# Boltzmann Transport Equation (with Delayed Neutron Neglected)

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# References


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# Neutron Density

- Angular neutron density = $N(\mathbf{r},E,\Omega,t) = N(\mathbf{r},\mathbf{v},t)$

$$\mathbf{v} \equiv (v\Omega_x, v\Omega_y, v\Omega_z), E = \frac{1}{2}m_n v^2$$

  - The expected number of neutrons at the position **r** with energy $E$ and direction Ω at time $t$, per unit volume per unit energy per unit solid angle, e.g. per $cm^3$ per steradian per MeV

    (By defining the angular density as the expected, rather than the actual, number of neutrons in an element of volume in the phase space, the possibility of describing the fluctuations in the neutron population can be excluded.)
  - $N(\mathbf{r},E,\Omega,t)d\mathbf{r}dEd\Omega$ is the number of neutrons in the volume element d**r** about **r**, having energies in d$E$ about $E$ and directions within dΩ about Ω at time $t$.

- Neutron Density = $n(\mathbf{r},E,t)$
  - The expected number of neutrons at **r** with energy $E$ at time $t$, per unit volume per unit energy

$$n(\mathbf{r},E,t) = \int_{4\pi} N(\mathbf{r},E,\mathbf{\Omega},t)d\mathbf{\Omega} = \int_{-1}^{1}\int_{0}^{2\pi} N(\mathbf{r},E,\mathbf{\Omega},t)d\varphi d\mu$$

$$d\mathbf{\Omega} = \sin\theta d\theta d\varphi = d\mu d\varphi; \quad \mu = \cos\theta$$

# Angular Current Density

- Neutron angular current, or vector flux, or angular current density

  $= \mathbf{v}N(\mathbf{r},E,\mathbf{\Omega},t) = v\mathbf{\Omega}N(\mathbf{r},E,\mathbf{\Omega},t)$

  - The expected number of neutrons passing through a unit area at $\mathbf{r}$ having the normal vector of $\mathbf{\Omega}$ with energy $E$ and direction $\mathbf{\Omega}$ at time $t$, per unit energy per unit solid angle in unit time
  - $\underline{\mathbf{v}N(\mathbf{r},E,\mathbf{\Omega},t)\cdot d\mathbf{A}dEd\Omega dt}$ is the expected net number of neutrons passing through an area $dA$ with energy $E$ in $dE$, direction $\mathbf{\Omega}$ in $d\Omega$ during $dt$ at time $t$.

    ($d\mathbf{A} = \mathbf{n}_s dA$ where $\mathbf{n}_s$ is the unit vector normal to the surface)

  - $\underline{\mathbf{v}N(\mathbf{r},E,\mathbf{\Omega},t)\cdot d\mathbf{A}}$ is the number of neutrons crossing the surface element per unit solid angle per unit energy in unit time. (A crossing is counted as negative if $\mathbf{v}\cdot d\mathbf{A}<0$.) – Bell and Glasstone
  - $\underline{\mathbf{v}N(\mathbf{r},E,\mathbf{\Omega},t)\cdot d\mathbf{A}dEd\Omega}$ is the expected number of neutrons passing through an area $dA$ per unit time with energy $E$ in $dE$, direction $\mathbf{\Omega}$ in $d\Omega$ at time $t$. – Duderstadt and Hamilton

# Questions about Current Density

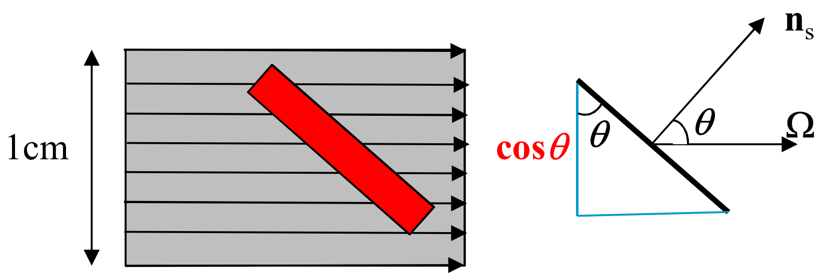
Q#1 Suppose that neutrons are generated from an accelerator unidirectionally with strength of $S_0$ neutron/cm$^2$·sec and speed of $v$.

a) What is the number of neutrons passing a unit surface (1cm$^2$) perpendicular to the neutron direction in unit time?


