\mathbf{d}^-_{(2\times1)}\\\mathbf{u}_{(3\times1)}\\\mathbf{s}_{(3\times1)}\end{bmatrix}}_{=\mathbf{X}_{(10\times1)}}=\underbrace{\begin{bmatrix}-\mathbf{c}_{(2\times1)}\\\mathbf{b}_{(3\times1)}\end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

where, $\mathbf{d}^+_{(2\times1)}=\begin{bmatrix}d_1^+\\d_2^+\end{bmatrix},\mathbf{d}^-_{(2\times1)}=\begin{bmatrix}d_1^-\\d_2^-\end{bmatrix},\mathbf{c}_{(2\times1)}=\begin{bmatrix}-1\\-1\end{bmatrix},\mathbf{H}_{(2\times2)}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\mathbf{A}_{(2\times3)}=\begin{bmatrix}\frac{1}{3}&-1&0\\\frac{1}{3}&0&-1\end{bmatrix},\mathbf{b}_{(3\times1)}=\begin{bmatrix}\frac{2}{3}\\1\\1\end{bmatrix}$

$$\mathbf{B}_{(5\times10)}=\left[\begin{array}{cc|cc|ccc|ccc}1&0&-1&0&\frac{1}{3}&-1&0&0&0&0\\0&1&0&-1&\frac{1}{3}&0&-1&0&0&0\\\hline\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}&-\frac{1}{3}&0&0&0&1&0&0\\-1&0&1&0&0&0&0&0&1&0\\0&-1&0&1&0&0&0&0&0&1\end{array}\right]$$

$$\mathbf{X}^T_{(1\times10)}=\begin{bmatrix}d_1^+&d_2^+&d_1^-&d_2^-&u_1&u_2&u_3&s_1&s_2&s_3\end{bmatrix},\mathbf{D}^T_{(1\times5)}=\begin{bmatrix}1&1&\frac{2}{3}&1&1\end{bmatrix}$$

Formulation of the Quadratic Programming Problem to Find the Search Direction by Using the Simplex Method

Kuhn-Tucker necessary condition (matrix form)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} = \mathbf{D}_{(5\times1)}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \frac{1}{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{3} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}}_{\text{We want to find.}} = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

➡ This problem is to find $\mathbf{X}$ in the linear programming problem only having the equality constraints.

➡ $u_i s_i = 0;\ i = 1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (1/6)

Simplex method to solve the quadratic programming problem

1. The problem to solve the Kuhn-Tucker necessary condition is the same with the problem having only the equality constraints (Linear Programming problem).
2. To solve the linear indeterminate equation, we introduce the artificial variables, define the artificial objective function, and then determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} + \underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}} = \mathbf{D}_{(5\times1)}$$

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{5} Y_i = \sum_{i=1}^{5} D_i - \sum_{j=1}^{10}\sum_{i=1}^{5} B_{ij}X_j = w_0 + \sum_{j=1}^{10} C_j X_j$$

where $C_j = -\sum_{i=1}^{5} B_{ij}$: Add the elements of the jth column of the matrix B and change the its sign (initial relative objective coefficient).

$$w_0 = \sum_{i=1}^{5} D_i = 1 + 1 + \tfrac{2}{3} + 1 + 1 = \tfrac{14}{3}$$: Initial value of the artificial objective function (summation of the all elements of the matrix D)

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following nonlinear equation.

$u_i s_i = 0;\ i = 1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (2/6)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)}+\underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}}=\mathbf{D}_{(5\times1)} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & \frac{1}{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{3} & 0 & -1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+(=X_1) \\ d_2^+(=X_2) \\ d_1^-(=X_3) \\ d_2^-(=X_4) \\ u_1(=X_5) \\ u_2(=X_6) \\ u_3(=X_7) \\ s_1(=X_8) \\ s_2(=X_9) \\ s_3(=X_{10}) \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

Define the artificial objective function for using the Simplex method:

Sum all the rows (1~5): $\frac{1}{3}X_1+\frac{1}{3}X_2-\frac{1}{3}X_3-\frac{1}{3}X_4+\frac{2}{3}X_5-X_6-X_7+X_8+X_9+X_{10}+\underbrace{Y_1+Y_2+Y_3+Y_4+Y_5}_{w}=\frac{14}{3}$

Express the artificial function as w and rearrange: $-\frac{1}{3}X_1-\frac{1}{3}X_2+\frac{1}{3}X_3+\frac{1}{3}X_4-\frac{2}{3}X_5+X_6+X_7-X_8-X_9-X_{10}=w-\frac{14}{3}$

1

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
Y3	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	2/3
Y4	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	-1/3	-1/3	1/3	1/3	-2/3	1	1	-1	-1	-1	0	0	0	0	0	w-14/3	-

Artificial objective function

Sum all the elements of the row and change the its sign (ex. Row 1: -(1+0+1/3-1+0)=-1/3)

