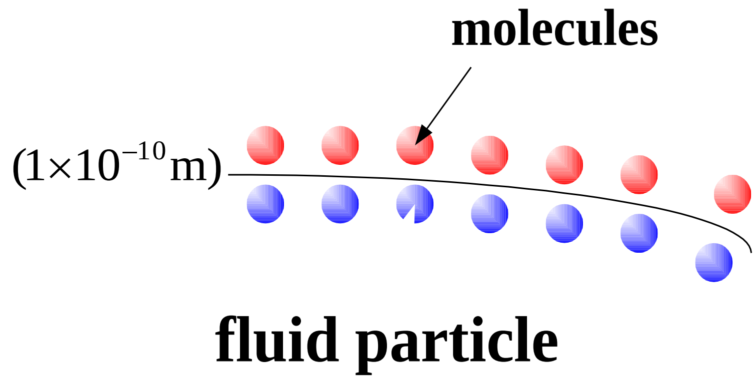
A Quick Look

- Forces on a Fluid Particle
- Stress & Pressure in a Fluid
  - Pascal's law, Pressure force & measurement
- Pressure Forces on Solid Surfaces
  - Plane & curved surfaces
- Pressure Forces on Bodies Immersed in Fluids
  - Archimedes' principle
  - Equilibrium (static & stable)
- Stratified Fluids
  - Stability, pressure distribution, Earth's atmosphere
- Surface Tension & Capillarity
- Hydraulic Force Transmission

Forces on a Fluid Particle

- Forces on a Fluid Particle
  - Surface force (stress): $\boldsymbol{\sigma}$
    intermolecular forces, short range
  - Body force (gravity): $\boldsymbol{g}$
    acts on the entire particle, long range

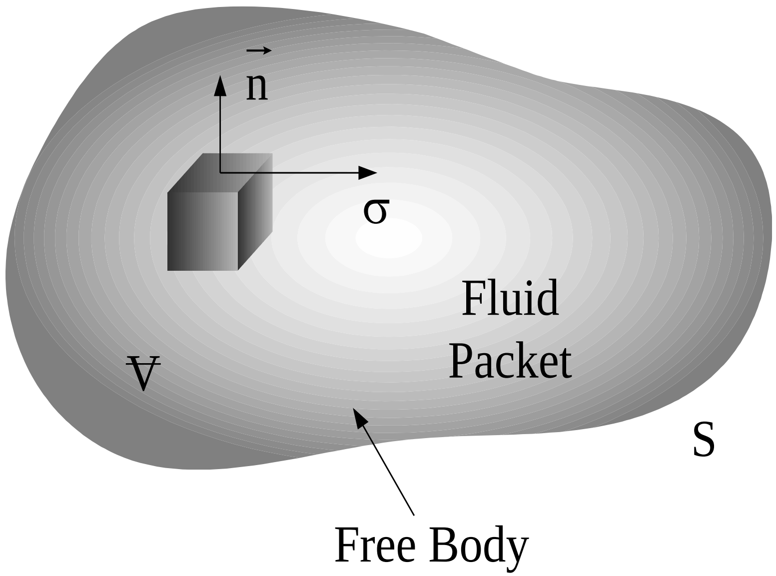


- Surface force depends on
  - Relative position of the molecules
    - pressure is normal to the surface: $p \approx z$
  - Relative average motion of theses particles
    - viscous stress: $\tau \approx \mu$
- Body force affects the motion of fluids
  - Gravity $\boldsymbol{g}$ is exerted by the entire earth
  - Electrostatic force acts between electrically charged particles
  - Not considered in this class, though

Stress in the Fluid

- How the stress is generated in the fluid
  - Compress the fluid $\Rightarrow$ force the molecules closer together $\Rightarrow$ inward force on fluid boundaries $\Rightarrow$ transmitted throughout the fluid as the molecules repel each other more strongly $\Rightarrow$ internal stress is generated
- Stress $\boldsymbol{\sigma}$ on the surface (vector)
  - Normal stress
  - Shear stress
- Stress $\underline{\boldsymbol{\tau}}$ within the fluid (tensor)

$$\underline{\boldsymbol{\tau}} = \begin{matrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{matrix}$$



Pressure in a Static Fluid - *Pascal's Law*

"At a point P in a static fluid, $\boldsymbol{\sigma}$ must have the direction of $\vec{n}$ and have the same magnitude for all directions of $\vec{n}$ ".

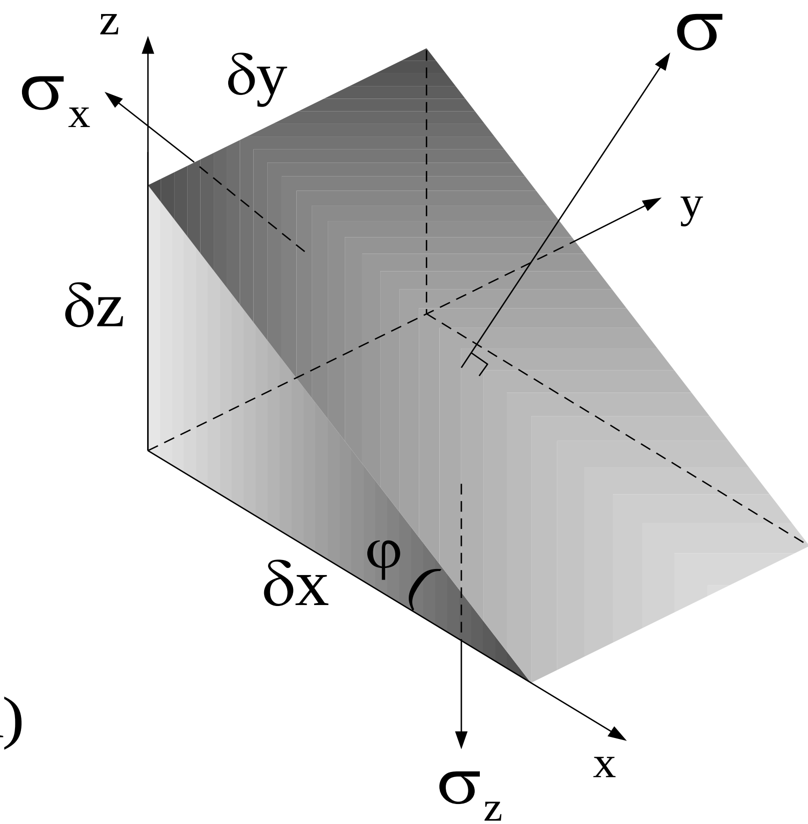
$$-\sigma_x \delta y \delta z + (\sigma \sin\varphi)\delta y \frac{\delta z}{\sin\varphi} = 0$$