\# of neutrons passing the slab in unit time

$= S_0$ [neutron/sec]

b) When an angle between the normal direction of the slab and the neutron direction becomes $\theta$, what is the number of neutrons passing the artificial slab in unit time?




\# of neutrons passing the slab in unit time

$= S_0 \cdot \cos\theta$ [neutron/sec]

# Angular Flux

- Neutron angular flux = $vN(\mathbf{r},E,\Omega,t) = \Phi(\mathbf{r},E,\Omega,t)$
  - The expected number of neutrons passing through a unit area at **r** having the normal vector of **Ω** with energy *E* and direction **Ω** at time *t*, per unit energy per unit solid angle in unit time

  *Cf*. Neutron angular current = $\mathbf{v}N(\mathbf{r},E,\Omega,t)$:

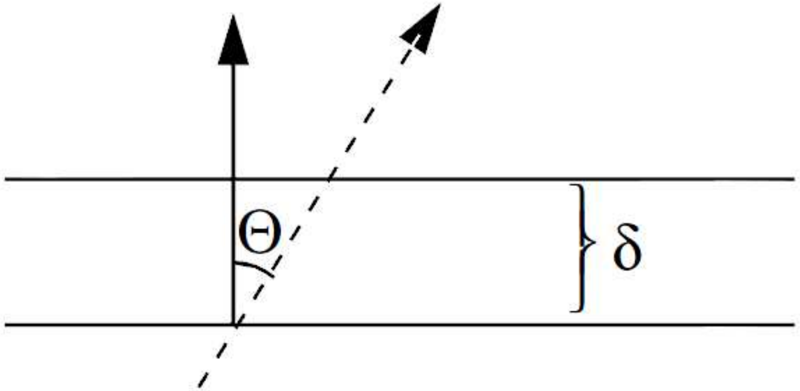
  The expected number of neutrons passing through a unit area at **r** having the normal vector of **Ω** with energy *E* and direction **Ω** at time *t*, per unit energy per unit solid angle in unit time

  - $\Phi(\mathbf{r},E,\Omega,t)dA\,dE\,d\Omega\,dt$ is the expected **effective** number of neutrons passing through an area d*A* with energy *E* in d*E*, direction **Ω** in d**Ω** during d*t* at time *t*.
  - $\Phi(\mathbf{r},E,\Omega,t)d\mathbf{r}\,dE\,d\Omega$ is **amount of neutron track length** in a differential volume d**r** about **r**, associated with particles of a differential energy in d*E* about *E*, moving in a differential solid angle in d**Ω** about **Ω**, at time *t*. – Wikipedia

# Estimation of Flux from Surface Crossing Particles

- The volume average flux is defined as the sum of track lengths in a volume divided by the volume.

- Suppose that we are estimating the volume flux of a slab with area $A$ and thickness $\delta$. When a particle having a direction of $\mathbf{\Omega}$ passes the slab, the volume flux becomes




$$\Phi_V(\mathbf{\Omega}) = \frac{\text{Track Lengths}}{\text{Volume}} = \frac{W \cdot L}{V}$$

where $W$ is the number of neutrons passing the slab in unit time.

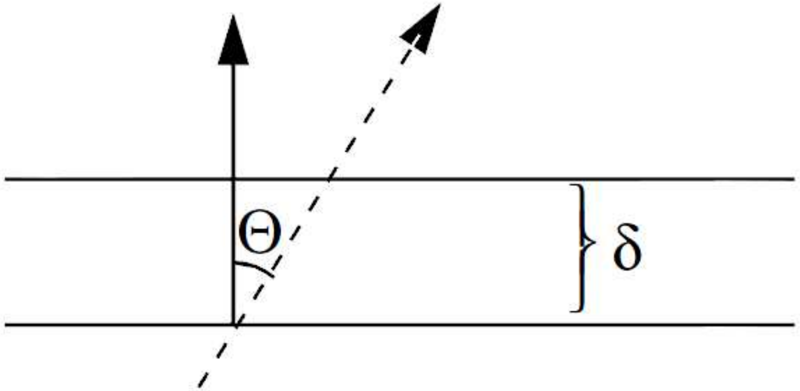
- The surface flux can be regarded as the limiting case of the volume flux when the slab becomes infinitely thin. Therefore the surface flux can be expressed by

$$\Phi_S(\mathbf{\Omega}) = \lim_{\delta \to 0} \frac{W \cdot L}{A \cdot \delta} = \lim_{\delta \to 0} \frac{W \cdot \left( \cancel{\delta} / \cos\theta \right)}{A \cdot \cancel{\delta}} = \frac{W / \cos\theta}{A}$$

# Estimation of Current from Surface Crossing Particles

- The angular current of a neutron of $\boldsymbol{\Omega}$ direction on a surface of which normal direction is $\mathbf{n}_s$ is defined by the number of particles that pass through a surface per unit area times sign($\mathbf{n}_s \cdot \boldsymbol{\Omega}$).