105

Determine the Search Direction by Using the Simplex Method - Iteration 1 (3/6)

2

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	-
Y4	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	1
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	0	0	0	0	-2/3	1	1	0	-1	-1	0	0	1	0	0	w-4	-

3

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	-
X9	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	1
A. Obj.	-1	0	1	0	-2/3	1	1	0	0	-1	0	0	1	1	0	w-3	-

4

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	1
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	1/3	1/3	-1/3	-1/3	0	0	0	1	0	0	0	0	1	0	0	2/3	2
X9	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1	-
X10	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	-1	-1	1	1	-2/3	1	1	0	0	0	0	0	1	1	1	w-2	-

106

Determine the Search Direction by Using the Simplex Method - Iteration 1 (4/6)

5

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
Y2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	1
X8	0	1/3	0	-1/3	-1/9	1/3	0	1	0	0	-1/3	0	1	0	0	1/3	1
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1	-
A. Obj.	0	-1	0	1	-1/3	0	1	0	0	0	1	0	1	1	1	w-1	-

6

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
X2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	0	0	0	0	-2/9	1/3	1/3	1	0	0	-1/3	-1/3	1	0	0	0	-
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	0	0	0	1/3	0	-1	0	0	1	0	1	0	0	1	2	-
A. Obj.	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	w-0	-

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (5/6)

6

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X1	1	0	-1	0	1/3	-1	0	0	0	0	1	0	0	0	0	1	-
X2	0	1	0	-1	1/3	0	-1	0	0	0	0	1	0	0	0	1	-
X8	0	0	0	0	-2/9	1/3	1/3	1	0	0	-1/3	-1/3	1	0	0	0	-
X9	0	0	0	0	1/3	-1	0	0	1	0	1	0	0	1	0	2	-
X10	0	0	0	0	1/3	0	-1	0	0	1	0	1	0	0	1	2	-
A. Obj.	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	w-0	-

Since the value of the objective function becomes zero, the initial basic feasible solution is obtained.

$$\mathbf{X}^T_{(1\times10)} = \begin{bmatrix} d_1^+ & d_2^+ & d_1^- & d_2^- & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}$$

Basic solution:

$$X_1 = 1,\quad X_2 = 1,\quad X_8 = 0,\quad X_9 = 2,\quad X_{10} = 2$$

Nonbasic solution:

$$X_3 = X_4 = X_5 = X_6 = X_7 = 0$$

This solution satisfies the nonlinear indeterminate equation ($X_i X_{3+i} = 0; i = 5\ to\ 7, X_i \geq 0; i = 1\ to\ 10$)

So, the optimal solution is $d_1 = d_2 = 1, u_1 = u_2 = u_3 = 0, s_1 = 0, s_2 = s_3 = 2$.

➨ Caution: In the Pivot step, if the smallest (i.e., the most negative) coefficient of the artificial objective function or the smallest positive ratio "b_i/a_i" appears more than one time, the initial basic feasible solution can be changed depending on the selection of the pivot element in the pivot procedure.

➨ We have to check the solution until the nonlinear indeterminate equation ($u_i \times s_i = 0$) are satisfied.

Determine the Search Direction by Using the Simplex Method - Iteration 1 (6/6)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

The optimal solution for this problem is $d_1 = d_2 = 1, u_1 = u_2 = u_3 = 0, s_1 = 0, s_2 = s_3 = 2.$

Why are the values of u_1 and s_1 are zero at the same time?

Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = (-d_1 - d_2) + 0.5(d_1^2 + d_2^2)$$

$$\textit{Subject to} \quad \tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 \le \tfrac{2}{3}$$

$$-d_1 \le 1$$

$$-d_2 \le 1$$

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \underline{\sum_{i=1}^{3} u_i (g_i(\mathbf{x}) + s_i')}$$

Zero at optimum

This example is graphically represented in the right side.

$s_1 = 0 \longrightarrow$

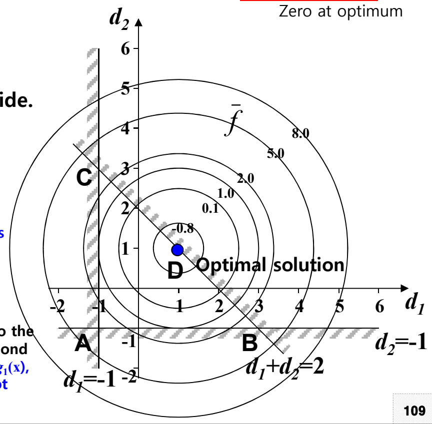
However, fortunately, the optimal solution is on the linearized inequality constraint ($g_1(\mathbf{x})$, $d_1+d_2=2$).

$u_2 = u_3 = 0 \longrightarrow$

The optimal solution is in the region satisfying the inequality constraints of $d_1 = -1 > 0$, $d_2 = -1 > 0$ which are inactive.