$$(-\sigma_x + \sigma)\delta y \delta z = 0 \quad \text{or} \quad \sigma = \sigma_x$$

$$\vec{\sigma} = (-p)\vec{n}$$

$$\frac{\delta \vec{F}}{\delta \forall} = \text{pressue Force / volume} = -\nabla p$$

$$\iint_S (-p)\vec{n}\, dS = \iiint_\forall (-\nabla p)\, d\forall \quad \text{(Gauss' theorem)}$$



Pressure in a Static Fluid - Example

Near the surface of the earth, the pressure decreases with altitude as

$$p = p_o \exp(-\alpha z)$$

where $p_o = 1.0133 \times 10^5$ Pa at sea level

$\alpha = 1.2 \times 10^{-4}$ / m

Calculate the pressure force/volume at

i) $z = 0$ $\quad -\nabla p = 12.16\ [\mathrm{N/m^3}]$

ii) $z = 5$ km $\quad -\nabla p = 6.805\ [\mathrm{N/m^3}]$

Pressure in a Fluid in a Gravity Field

Gravitational force / unit volume $= \rho \vec{g}$

In order for a fluid to remain motionless $-\nabla p + \rho \vec{g} = 0$

Integrate between points 1 and 2

$$\int_1^2 (-\nabla p) \cdot d\vec{C} + \int_1^2 \rho \vec{g} \cdot d\vec{C} = 0$$

$$\int_1^2 (-\nabla p) \cdot d\vec{C} = -\int_1^2 \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz \right) = -\int_1^2 dp = p_1 - p_2$$

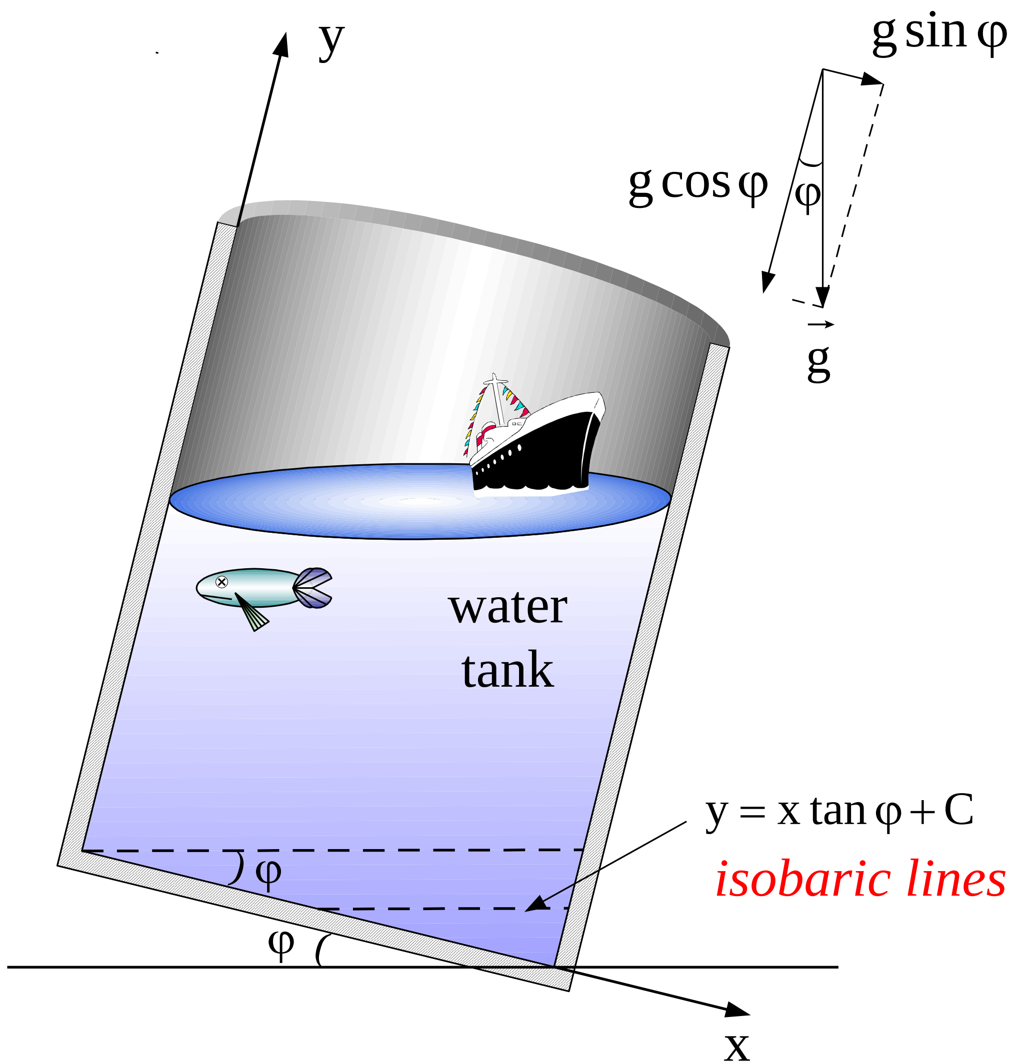
$$\int_1^2 \rho \vec{g} \cdot d\vec{C} = \rho \int_1^2 \nabla(\vec{g} \cdot \vec{R}) \cdot d\vec{C} = \rho \vec{g} \cdot \vec{R}_2 - \rho \vec{g} \cdot \vec{R}_1$$

$$\therefore\ p - \rho \vec{g} \cdot \vec{R} = \text{constant}$$

$$p_1 + \rho g z_1 = p_2 + \rho g z_2 = p + \rho g z = C$$

$$p(z) = p_1 + \rho g z_1 - \rho g z \qquad \text{or} \qquad p + \rho g z = \text{constant}$$