- Note that the surface angular flux is expressed as




$$\Phi_S(\boldsymbol{\Omega}) = \lim_{\delta \to 0} \frac{W \cdot L}{A \cdot \delta} = \lim_{\delta \to 0} \frac{W \cdot (\delta / \cos\theta)}{A \cdot \delta} = \frac{W/\cos\theta}{A}$$

- Then the angular current on the surface becomes

$$J_S(\boldsymbol{\Omega}) = \frac{W/\cos\theta}{A} \cdot (\boldsymbol{\Omega} \cdot \mathbf{n}_s)$$

$$= \frac{W/\cos\theta}{A} \cdot \cos\theta \cdot \text{sign}(\boldsymbol{\Omega} \cdot \mathbf{n}_s) = \frac{W}{A} \text{sign}(\boldsymbol{\Omega} \cdot \mathbf{n}_s)$$

# Current & Flux

- Neutron current or neutron current density = $\mathbf{J}(\mathbf{r},E,t)$

$$\mathbf{J}(\mathbf{r},E,t)=\int_{4\pi}\mathbf{v}N(\mathbf{r},E,\mathbf{\Omega},t)d\mathbf{\Omega}=v\int_{4\pi}\mathbf{\Omega}N(\mathbf{r},E,\mathbf{\Omega},t)d\mathbf{\Omega}$$

  - The expected net number of neutrons passing through a unit area with energy $E$ at time $t$, per unit energy in unit time
  - $\underline{\mathbf{J}(\mathbf{r},E,t)\cdot d\mathbf{A}dE}$ is the expected net number of neutrons crossing the surface element dA per unit time with energy $E$ in d$E$ at time $t$.

    (d**A** = $\mathbf{n}_s$dA where $\mathbf{n}_s$ is the unit vector normal to the surface)

- Neutron flux = $\phi(\mathbf{r},E,t)$

$$\phi(\mathbf{r},E,t)=\int_{4\pi}\Phi(\mathbf{r},E,\mathbf{\Omega},t)d\mathbf{\Omega}$$

  - The expected number of neutrons effectively passing through a unit area with energy $E$ at time $t$, per unit energy in unit time
  - $\underline{\phi(\mathbf{r},E,t)\,dAdE}$ is the expected effective number of neutrons passing through an area d$A$ per unit time with energy $E$ in d$E$, at time $t$.
  - $\underline{\phi(\mathbf{r},E,t)\,d\mathbf{r}dE}$ is the amount of track length of neutrons in d**r** about **r** with energies in d$E$ about $E$ at time $t$.

# Macroscopic cross section

- Macroscopic cross section = $\Sigma_x(\mathbf{r},E,\Omega,t)$
  - The probability that a neutron located at **r** with energy *E*, direction **Ω** at time *t* will undergo a particular reaction, indicated by *x*, while it travels in unit distance

- $v(E)\Sigma_x(\mathbf{r},E,\Omega,t)$
  - The probability that a neutron located at **r** with energy *E*, direction **Ω** at time *t* will undergo a reaction of type x while it travels in unit time

# Time Dependence of Atomic Density, $X_i(\mathbf{r},t)$

- They may change with time for two reasons except the changes by external means such as the control rod movements and the soluble boron injection nor those due to phase transition, i.e., liquid to vapor.
  1. $X_i(\mathbf{r},t)$ are **temperature dependent**. Even in the solid state, the decrease in the atomic densities with increase in temperature due to thermal expansion can have a significant effect upon reactor operation.
  2. Number densities change because of **the continual occurrence of nuclear reactions and decays** in the reactor.
     - ✓ For example, the production of fission fragments of high-absorption-cross section such as $^{135}Xe$ leads to important, temporal variations in the reactor.
     - ✓ Also, fuel nuclei are "burning up," and low-absorption-cross section, relatively stable fission products are accumulating so long as the reactor is in operation.