$u_1 = 0 \longrightarrow$

The optimal solution is on the inequality constraint ($g_1(\mathbf{x})$) and is equal to the optimal solution of the objective function to be approximated to the second order. Therefore, although we do not consider the inequality constraint $g_1(\mathbf{x})$, the optimal solution of QP problem is not changed. That is, $g_1(\mathbf{x})$ does not affect the optimal solution of this problem.



Determination of the Step Size Using the Golden Section Search Method

$f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

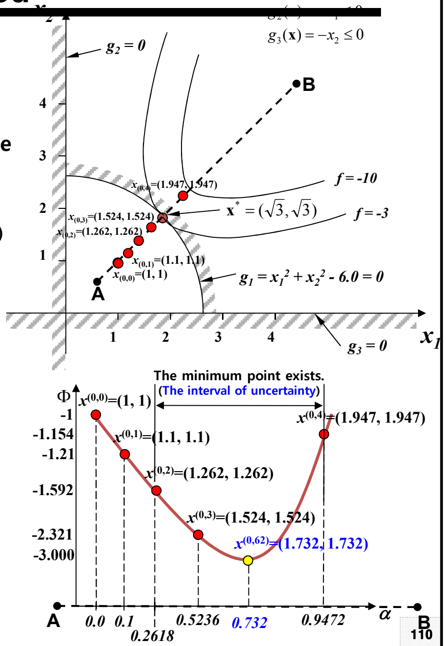
$g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_3(\mathbf{x}) = -x_2 \le 0$

The value of the $\alpha_0 = 0.732$ is found at which the penalty function is minimized in the interval between $\mathbf{x}^{(0,2)}$ and $\mathbf{x}^{(0,4)}$.

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha_0 \cdot \mathbf{d}^{(0)} = (1,1) + 0.732 \cdot (1,1) = (1.732, 1.732)$$

$$f\left(\mathbf{x}^{(1)}\right) = f\left(1.732, 1.732\right) = -3$$



Determine the Search Direction by Using the Simplex Method - Iteration 2 (1/10)

Quadratic programming problem
- Objective function: quadratic form
- Constraint: linear form

Solve the QP problem to determine the search direction ($\mathbf{d}^{(0)}$).

Constrained Optimal Design Problem (Original problem)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$

Subject to $g_1(\mathbf{x}) = \frac{1}{6}x_1^2 + \frac{1}{6}x_2^2 - 1.0 \le 0$

$g_2(\mathbf{x}) = -x_1 \le 0$

$g_3(\mathbf{x}) = -x_1 \le 0$

①

$f(1.732, 1.732) = -3, \nabla f = (-1.732, -1.732)$

$g_1(1.732, 1.732) = -5.866\times10^{-5}, \nabla g_1 = (0.577, 0.577)$

$g_2(1.732, 1.732) = -1.732, \nabla g_2 = (-1, 0)$

$g_3(1.732, 1.732) = -1.732, \nabla g_3 = (0, -1)$

Quadratic Programming Problem

Minimize $\bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $0.577d_1 + 0.577d_2 \le 5.866\times10^{-5}$

$-d_1 \le 1.732$

$-d_2 \le 1.732$

where, $d_1 = x_1 - 1.732$, $d_2 = x_2 - 1.732$

②

Lagrange function

$$L = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2) + u_1[0.577(d_1 + d_2) - 5.866\times10^{-5} + s_1^2] + u_2(-d_1 - 1.732 + s_2^2) + u_3(-d_2 - 1.732 + s_3^2)$$

③

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$

$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$

$\frac{\partial L}{\partial u_1} = 0.577(d_1 + d_2) - 5.866\times10^{-6} + s_1^2 = 0$

$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2^2 = 0$

$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3^2 = 0$

$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u \ge 0,\ i = 1,2,3$

1. Multiply the both side by s_i and replace s_i^2 with s_i'.
2. Represent s_i' to s_i for the convenience.

111

Determine the Search Direction by Using the Simplex Method - Iteration 2 (2/10)

Quadratic Programming Problem

Minimize $\bar{f} = (-1.732d_1 - 1.732d_2) + 0.5(d_1^2 + d_2^2)$

Subject to $0.577d_1 + 0.577d_2 \le 0$

$-d_1 \le 1.732$

$-d_2 \le 1.732$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times2)}\mathbf{d}_{(2\times1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times2)}\mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)}$

Subject to $\mathbf{A}^T{}_{(3\times2)}\mathbf{d}_{(2\times1)} \le \mathbf{b}_{(3\times1)}$

Kuhn-Tucker necessary condition

$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$

$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$

$\frac{\partial L}{\partial u_1} = 0.577(d_1 + d_2) - 5.866\times10^{-6} + s_1 = 0$