Pressure in a Fluid in a Gravity Field - Example



$$\vec{g} = g\sin\varphi\,\vec{i}_x - g\cos\varphi\,\vec{i}_y$$

Invariant :

$$p - \rho\vec{g}\cdot\vec{R}$$

$$= p - \rho(g\sin\varphi\,\vec{i}_x - g\cos\,\vec{i}_y)$$

$$\cdot(x\,\vec{i}_x + y\,\vec{i}_y + z\,\vec{i}_z)$$

$$= p - \rho g(x\sin\varphi - y\cos\varphi)$$

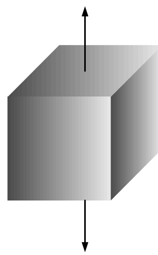
$$\therefore\ p(x,y)$$

$$= p_1 - \rho g(x_1\sin\varphi - y_1\cos\varphi)$$

$$+\rho g(x\sin\varphi - y\cos\varphi)$$

Hydrostatic Balance Equation

- Differential Form



$$-\nabla p + \rho \vec{g} = 0$$

↓ Integrate with ¢£

$$\iiint_{\forall} (-\nabla p)\, d\forall + \iiint_{\forall} \rho \vec{g}\, d\forall = 0$$

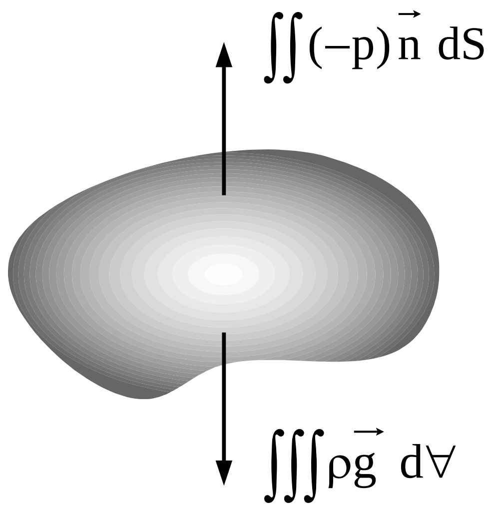
↓ Gauss' theorem

$$\iint_{S} (-p)\vec{n}\, dS + \iiint_{\forall} \rho \vec{g}\, d\forall = 0$$

pressure force acting on the surface

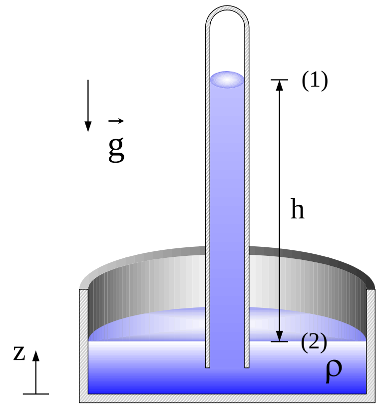
gravity force acting on the whole body

- Integral Form



Pressure Measurement

➢ Hg Barometer



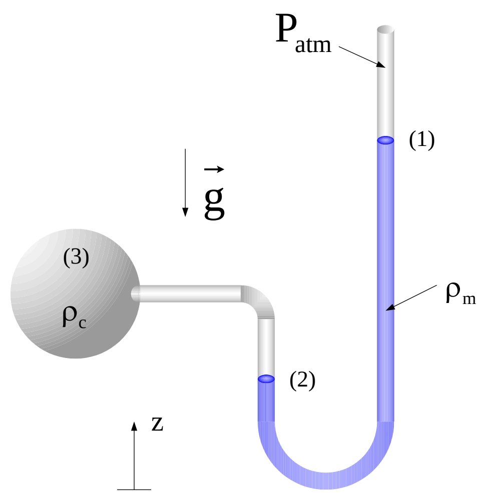
$$p_1 + \rho g z_1 = p_2 + \rho g z_2$$

$$p_1 \approx 0$$

$$\therefore\ p_2 = \rho g h$$

$$= 1.0133 \times 10^5\ [\mathrm{Pa}]$$

➢ U Tube Manometer



$$p_2 + \rho_m g z_2 = \underbrace{p_1}_{p_{atm}} + \rho_m g z_1$$

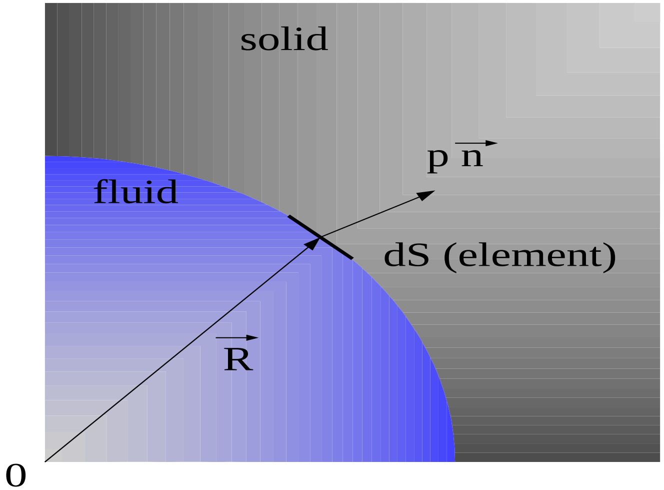
$$p_3 + \rho_c g z_3 = p_2 + \rho_c g z_2$$

$$p_3 = p_2 - \rho_c g (z_3 - z_2)$$

$$\therefore\ p_3 = p_{atm} + \rho_m g (z_1 - z_2) - \rho_c g (z_3 - z_2)$$

used in measuring blood pressure!