- *Then, on which variables the microscopic cross section is dependent?*

# Reaction Rate Density

- Angular reaction rate density = $R_x(\mathbf{r},E,\mathbf{\Omega},t)$

$$R_x(\mathbf{r},E,\mathbf{\Omega},t) = v(E)\Sigma_x(\mathbf{r},E,\mathbf{\Omega},t)\cdot N(\mathbf{r},E,\mathbf{\Omega},t)$$

  - The expected number of interactions of type $x$ made with nuclei by neutrons of speed $v$ corresponding energy E, at position **r**, direction $\mathbf{\Omega}$ and time $t$, per unit volume per unit energy per unit solid angle <u>per unit time</u>
  - <u>$v(E)\Sigma_x(\mathbf{r},E,\mathbf{\Omega},t)N(\mathbf{r},E,\mathbf{\Omega},t)\mathrm{d}\mathbf{r}\mathrm{d}E\mathrm{d}\mathbf{\Omega}$</u> is the reaction rate of type $x$ in the volume element d**r** about **r** by neutrons with energies in d$E$ about $E$ and directions within d$\mathbf{\Omega}$ about $\mathbf{\Omega}$ at time $t$.

- Reaction rate density = $R_x(\mathbf{r},E,t)$

$$R_x(\mathbf{r},E,t) = \int_{4\pi} v(E)\Sigma_x(\mathbf{r},E,\mathbf{\Omega},t)N(\mathbf{r},E,\mathbf{\Omega},t)d\mathbf{\Omega}$$

  - The expected number of interactions of type $x$ made with nuclei by neutrons of speed $v$ corresponding energy $E$, at position **r** and time $t$, per unit volume per unit energy <u>per unit time</u>

# Neutron Density vs. Neutron Flux

- The product of $vN$ arising in the definitions of the reaction rate densities occurs very frequently in reactor theory, and therefore it is given a special name [2]:

$$\phi = vN = \text{ neutron flux [cm}^{-2}\text{ sec}^{-1}]$$

- Although it will certainly prove covenient to work with $\phi$ rather than $N$ (since then one does not have to worry about including the neutron speed $v$ in the reaction rate densities), the tradition in nuclear engineering of referring to this quantity as the neutron "flux" is very misleading. For $\phi$ is not at all like the fluxes encountered in electromagnetic theory or heat conduction, since these latter fluxes are vector quantities, whereas $\phi$ is a scalar quantity. Actually the "neutron current" $J$ corresponds more closely to the conventional flux.
- Think of the neutron flux as simply a convenient mathematical variable (speed $\times$ density) to use in computing reaction rates:

$$R_x = v\Sigma_x N = \Sigma_x \phi$$

# Derivation of Linear Boltzmann Transport Equation

- The *linear Boltzmann transport equation* serves to precisely describe **particle balance in which the rate of accumulation of particles is equal to the difference between their rates of production and removal**.
- If $N(\mathbf{r},E,\Omega,t)$ is the distribution of particles as a function of the seven phase-space variables,

$N(\mathbf{r},E,\Omega,t)\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t \equiv$ the number of particles
in volume $\Delta\mathbf{r}$ about $\mathbf{r}$,
with energy in $\Delta E$ about $E$,
moving in direction $\Delta\Omega$ about $\Omega$,
in time interval $\Delta t$ about $t$.

- A pseudo equation for particle balance in a phase space volume of $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$ can be written as

[① Accumulation of particles in a phase space volume] = − [Amounts of removal by leakage ② and collisions ③] + [Amounts of production by scattering, ④ fission ⑤ and fixed source ⑥] ………… (1)

# Derivation of BTE (Contd.)

| | | | | |
|---|---|---|---|---|
| ① Accumulation of particles in a phase space volume | = − | Amounts of removal by leakage ② and collisions ③ | + | Amounts of production by scattering, ④ fission ⑤ and fixed source ⑥ |

…… (1)

- By dividing Eq. (1) by $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$, a balance equation for the angular neutron density can be obtained.

| | | | | |
|---|---|---|---|---|
| Accumulation rate of particles at the phase space ①' | = − | Rate of removal by leakage ②' and collisions ③' | + | Rate of production by scattering, ④' fission ⑤' and fixed source ⑥' |

# Questions

***What if the medium is vacuum?***

| ①Accumulation of particles in a phase space volume | = − | Amounts of removal by leakage ② and collisions ✗ | + | Amounts of production by scattering, ✗ fission ✗ and fixed source ⑥ |
|---|---|---|---|---|