$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2 = 0$

$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3 = 0$

$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u_i, s_i \ge 0,\ i = 1,2,3$

Assume that $\mathbf{H}_{(2\times2)}$ is equal to $\mathbf{I}_{(2\times2)}$.

where, $\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$

$\mathbf{A}_{(2\times3)} = \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$

Topics in Ship Design Automation, Fall 2016, Myung-Il Roh

sydlab SEOUL NAT'L UNIV. 112

Determine the Search Direction by Using the Simplex Method - Iteration 2 (3/10)

$$\mathbf{d}_{(2\times1)} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_{(2\times3)} = \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

Kuhn-Tucker necessary condition

$$\frac{\partial L}{\partial d_1} = -1.732 + d_1 + 0.577u_1 - u_2 = 0$$
$$\frac{\partial L}{\partial d_2} = -1.732 + d_2 + 0.577u_1 - u_3 = 0$$
$$\frac{\partial L}{\partial u_1} = 0.577(d_1 + d_2) - 5.866\times10^{-6} + s_1 = 0$$
$$\frac{\partial L}{\partial u_2} = -d_1 - 1.732 + s_2 = 0$$
$$\frac{\partial L}{\partial u_3} = -d_2 - 1.732 + s_3 = 0$$
$$\frac{\partial L}{\partial s_i} = u_i s_i = 0,\ u_i, s_i \ge 0,\ i = 1,2,3$$

$$\begin{bmatrix} -1.732 + d_1 + 0.577u_1 - u_2 \\ -1.732 + d_2 + 0.577u_1 - u_3 \end{bmatrix} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
$$= \mathbf{c}_{(2\times1)} + \mathbf{H}_{(2\times2)}\mathbf{d}_{(2\times1)} + \mathbf{A}_{(2\times3)}\mathbf{u}_{(3\times1)} = \mathbf{0}$$

$$\begin{bmatrix} -0.577d_1 - 0.577d_2 + s_1 \\ -d_1 - 1.732 + s_2 \\ -d_2 - 1.732 + s_3 \end{bmatrix} = \begin{bmatrix} 0.577 & 0.577 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$
$$= \mathbf{A}^T_{(3\times2)}\mathbf{d}_{(2\times1)} + \mathbf{s}_{(3\times1)} - \mathbf{b}_{(3\times1)} = \mathbf{0}$$

Since the design variables $\mathbf{d}_{(n\times1)}$ are free in sign, we may decompose them as follows for using the Simplex method.

$$\mathbf{d}_{(2\times1)} = \mathbf{d}^+_{(2\times1)} - \mathbf{d}^-_{(2\times1)}$$

Matrix form

$$\underbrace{\begin{bmatrix} \mathbf{H}_{(2\times2)} & -\mathbf{H}_{(2\times2)} & \mathbf{A}_{(2\times3)} & \mathbf{0}_{(2\times3)} \\ \mathbf{A}^T_{(3\times2)} & -\mathbf{A}^T_{(3\times2)} & \mathbf{0}_{(3\times3)} & \mathbf{I}_{(3\times3)} \end{bmatrix}}_{=\mathbf{B}_{(5\times10)}} \underbrace{\begin{bmatrix} \mathbf{d}^+_{(2\times1)} \\ \mathbf{d}^-_{(2\times1)} \\ \mathbf{u}_{(3\times1)} \\ \mathbf{s}_{(3\times1)} \end{bmatrix}}_{=\mathbf{X}_{(10\times1)}} = \underbrace{\begin{bmatrix} -\mathbf{c}_{(2\times1)} \\ \mathbf{b}_{(3\times1)} \end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

Determine the Search Direction by Using the Simplex Method - Iteration 2 (4/10)

Kuhn-Tucker necessary condition: $\nabla L(\mathbf{d}^+, \mathbf{d}^-, \mathbf{u}, \mathbf{s}) = \mathbf{0}$

$$\underbrace{\begin{bmatrix} \mathbf{H}_{(2\times2)} & -\mathbf{H}_{(2\times2)} & \mathbf{A}_{(2\times3)} & \mathbf{0}_{(2\times3)} \\ \mathbf{A}^T_{(3\times2)} & -\mathbf{A}^T_{(3\times2)} & \mathbf{0}_{(3\times3)} & \mathbf{I}_{(3\times3)} \end{bmatrix}}_{=\mathbf{B}_{(5\times10)}} \underbrace{\begin{bmatrix} \mathbf{d}^+_{(2\times1)} \\ \mathbf{d}^-_{(2\times1)} \\ \mathbf{u}_{(3\times1)} \\ \mathbf{s}_{(3\times1)} \end{bmatrix}}_{=\mathbf{X}_{(10\times1)}} = \underbrace{\begin{bmatrix} -\mathbf{c}_{(2\times1)} \\ \mathbf{b}_{(3\times1)} \end{bmatrix}}_{=\mathbf{D}_{(5\times1)}}$$