Pressure Force on Solid Surfaces



$$d\vec{F} = p\vec{n}\ dS \qquad : \text{force}$$

$$d\vec{T} = \vec{R} \times p\vec{n}\ dS \quad : \text{torque}$$

$$\vec{F} = \iint p\vec{n}\ dS$$

$$\vec{T} = \iint \vec{R} \times p\vec{n}\ dS = \vec{R}_P \times \vec{F}$$

The pressure forces acting on the surface may be replaced by a single force $\vec{F}$ acting at a point $\vec{R}_P$, center of pressure, located so as to give the same moment $\vec{T}$ as the pressure forces.

The moment of the pressure forces about $\vec{R}_P = 0$

$$\iint (\vec{R} - \vec{R}_P) \times p\vec{n}\ dS = \iint (\vec{R} \times p\vec{n})\ dS - \vec{R}_P \times \iint p\vec{n}\ dS = \vec{T} - \vec{T} = 0$$

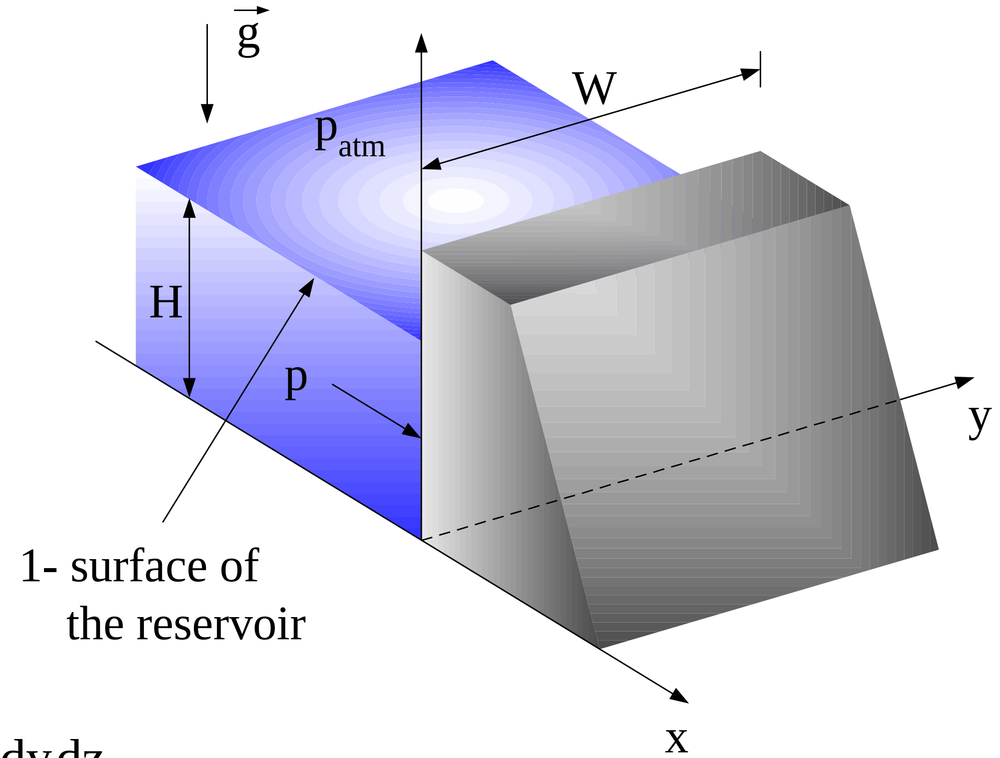
$$\vec{F} = \iint p\vec{n}\, dS = \iint (p - p_{atm})\vec{n}\, dS \qquad (\because \iint p_{atm}\vec{n}\, dS = \iiint \nabla p_{atm}\, d\forall = 0)$$

Pressure Force on Solid Surfaces - *Example*

$$\vec{F} = \iint p\,\vec{n}\;dS$$

$$= \int_0^W \int_0^H \rho g (H - z)\,\vec{i}_x\;dy\,dz$$

$$= \underbrace{\left(\rho g \frac{H}{2}\right)}_{\text{avg p}}\;\underbrace{(WH)}_{\text{area}}\;\vec{i}_x$$