***What if the medium is pure absorber?***

| ①Accumulation of particles in a phase space volume | = − | Amounts of removal by leakage ② and absorptions ③ | + | Amounts of production by scattering, ✗ fission ✗ and fixed source ⑥ |
|---|---|---|---|---|

# ① Accumulation of Particles in $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$

- Accumulation of particles in the phase space volume of $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$ about $(\mathbf{r},E,\Omega,t)$, or change of particle numbers during $\Delta t$ in the volume of $\Delta\mathbf{r}\Delta E\Delta\Omega$, can be represented as

$$\text{①} = N(\mathbf{r},E,\mathbf{\Omega},t+\Delta t)\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega} - N(\mathbf{r},E,\mathbf{\Omega},t)\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}$$

$$= \left[N(\mathbf{r},E,\mathbf{\Omega},t+\Delta t) - N(\mathbf{r},E,\mathbf{\Omega},t)\right]\cdot\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega} \qquad \text{(2)}$$

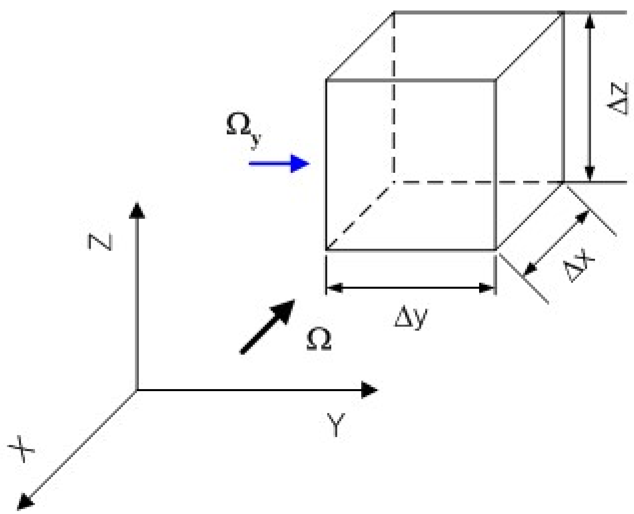
- Then the **accumulation rate** of particles at $(\mathbf{r},E,\Omega,t)$ per unit volume per unit energy per unit steradian is obtained, by dividing Eq. (2) by $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$ and taking the limit as the phase space volume approaches zero, as

$$\text{①'} = \lim_{\Delta t\to 0}\frac{\left[N(\mathbf{r},E,\mathbf{\Omega},t+\Delta t) - N(\mathbf{r},E,\mathbf{\Omega},t)\right]\cdot \cancel{\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}}}{\cancel{\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}}\,\Delta t} \qquad \text{(2)'}$$

$$= \frac{\partial}{\partial t}N(\mathbf{r},E,\mathbf{\Omega},t)$$

# ② Leakage Amount in ΔrΔ*E*ΔΩΔ*t*

- Let us consider a Cartesian incremental volume $\Delta\mathbf{r}=\Delta x\Delta y\Delta z$ as below.




$$\mathbf{\Omega} \equiv \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$$

- Then, the particle amount **to enter the volume** through the face of area $\Delta x\Delta z$ at $y$ becomes

$$vN(x,y,z,E,\mathbf{\Omega},t)\mathbf{\Omega}\cdot\Delta\mathbf{A}\Delta E\Delta\mathbf{\Omega}\Delta t = vN(x,y,z,E,\mathbf{\Omega},t)\left(\mathbf{\Omega}\cdot\mathbf{j}\Delta x\Delta z\right)\Delta E\Delta\mathbf{\Omega}\Delta t$$

$$= v\Omega_y N(x,y,z,E,\mathbf{\Omega},t)\Delta x\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t$$

- Similarly, the number of particles to leave the volume through the face at $y+\Delta y$ is $v\Omega_y N(x,y+\Delta y,z,E,\mathbf{\Omega},t)\Delta x\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t$.