where, $\mathbf{d}^+_{(2\times1)} = \begin{bmatrix} d_1^+ \\ d_2^+ \end{bmatrix}, \mathbf{d}^-_{(2\times1)} = \begin{bmatrix} d_1^- \\ d_2^- \end{bmatrix}, \mathbf{c}_{(2\times1)} = \begin{bmatrix} -1.732 \\ -1.732 \end{bmatrix}, \mathbf{H}_{(2\times2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_{(2\times3)} = \begin{bmatrix} 0.577 & -1 & 0 \\ 0.577 & 0 & -1 \end{bmatrix}, \mathbf{b}_{(3\times1)} = \begin{bmatrix} 0 \\ 1.732 \\ 1.732 \end{bmatrix}$

$$\mathbf{B}_{(5\times10)} = \left[\begin{array}{cc|cc|ccc|ccc} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ \hline 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{X}^T_{(1\times10)} = \begin{bmatrix} d_1^+ & d_2^+ & d_1^- & d_2^- & u_1 & u_2 & u_3 & s_1 & s_2 & s_3 \end{bmatrix}, \mathbf{D}^T_{(1\times5)} = \begin{bmatrix} 1.732 & 1.732 & 0 & 1.732 & 1.732 \end{bmatrix}$$

Determine the Search Direction by Using the Simplex Method - Iteration 2 (5/10)

Kuhn-Tucker necessary condition (matrix form)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} = \mathbf{D}_{(5\times1)}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1.732 \\ 1.732 \\ 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

We want to find.

➡ This problem is to find $\mathbf{X}$ in the linear programming problem only having the equality constraints.

➡ $u_i s_i = 0;\ i = 1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 2 (6/10)

Simplex method to solve the quadratic programming problem

1. The problem to solve the Kuhn-Tucker necessary condition is the same with the problem having only the equality constraints (linear programming problem).
2. To solve the linear indeterminate equation, we introduce the artificial variables, define the artificial objective function and determine the initial basic feasible solution by using the Simplex method.

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} + \underbrace{\mathbf{Y}_{(5\times1)}}_{\text{Artificial variables}} = \mathbf{D}_{(5\times1)}$$

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{5} Y_i = \sum_{i=1}^{5} D_i - \sum_{j=1}^{10}\sum_{i=1}^{5} B_{ij} X_j = w_0 + \sum_{j=1}^{10} C_j X_j$$

where $C_j = -\sum_{i=1}^{5} B_{ij}$: Add the elements of the jth column of the matrix B and change the its sign (initial relative objective coefficient).

$w_0 = \sum_{i=1}^{5} D_i = 1 + 1 + \frac{2}{3} + 1 + 1 = \frac{14}{3}$: Initial value of the artificial objective function (summation of the all elements of the matrix D)

4. Solve the linear programming problem by using the Simplex and check whether the solution satisfies the following equation.

$u_i s_i = 0;\ i = 1\ to\ 3$: Check whether the solution obtained from the linear indeterminate equation satisfies the nonlinear indeterminate equation and determine the solution.

Determine the Search Direction by Using the Simplex Method - Iteration 2 (7/10)

$$\mathbf{B}_{(5\times10)}\mathbf{X}_{(10\times1)} + \underline{\mathbf{Y}_{(5\times1)}} = \mathbf{D}_{(5\times1)}$$

Artificial variables

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 0.577 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0.577 & 0 & -1 & 0 & 0 & 0 \\ 0.577 & 0.577 & -0.577 & -0.577 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^+ \\ d_2^+ \\ d_1^- \\ d_2^- \\ u_1 \\ u_2 \\ u_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1.732 \\ 1.732 \\ 0 \\ 1.732 \\ 1.732 \end{bmatrix}$$

Define the artificial objective function for using the Simplex method

Sum the all rows (1~5): $0.577X_1 + 0.577X_2 - 0.577X_3 - 0.577X_4 + 1.154X_5 - X_6 - X_7 + X_8 + X_9 + X_{10} + \underbrace{Y_1 + Y_2 + Y_3 + Y_4 + Y_5}_{w} = 6.928$

Replace the summation of the all artificial to w and rearrange: $-0.577X_1 - 0.577X_2 + 0.577X_3 + 0.577X_4 - 1.154X_5 + X_6 + X_7 - X_8 - X_9 - X_{10} = w - 6.928$

1

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
Y1	1	0	-1	0	0.577	-1	0	0	0	0	1	0	0	0	0	1.732	3
Y2	0	1	0	-1	0.577	0	-1	0	0	0	0	1	0	0	0	1.732	3
Y3	0.577	0.577	-0.577	-0.577	0	0	0	1	0	0	0	0	1	0	0	0	-
Y4	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	0	1.732	-
Y5	0	-1	0	1	0	0	0	0	0	1	0	0	0	0	1	1.732	-
A. Obj.	-0.577	-0.577	0.577	0.577	-1.154	1	1	-1	-1	-1	0	0	0	0	0	w-6.928	-