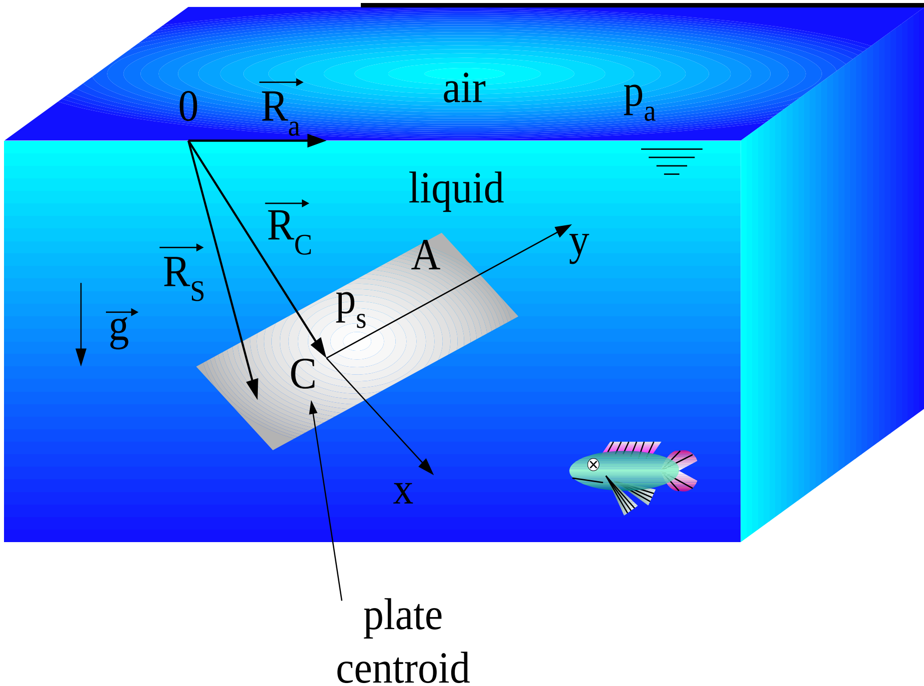
$$\vec{T} = \iint \vec{R} \times \vec{F}\;dS = \iint (\vec{R} \times p\vec{n})\;dS$$

$$= \int_0^H \int_0^W (x\vec{i}_x + y\vec{i}_y + z\vec{i}_z) \times \rho g (H - z)\vec{i}_x\;dy\,dz$$

$$= \rho g \frac{WH^3}{6}\vec{i}_y - \rho g \frac{W^2H^2}{4}\vec{i}_z$$

$$= T_y\vec{i}_y + T_z\vec{i}_z$$

Pressure Force on A Plane Surface



Find $\vec{F}=\iint p\,\vec{n}\,dS$ & $\vec{T}=\iint(\vec{R}\times p\,\vec{n})\,dS$

$$p_S = p_a + \rho\vec{g}\cdot\vec{R}_S = p_C + \rho(g_x x + g_y y)$$

$$\vec{F}=\vec{n}\iint p_S\,dS=(p_C\,A)\,\vec{n}$$

The center of pressure is located at (x_{cp}, y_{cp})

$$\iint p_S\,(x-x_P)\,dS=0,\ \iint p_S\,(y-y_P)\,dS=0$$

$$\iint[p_C+\rho(g_x x+g_y y)](x-x_P)\,dS=0 \quad\Rightarrow x_P$$

$$\iint[p_C+\rho(g_x x+g_y y)](y-y_P)\,dS=0 \quad\Rightarrow y_P$$

We may then compute the moment of the pressure force about the origin *O*.

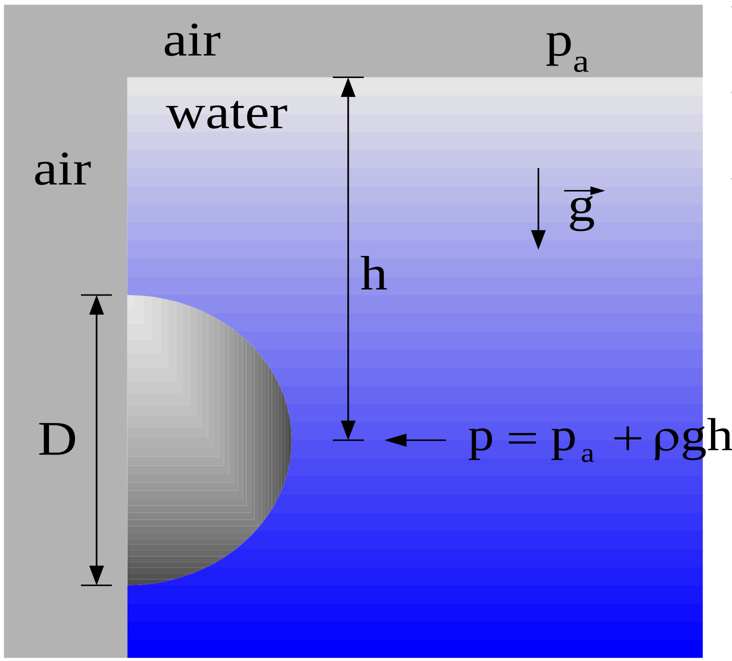
The position vector

$$\vec{R}_S=\vec{R}_C+x\,\vec{i}_x+y\,\vec{i}_y$$

The centroid position

$$\vec{R}_C=\frac{1}{S}\iint\vec{R}_S\,dS \ :\ \iint x\,dS=0,\ \iint y\,dS=0$$

Pressure Force on A Curved Surface



Forces exerted on the hemispherical surface by water

F_h : horizontal

F_v : vertical upward

$$F_h = (p_a + \rho g h)\frac{\pi D^2}{4}$$

$$F_v = \rho g\left[\frac{1}{2}\frac{4\pi}{3}\left(\frac{D}{2}\right)^3\right] = \frac{\pi \rho g D^3}{12}$$

First replace the curved structure and then take the balance on the pressure distribution on the right and left surfaces of the fluid.