# ② Leakage Amount (Contd.)

- The difference between outflowing and inflowing particles, or the particle leakage, though the area of $\Delta x \Delta z$ becomes

$$v\Omega_y \left[ N(x, y+\Delta y, z, E, \mathbf{\Omega}, t) - N(x, y, z, E, \mathbf{\Omega}, t) \right] \Delta x \Delta z \Delta E \Delta \mathbf{\Omega} \Delta t$$

- By similarly expressing the leakages through areas of $\Delta y \Delta z$ and $\Delta x \Delta y$ and summing all the leakage amounts, we can obtain

$$\begin{aligned} ② = \; & v\Omega_x \left[ N(x+\Delta x, y, z, E, \mathbf{\Omega}, t) - N(x, y, z, E, \mathbf{\Omega}, t) \right] \Delta y \Delta z \Delta E \Delta \mathbf{\Omega} \Delta t \\ & + v\Omega_y \left[ N(x, y+\Delta y, z, E, \mathbf{\Omega}, t) - N(x, y, z, E, \mathbf{\Omega}, t) \right] \Delta x \Delta z \Delta E \Delta \mathbf{\Omega} \Delta t \qquad \cdots\cdots (3) \\ & + v\Omega_z \left[ N(x, y, z+\Delta z, E, \mathbf{\Omega}, t) - N(x, y, z, E, \mathbf{\Omega}, t) \right] \Delta x \Delta y \Delta E \Delta \mathbf{\Omega} \Delta t \end{aligned}$$

# ② Leakage Rate

- Then the leakage rate at the seven-dimensional phase space point, ($\mathbf{r}$,$E$,$\mathbf{\Omega}$,$t$), in the limit of vanishingly small $\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}\Delta t$ becomes

②' =

$$\lim_{\Delta x\Delta y\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t\to 0}\frac{\begin{gathered}v\Omega_x\left[N(x+\Delta x,y,z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)\right]\Delta y\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t\\+v\Omega_y\left[N(x,y+\Delta y,z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)\right]\Delta x\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t\\+v\Omega_z\left[N(x,y,z+\Delta z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)\right]\Delta x\Delta y\Delta E\Delta\mathbf{\Omega}\Delta t\end{gathered}}{\Delta x\Delta y\Delta z\Delta E\Delta\mathbf{\Omega}\Delta t}$$

$$=\lim_{\Delta x\to 0}v\Omega_x\frac{N(x+\Delta x,y,z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)}{\Delta x}$$

$$+\lim_{\Delta y\to 0}v\Omega_y\frac{N(x,y+\Delta y,z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)}{\Delta y}$$

$$+\lim_{\Delta z\to 0}v\Omega_z\frac{N(x,y,z+\Delta z,E,\mathbf{\Omega},t)-N(x,y,z,E,\mathbf{\Omega},t)}{\Delta z}$$

$$=v\left(\Omega_x\frac{\partial}{\partial x}+\Omega_y\frac{\partial}{\partial y}+\Omega_z\frac{\partial}{\partial z}\right)N(x,y,z,E,\mathbf{\Omega},t) \quad \cdots\cdots\cdots\ (3)'$$

$$=v\mathbf{\Omega}\cdot\nabla N(\mathbf{r},E,\mathbf{\Omega},t)$$

$$=\mathbf{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\mathbf{\Omega},t) \quad \cdots\cdots\cdots\ (4)$$

# ③ Removal Rate by Collisions

- From the definition of the angular reaction rate density, the amount by which particles are lost due to collisions of any kind with the nuclei comprising the medium in the phase space volume $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$ can be expressed as

$$③ = v(E)\Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)N(\mathbf{r},E,\mathbf{\Omega},t)\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}\Delta t$$
$$= \Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t)\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}\Delta t \quad \cdots\cdots \quad (5)$$

- By dividing Eq. (5) by $\Delta\mathbf{r}\Delta E\Delta\Omega\Delta t$, the removal rate by collisions can be written as

Here $\Sigma_t(\mathbf{r},E,\Omega,t)$ is the macroscopic total cross section of the medium defined such that Σds is the probability of a collision in a path length, ds.

$$③' = \frac{v(E)\Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)N(\mathbf{r},E,\mathbf{\Omega},t)\cancel{\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}\Delta t}}{\cancel{\Delta\mathbf{r}\Delta E\Delta\mathbf{\Omega}\Delta t}}$$
$$= v(E)\Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)N(\mathbf{r},E,\mathbf{\Omega},t) = \Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t) \quad \cdots\cdots \quad (5)'$$

# ④ Production Rate by Scattering

- The production rate of neutrons by reactions except the fission reaction can be expressed as

$$\int_{4\pi} d\mathbf{\Omega}' \int_{E'} dE' \sum_{x \neq fis.} \nu_x(E)\Sigma_x(\mathbf{r}, E' \rightarrow E, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}, t)\Phi(\mathbf{r}, E', \mathbf{\Omega}', t) \quad \text{......} \quad (6)$$