Artificial objective function

Sum all the elements of the row and change the its sign (ex. 1 row: -(1+0+1/3-1+0)=-1/3)

117

Determine the Search Direction by Using the Simplex Method - Iteration 2 (8/10)

2

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	1.732	0.000	-1.732	0.000	1.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	0.000	3.000	-1.732
Y2	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	0.577	0.577	-0.577	-0.577	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
Y4	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	1.732	1.732
Y5	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-
A. Obj.	1.423	-0.577	-1.423	0.577	0.000	-1.000	1.000	-1.000	-1.000	-1.000	2.000	0.000	0.000	0.000	0.000	w-3.464	

3

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	-
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	-
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	-
Y4	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	-
Y5	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	1.732
A. Obj.	0.000	0.845	0.000	-0.845	0.000	0.423	-0.423	-1.000	-1.000	-1.000	0.577	1.423	0.000	0.000	0.000	w-3.464	

4

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	-
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	-
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	-
Y4	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	1.732
X10	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-
A. Obj.	0.000	-0.155	0.000	0.155	0.000	0.423	-0.423	-1.000	-1.000	0.000	0.577	1.423	0.000	0.000	1.000	w-1.732	

Determine the Search Direction by Using the Simplex Method - Iteration 2 (9/10)

5	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	1.732	0.000	-1.732	1.000	0.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	3.000	1.732
X3	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	0.000	1.155	0.000	-1.155	0.000	0.577	-0.577	1.000	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.000	0.000
X9	0.000	-1.000	0.000	1.000	0.000	-1.000	1.000	0.000	1.000	0.000	1.000	-1.000	0.000	1.000	0.000	1.732	-1.732
X10	0.000	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	1.732	-1.732
A. Obj.	0.000	-1.155	0.000	1.155	0.000	-0.577	0.577	-1.000	0.000	0.000	1.577	0.423	0.000	1.000	1.000	w-0.000	

6	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	1.732	0.000	-1.732	0.000	1.000	-1.732	0.000	0.000	0.000	0.000	1.732	0.000	0.000	0.000	0.000	3.000	1.732
X2	-1.000	1.000	1.000	-1.000	0.000	1.000	-1.000	0.000	0.000	0.000	-1.000	1.000	0.000	0.000	0.000	0.000	0.000
Y3	1.155	0.000	-1.155	0.000	0.000	-0.577	0.577	1.000	0.000	0.000	0.577	-0.577	1.000	0.000	0.000	0.000	0.000
X9	-1.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	1.000	0.000	1.732	-1.732
X10	-1.000	0.000	1.000	0.000	0.000	1.000	-1.000	0.000	0.000	1.000	-1.000	1.000	0.000	0.000	1.000	1.732	-1.732
A. Obj.	-1.155	0.000	1.155	0.000	0.000	0.577	-0.577	-1.000	0.000	0.000	0.423	1.577	0.000	1.000	1.000	w-0.000	

7	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	0.000	0.000	0.000	1.000	-0.866	-0.866	-1.500	0.000	0.000	0.866	0.866	-1.500	0.000	0.000	3.000	
X2	0.000	1.000	0.000	-1.000	0.000	0.500	-0.500	0.866	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.000	
X1	1.000	0.000	-1.000	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.500	-0.500	0.866	0.000	0.000	0.000	
X9	0.000	0.000	0.000	0.000	0.000	-0.500	0.500	0.866	1.000	0.000	0.500	-0.500	0.866	1.000	0.000	1.732	
X10	0.000	0.000	0.000	0.000	0.000	0.500	-0.500	0.866	0.000	1.000	-0.500	0.500	0.866	0.000	1.000	1.732	
A. Obj.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	1.000	1.000	1.000	w-0.000	

Determine the Search Direction by Using the Simplex Method - Iteration 2 (10/10)

7	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Y1	Y2	Y3	Y4	Y5	bi	bi/ai
X5	0.000	0.000	0.000	0.000	1.000	-0.866	-0.866	-1.500	0.000	0.000	0.866	0.866	-1.500	0.000	0.000	3.000	
X2	0.000	1.000	0.000	-1.000	0.000	0.500	-0.500	0.866	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.000	
X1	1.000	0.000	-1.000	0.000	0.000	-0.500	0.500	0.866	0.000	0.000	0.500	-0.500	0.866	0.000	0.000	0.000	
X9	0.000	0.000	0.000	0.000	0.000	-0.500	0.500	0.866	1.000	0.000	0.500	-0.500	0.866	1.000	0.000	1.732	
X10	0.000	0.000	0.000	0.000	0.000	0.500	-0.500	0.866	0.000	1.000	-0.500	0.500	0.866	0.000	1.000	1.732	
A. Obj.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	1.000	1.000	1.000	w-0.000	

$$\mathbf{X}^T_{(1\times10)} = \left[d_1^+ \quad d_2^+ \quad d_1^- \quad d_2^- \quad u_1 \quad u_2 \quad u_3 \quad s_1 \quad s_2 \quad s_3\right]$$