Pressure Force on A Body Immersed in A Fluid

- Archimedes' Principle

"The pressure force (buoyant force) on an immersed body is equal in magnitude, but opposite in direction to the force of gravity acting on the displaced fluid."

$$\boxed{\begin{array}{c}\text{pressure force integral}\\ \text{over surface of body}\end{array}} = \boxed{\begin{array}{c}\text{integral over volume of the}\\ \text{fluid displaced by structure}\end{array}}$$

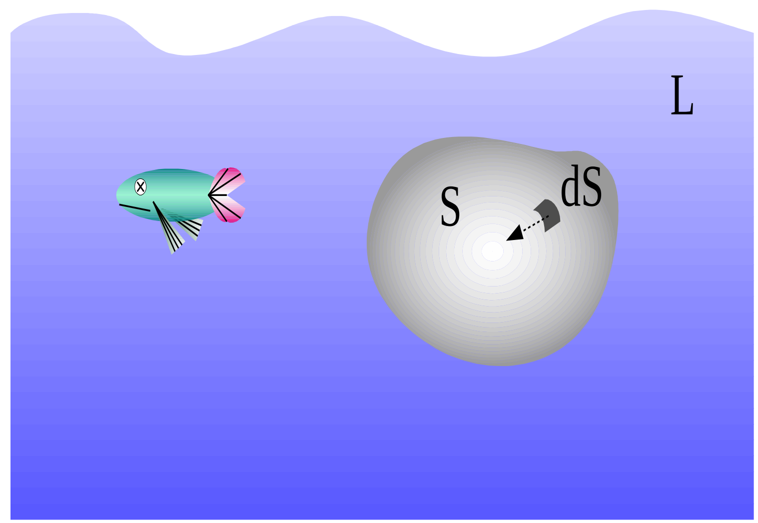
$$\vec{F}_B = \iint_S p\,\vec{n}\,dS$$

$$= -\iiint_{\forall} \nabla p\, d\forall = -\iiint_{\forall} \rho \vec{g}\, d\forall = -\rho \vec{g}\,\forall$$

$$\vec{R}_B = \frac{1}{\forall}\iiint_{\forall} \vec{R}\, d\forall \quad \text{(mass center of displaced fluid)}$$

$$\vec{T}_B = \vec{R}_B \times \vec{F}_B = -\iiint_{\forall} (\vec{R}\times\rho\vec{g})\, d\forall = \iiint_{\forall} \vec{R}\, d\forall \times (-\rho\vec{g})$$

$$= \vec{R}_B \forall \times \frac{\vec{F}_B}{\forall} = \vec{R}_B \times \vec{F}_B$$

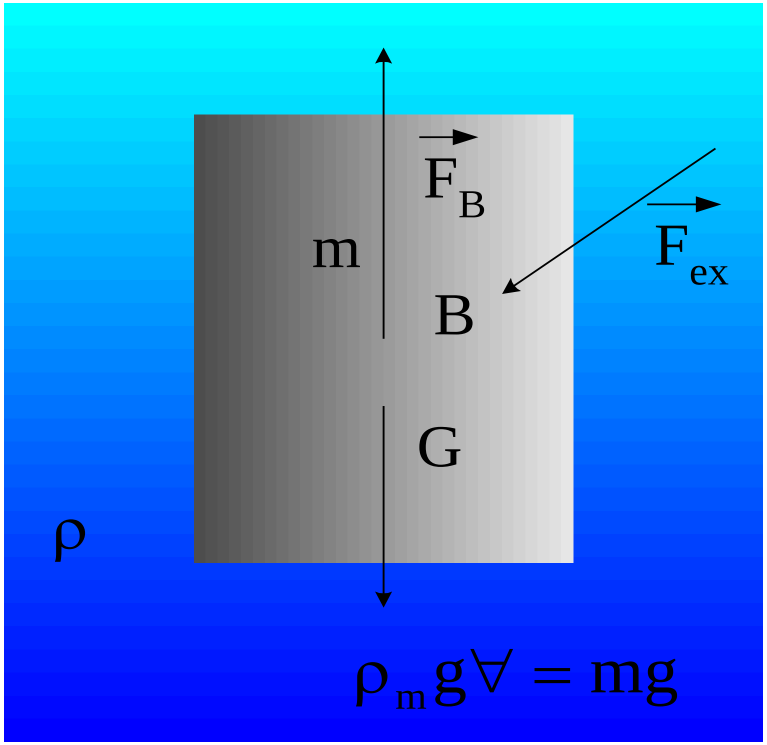


Equilibrium in Fluids - Static Equilibrium

1) Force

$$m\vec{g} + \vec{F}_{ex} - \rho\forall\vec{g} = 0$$

$$\text{If } \vec{F}_{ex} = 0, \quad m = \rho\forall$$



2) Moment

$$\vec{R}_G \times m\vec{g} - \vec{R}_B \times \rho\forall\vec{g} + \vec{R}_{ex} \times \vec{F}_{ex} = 0$$

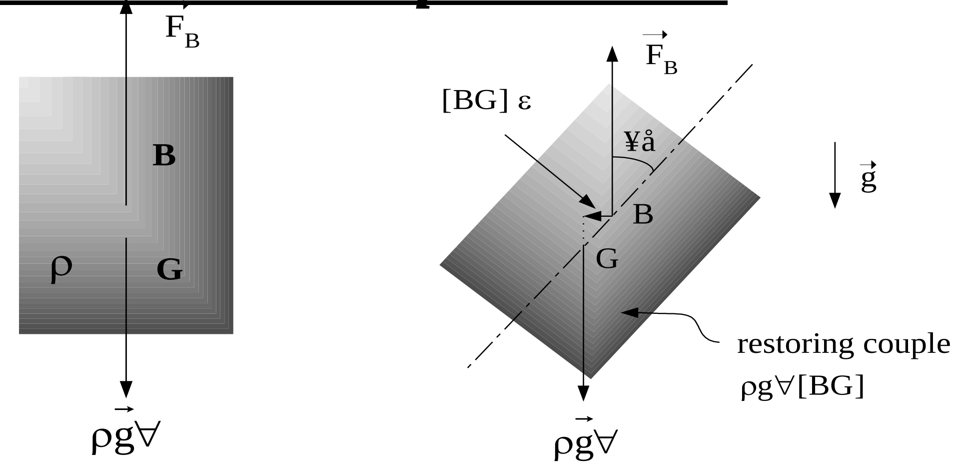
$$\text{If } \vec{F}_{ex} = 0, \ (\vec{R}_G - \vec{R}_B)\times\vec{g} = 0 \text{ or } (\vec{R}_G - \vec{R}_B) // \vec{g}$$

The center of gravity and the center of buoyancy lie on the same vertical line.