$\nu_x$ = the average number of neutrons produced from a reaction of type $x$,

$\Sigma_x(\mathbf{r}, E' \rightarrow E, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}, t)$ = the macroscopic differential cross section of a reaction of type $x$ by which a neutron transferred from energy $E$' and direction Ω' to energy $E$ and direction Ω

- By using the scattering cross section, Eq. (6) can be written briefly as

④' = $$S_s(\mathbf{r}, E, \mathbf{\Omega}, t) = \int_{4\pi} d\mathbf{\Omega}' \int_{E'} dE' \, \Sigma_s(\mathbf{r}, E' \rightarrow E, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}, t)\Phi(\mathbf{r}, E', \mathbf{\Omega}', t) \quad \text{......} \quad (7)$$

$\Sigma_s(\mathbf{r}, E' \rightarrow E, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}, t)$ = the macroscopic differential cross section for scattering neutrons from energy $E$' and direction Ω' to energy $E$ and direction Ω.

$\Sigma_s(\mathbf{r}, E' \rightarrow E, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}, t) d\mathbf{r} dE d\mathbf{\Omega} dE' d\mathbf{\Omega}'$ = the expected number of neutrons in d**r** at **r** scattered into d$E$ at $E$ and dΩ at Ω per unit time at time $t$ collisions of neutrons with energy in d$E'$ at $E'$ and direction in dΩ′ at Ω′ in the same volume element at **r**

# ⑤ Production Rate by Fission

- The fission contribution to the source be given by

⑤' = $S_F(\mathbf{r},E,\mathbf{\Omega},t) =$

$$\int_{4\pi} d\mathbf{\Omega}' \int_{E'} dE' \chi(\mathbf{r},E' \rightarrow E)\nu(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\mathbf{\Omega}' \rightarrow \mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t) \quad \cdots\cdots \quad (8)$$

$\chi(\mathbf{r},E' \rightarrow E)$ = the probability of neutrons appearing at energy $E$ as a result of a fission caused by a particle of energy $E$' at point **r**.

$\nu(\mathbf{r},E)$ = the average number of neutrons emerging from a fission at point **r**

$\Sigma_f(\mathbf{r},E',\mathbf{\Omega}' \rightarrow \mathbf{\Omega},t)$ = the macroscopic cross section for fission induced by neutrons with energy $E$' and direction **Ω**' with emergent neutrons from the fission having direction **Ω**.

# ⑥ Fixed Source

- The inhomogeneous (or fixed) source density is denoted by $Q(\mathbf{r},E,\mathbf{\Omega},t)$

$$⑥' = Q(\mathbf{r},E,\mathbf{\Omega},t) \qquad \cdots\cdots (9)$$

# Time-Dependent NTE with Delayed Neutron Neglected

- Inserting the derived terms into Eq. (1), the neutron transport equation can be written as

$$\frac{\partial N(\mathbf{r},E,\mathbf{\Omega},t)}{\partial t} = -\mathbf{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\mathbf{\Omega},t) - \Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t) + S(\mathbf{r},E,\mathbf{\Omega},t); \quad \cdots\cdots \quad (10)$$

$$S(\mathbf{r},E,\mathbf{\Omega},t) = S_s(\mathbf{r},E,\mathbf{\Omega},t) + S_F(\mathbf{r},E,\mathbf{\Omega},t) + Q(\mathbf{r},E,\mathbf{\Omega},t),$$

$$S_s(\mathbf{r},E,\mathbf{\Omega},t) = \int_{4\pi} d\mathbf{\Omega}' \int_{E'} dE'\, \Sigma_s(\mathbf{r},E' \to E,\mathbf{\Omega}' \to \mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t),$$

$$S_F(\mathbf{r},E,\mathbf{\Omega},t) = \int_{4\pi} d\mathbf{\Omega}' \int_{E'} dE'\, \chi(\mathbf{r},E' \to E)\nu(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\mathbf{\Omega}' \to \mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t)$$

| Accumulation rate of particles at the phase space ①' | = − | Rate of removal by leakage ②' and collisions ③' | + | Rate of production by scattering, ④' fission ⑤' and fixed source ⑥' |
|---|---|---|---|---|

······ (1)