Basic solution:

$$X_5 = 3, \quad X_2 = 0, \quad X_1 = 0, \quad X_9 = 1.732, \quad X_{10} = 1.732$$

Nonbasic solution:

$$X_3 = X_4 = X_6 = X_7 = X_8 = 0$$

This solution satisfy the nonlinear indeterminate equation $(X_i X_{3+i} = 0; i = 5\ to\ 7, X_i \geq 0; i = 1\ to\ 10)$.

So, the optimal solution is $d_1 = d_2 = 0,\ u_1 = 3,\ u_2 = u_3 = 0,\ s_1 = 0,\ s_2 = s_3 = 1.732$.

- **In the Pivot step, if the smallest (i.e., the most negative) coefficient of the artificial objective function or the smallest positive ratio "b_i/a_i" appears more than one time, the initial basic feasible solution can be changed by depending on the selection of the pivot element in the pivot procedure.**
- **We have to find and check the solution until the nonlinear indeterminate equation ($u_i \times s_i = 0$) is satisfied.**

6.6 Summary of the Sequential Quadratic Programming (SQP)

Formulation of the Quadratic Programming Problem to Determine the Search Direction

Minimize $f(\mathbf{x}+\Delta\mathbf{x}) \cong f(\mathbf{x}) + \nabla f^T(\mathbf{x})\Delta\mathbf{x} + 0.5\Delta\mathbf{x}^T\mathbf{H}\Delta\mathbf{x}$

The second-order Taylor series expansion of the objective function

Subject to $h_j(\mathbf{x}+\Delta\mathbf{x}) \cong h_j(\mathbf{x}) + \nabla h_j^T(\mathbf{x})\Delta\mathbf{x} = 0;\ j = 1\ to\ p$

The first-order (linear) Taylor series expansion of the equality constraints

$g_j(\mathbf{x}+\Delta\mathbf{x}) \cong g_j(\mathbf{x}) + \nabla g_j^T(\mathbf{x})\Delta\mathbf{x} \le 0;\ j = 1\ to\ m$

The first-order (linear) Taylor series expansion of the inequality constraints

Define: $\bar{f} = f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x}),\ e_j = -h_j(\mathbf{x}),\ b_j = -g_j(\mathbf{x}),$

$c_i = \partial f(\mathbf{x})/\partial x_i,\ n_{ij} = \partial h_j(\mathbf{x})/\partial x_i,\ a_{ij} = \partial g_j(\mathbf{x})/\partial x_i,$

$d_i = \Delta x_i$

Matrix form

Minimize $\bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$: Quadratic objective function

Subject to $\mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$: Linear equality constraints

$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$: Linear inequality constraints

Determination of the Step Size by Using the Golden Section Search Method

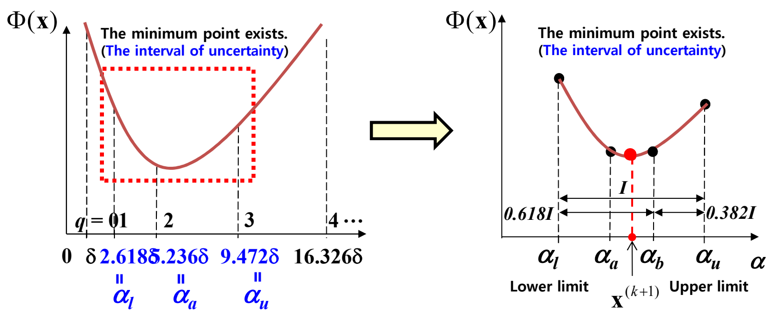
Trial design point for which the descent condition is checked

$\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + \alpha_{(k,j)}\mathbf{d}^{(k)}$ ➡ How can we determine the value of the $\alpha_{(k,j)}$ to find the improved design point?

Find the improved design point which minimizes the descent function more than the current point by changing $\alpha_{(k,j)}$. (One dimensional search method, such as the Golden section search method, can be used.)

Determination of the improved design point $\mathbf{x}^{(k+1)}$ by using the one dimensional search method such as the Golden section search method ($\mathbf{x}^{(k,j)}$ is changed to $\mathbf{x}^{(k+1)}$.)

After finding the interval in which the minimum lies, find the minimum point, $\mathbf{x}$, by reducing the interval (Golden section search method).