Equilibrium in Fluids - Stable Equilibrium

1) Totally Submerged

When disturbed slightly, the body tends to return to the equilibrium position. G should lie below B, i.e. B > G.



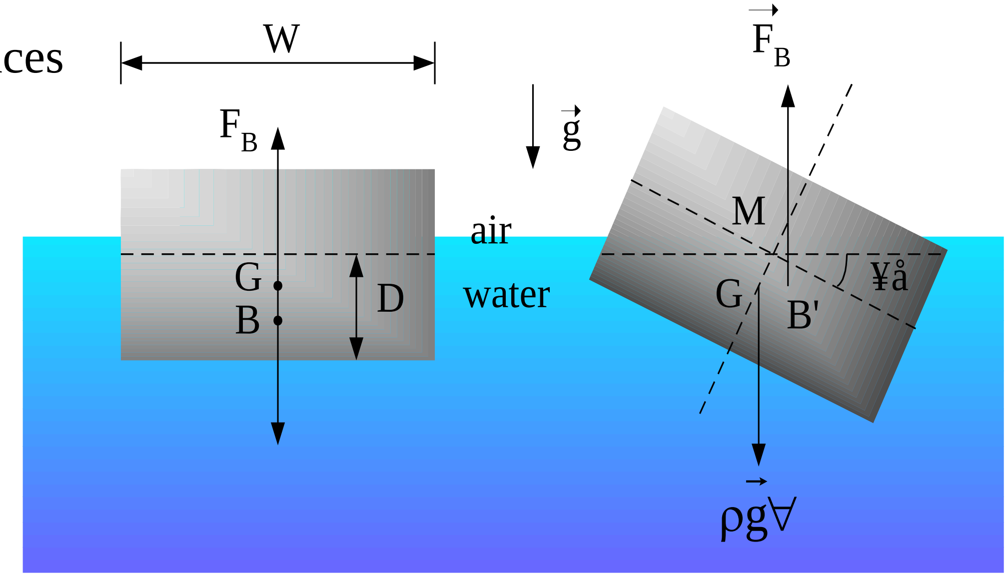
2) Partially Submerged

G: halfway between the upper & lower surfaces
B: halfway between the water line & lower surface

$$-DWL\times\overline{BB'}+\varepsilon\frac{W^2L}{8}\frac{2W}{3}=0$$

$$\rightarrow \overline{BB'}=\varepsilon\frac{W^2}{12D}$$

$$\therefore \overline{BM}=\frac{\overline{BB'}}{\varepsilon}=\frac{W^2}{12D}$$



Department of Nuclear Engineering, Seoul National University

Stratified Fluids - Static Stability

Stability Criterion: start from hydrostatic equilibrium

$$-\nabla p + \rho \vec{g} = 0$$

$$\nabla \times (-\nabla p + \rho \vec{g}) = 0$$

$$-\nabla \times (\nabla p) + \nabla \times (\rho \vec{g}) = 0$$

$$\rho(\nabla \times \vec{g}) + (\nabla \rho) \times \vec{g} = 0 \quad (\vec{g} = \text{constant})$$