# Cf. Exact Time-Dependent NTE

- Considering the delayed neutron generation, the time-dependent neutron transport equation (NTE) can be written by

$$
\begin{aligned}
\frac{1}{v}\frac{\partial \Phi(\mathbf{r},E,\mathbf{\Omega},t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\mathbf{\Omega},t) - \Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t) \\
&+ \int_{4\pi} d\mathbf{\Omega}'\int_{E'} dE'\, \Sigma_s(\mathbf{r},E'\to E,\mathbf{\Omega}'\to\mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t) \\
&+ \sum_j \int_{4\pi} d\mathbf{\Omega}'\int_{E'} dE'\,\chi_p^j(E'\to E)\left(1-\beta^j\right)\nu^j(E')\Sigma_f^j(\mathbf{r},E',\mathbf{\Omega}'\to\mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t) \\
&+ \sum_{i=1}^{I}\frac{\chi_i(E)}{4\pi}\lambda_i C_i(\mathbf{r},t) \\
&+ Q(\mathbf{r},E,\mathbf{\Omega},t) \qquad \equiv S(\mathbf{r},E,\Omega,t)
\end{aligned}
$$

$$
\frac{\partial C_i(\mathbf{r},t)}{\partial t} = \sum_j \beta_i^j \int_{4\pi} d\mathbf{\Omega}\int_E dE\;\nu^j(E)\Sigma_f^j(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t) - \lambda_i C_i(\mathbf{r},t)
$$

# Time-Independent Boltzmann Transport Equation

- From the Eq. (10), the time-independent (or steady-state) neutron transport equation can be written as

$$\begin{aligned}
&\mathbf{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\mathbf{\Omega})+\Sigma_t(\mathbf{r},E,\mathbf{\Omega})\Phi(\mathbf{r},E,\mathbf{\Omega}) \\
&\quad=\int_{4\pi}d\mathbf{\Omega}'\int_{E'}dE'\,\Sigma_s(\mathbf{r},E'\to E,\mathbf{\Omega}'\to\mathbf{\Omega})\Phi(\mathbf{r},E',\mathbf{\Omega}') \\
&\quad+\int_{4\pi}d\mathbf{\Omega}'\int_{E'}dE'\,\chi(\mathbf{r},E'\to E)\nu(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\mathbf{\Omega}'\to\mathbf{\Omega})\Phi(\mathbf{r},E',\mathbf{\Omega}') \\
&\quad+Q(\mathbf{r},E,\mathbf{\Omega}) \qquad\cdots\cdots (11)
\end{aligned}$$

# (Steady-State) Eigenvalue Equation

- For a nuclear system with no external sources such as a commercial reactor core, it is obvious that the time-independent NTE of Eq. (11) has a trivial solution of $\Phi(\mathbf{r},E,\boldsymbol{\Omega})=0$ for all $\mathbf{r}$, $E$, and $\boldsymbol{\Omega}$.

$$\begin{aligned}\boldsymbol{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\boldsymbol{\Omega})+\Sigma_t(\mathbf{r},E,\boldsymbol{\Omega})\Phi(\mathbf{r},E,\boldsymbol{\Omega})\\ =\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\,\Sigma_s(\mathbf{r},E'\to E,\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\\ +\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\chi(\mathbf{r},E'\to E)\nu(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\end{aligned} \quad \text{(12)}$$

- In order to obtain a non-trivial solution for the steady-state nuclear reactor core, $\lambda$ eigenvalue or $k$ eigenvalue ($\lambda=1/k$) should be introduced into Eq. (12) as

$$\begin{aligned}\boldsymbol{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\boldsymbol{\Omega})+\Sigma_t(\mathbf{r},E,\boldsymbol{\Omega})\Phi(\mathbf{r},E,\boldsymbol{\Omega})\\ =\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\,\Sigma_s(\mathbf{r},E'\to E,\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\\ +\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\chi(\mathbf{r},E'\to E)\nu^{fictitous}(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\end{aligned} \quad \text{(13a)}$$

$$\begin{aligned}\boldsymbol{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\boldsymbol{\Omega})+\Sigma_t(\mathbf{r},E,\boldsymbol{\Omega})\Phi(\mathbf{r},E,\boldsymbol{\Omega})\\ =\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\,\Sigma_s(\mathbf{r},E'\to E,\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\\ +\frac{1}{k}\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\chi(\mathbf{r},E'\to E)\nu(\mathbf{r},E')\Sigma_f(\mathbf{r},E',\boldsymbol{\Omega}'\to\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}')\end{aligned} \quad \text{(13b)}$$