Formulation of the Quadratic Programming Problem

$$\textit{Minimize} \quad \bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{H}_{(n\times n)}\mathbf{d}_{(n\times 1)}$$

$$\textit{Subject to} \quad \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$$

$$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$$

Assumption: $\mathbf{H}_{(n\times n)} = \mathbf{I}_{(n\times n)}$

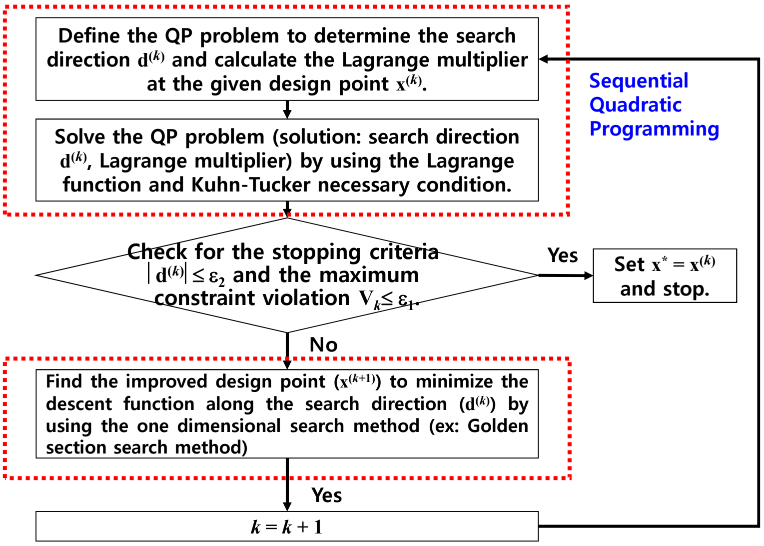
$$\textit{Minimize} \quad \bar{f} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{I}_{(n\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{c}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)} + \frac{1}{2}\mathbf{d}^T{}_{(1\times n)}\mathbf{d}_{(n\times 1)}$$

$$\textit{Subject to} \quad \mathbf{N}^T{}_{(p\times n)}\mathbf{d}_{(n\times 1)} = \mathbf{e}_{(p\times 1)}$$

$$\mathbf{A}^T{}_{(m\times n)}\mathbf{d}_{(n\times 1)} \le \mathbf{b}_{(m\times 1)}$$

- ➡ Since $\mathbf{H}_{(n\times n)} = \mathbf{I}_{(n\times n)}$, the objective function is a quadratic form.
- ➡ All constraints are linear.
- ➡ This problem is called the convex programming problem and any local optimum solution is also a global optimum solution.

Flow Diagram of the SQP Method



Summary of the SQP (CSD) Method (1/2)

☑ Step 1: Set $k = 0$. Estimate the initial value for the design variables as $x^{(0)}$. Select an appropriate initial value for the penalty parameter R_0, and two small number ε_1, ε_2 that define the permissible constraint violations and convergence parameter values, respectively.

☑ Step 2: At $x^{(k)}$, compute the objective and constraint functions and their gradient. Calculate the maximum constraint violation V_k.

☑ Step 3: Using the objective and constraints function values and their gradients, define the QP problem. Solve the QP problem to obtain the search direction $d^{(k)}$ $(= x^{(k+1)} - x^{(k)})$ and Lagrange multiplier $v^{(k)}$ and $u^{(k)}$.

Summary of the SQP (CSD) Method (2/2)

- ☑ Step 4: Check for the stopping criteria $|\mathbf{d}^{(k)}| \le \varepsilon_2$ and the maximum constraint violation $V_k \le \varepsilon_1$. If these criteria are satisfied then stop. Otherwise continue.
- ☑ Step 5: Calculate the sum r_k of the Lagrange multiplier. Set $R = \max\{R_k, r_k\}$.
- ☑ Step 6: Set $\mathbf{x}^{(k,j)} = \mathbf{x}^{(k)} + \alpha_{(k,j)}\mathbf{d}^{(k)}$, where $\alpha = \alpha_{(k,j)}$ is a proper step size. As for the unconstrained problems, the step size can be obtained by minimizing the descent function along the search direction $\mathbf{d}^{(k)}$. The one dimensional search method, such as the Golden section search method, can be used to determine the optimum step size.
(If the one dimensional search method is completed, the current design point $\mathbf{x}^{(k,j)}$ is changed to $\mathbf{x}^{(k+1)}$.)
- ☑ Step 7: Save the current penalty parameter as $R_k = R$. Update the iteration counter as $k = k+1$ and go to Step 2.

Effect of the Starting Point in the SQP Method

The starting point can affect performance of the method.

For example, at some points, the Quadratic Programming problem defined to determine the search direction may not have any solution.
This does not mean that the original problem is infeasible.
The original problem may be highly nonlinear, so that the linearized constraints may be inconsistent giving infeasible Quadratic Programming problem.

This situation can be handled by either temporarily deleting the inconsistent constraints or starting from another point.