$$\therefore (\nabla \rho) \times \vec{g} = 0$$

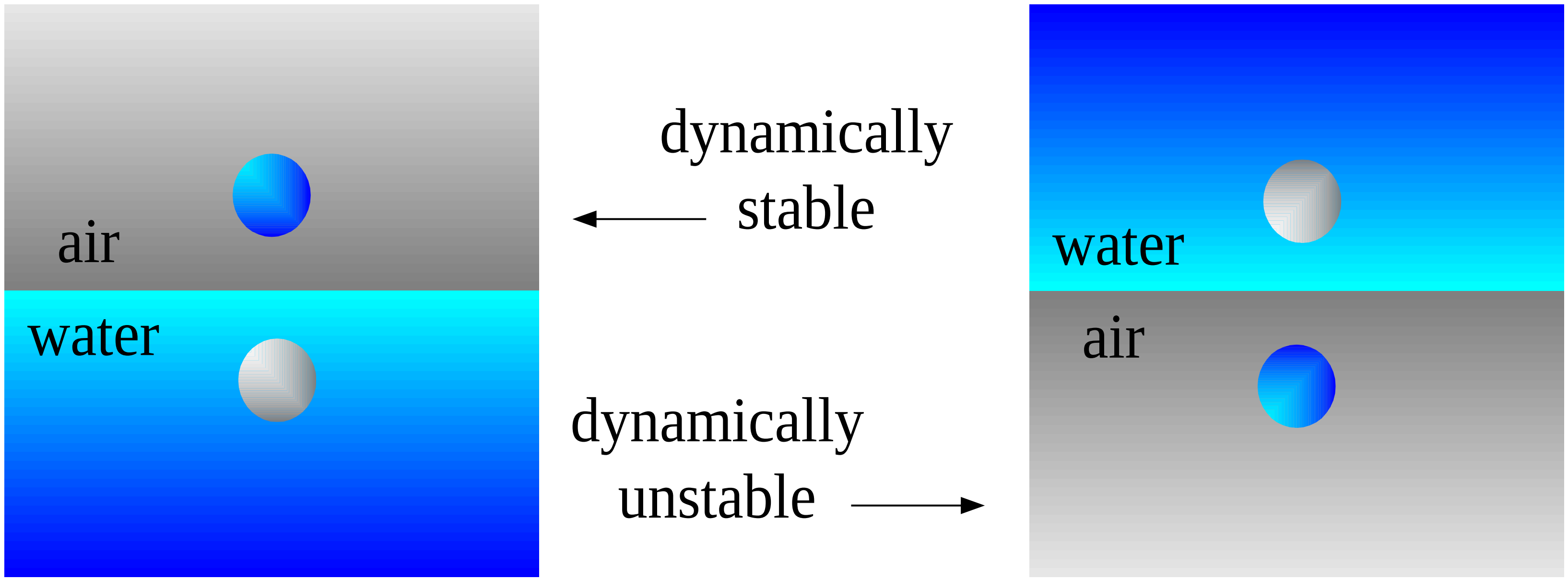
$$i)\ \nabla \rho = 0 \quad (trivial)$$

$$ii)\ \nabla \rho \text{ has the same direction as } \vec{g}$$

$$\Rightarrow \rho \text{ must be function of vertical height only}$$

$$\Rightarrow \text{Condition of static stability}$$

Stratified Fluids - Dynamic Stability



“Stability” is clearly different from “static” .
If a small perturbation is given to a certain object, it must not be influenced by that.

Pressure in Stratified Fluids

Model

$$-\nabla p + \rho \vec{g} = 0$$

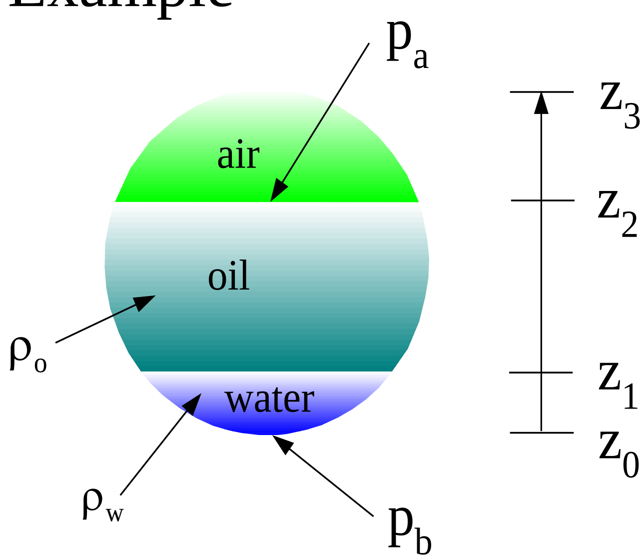
$$z - \text{component} \ : \ -(\nabla p)\cdot \vec{i_z} + \vec{g}\cdot\vec{i_z}\,\rho(z) = 0$$

$$-\frac{dp}{dz} - g\rho(z) = 0$$

$$\int_{p_o}^{p(z)} dp + \int_{z_o}^{z} g\rho(z)\,dz = 0$$

$$p(z) = p_o - g\int_{z_o}^{z} \rho(z)\,dz$$

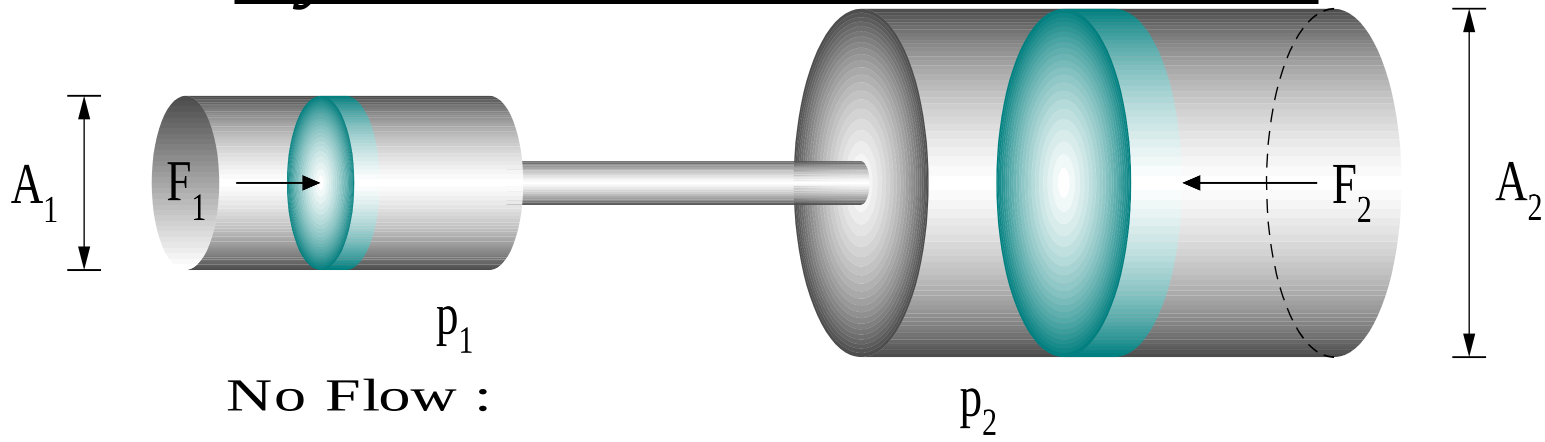
Example



$$p_b = p_a - g\int_{z_2}^{z_1} \rho_o\,dz - g\int_{z_1}^{z_o} \rho_w\,dz$$

$$\therefore\ p_b = p_o + g[\rho_o(z_2 - z_1) + \rho_w(z_1 - z_o)]$$

Hydraulic Force Transmission



No Flow :

$$\frac{F_1}{A_1} = p_1 = p_2 = \frac{F_2}{A_2}$$

$$\text{or } F_2 = \frac{A_2}{A_1} F_1$$

Flow :

$$A_2 \delta x_2 = A_1 \delta x_1 \quad \text{or} \quad \frac{A_2}{A_1} = \frac{\delta x_1}{\delta x_2}$$

$$F_2 = \frac{\delta x_1}{\delta x_2} F_1$$

$$\text{or } F_1 \delta x_1 = F_2 \delta x_2 \quad \text{(i.e. the same work)}$$