Phase and group velocity

SUPERPOSED WAVE OF DIFFERENT FREQUENCIES (VERY SMALL DIFFERENCE) AND BEATS

We will now discuss the superposition of two waves that have same vibration direction, same amplitude **a**, but different frequency and wave number(ω_1, k_1, ω_2, k_2). However, the frequency difference is very small. This will generate the very interesting "beat" phenomenon.

Since the phase difference between the vibrations is continually changing, the specification of some initial nonzero phase difference is in general not of major significance in this case.

So we can suppose that the individual vibrations have an initial phase of 0, and hence can be written as:

$$E_1 = a\cos(\omega_1 t - k_1 z)$$

$$E_2 = a\cos(\omega_2 t - k_2 z)$$

Then the sum of these two waves is:

$$E = E_1 + E_2 = a[\cos(\omega_1 t - k_1 z) + \cos(\omega_2 t - k_2 z)]$$

Using the following triangular formula

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

We get

$$E = 2a\cos\underline{\frac{1}{2}[(\omega_1-\omega_2)t-(k_1-k_2)z]}\cos\underline{\frac{1}{2}[(\omega_1+\omega_2)t-(k_1+k_2)z]}$$

We then introduce the notation of average angular frequency $\bar{\omega}$ and average wave number $\bar{k}$

$$\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$$

$$\bar{k} = \frac{1}{2}(k_1 + k_2)$$

And modulation frequency ω_m and modulation wave number k_m

$$\omega_m = \frac{1}{2}(\omega_1 - \omega_2)$$

$$k_m = \frac{1}{2}(k_1 - k_2)$$

We then get

$$E = \underbrace{2a\cos(\omega_m t - k_m z)}_{A}\cos(\bar{\omega} t - \bar{k} z)$$

We can make

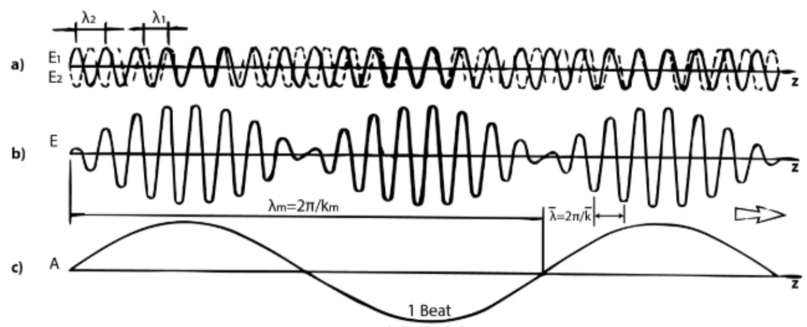
$$\boxed{A = 2a\cos(\omega_m t - k_m z)} \quad \textit{Amplitude}$$

Then we get

$$\boxed{E = A\cos(\bar{\omega} t - \bar{k} z)}$$

This means that the resultant superposed wave has an angular frequency $\bar{\omega}$, and its amplitude varies between **0** and **2a** with time **t** and position **z**.

The following picture shows the superposition result. Since light waves have very high frequency, if $\omega_1 \approx \omega_2$, then $\bar{\omega} \gg \omega_m$, which means that **A** varies slowly but **E** varies extremely fast.



Phase and group velocity

PHASE VELOCITY AND GROUP VELOCITY

Clearly the velocity of a monochrome light equals its equiphase surface propagation velocity. However, in the case of a superposed wave, we need to carefully define its propagation velocity.

Let's continue using the superposed wave equation from above:

$$E=2\,a\cos(\omega_m t-k_m z)\cos(\bar{\omega} t-\bar{k} z)$$

The superposed wave has two propagation velocities: equiphase surface propagation velocity (called Phase Velocity **Vp**), and equiamplitude surface propagation velocity (called Group Velocity **Vg** as defined by Rayleigh).

$$\boxed{E=A\cos(\bar{\omega} t-\bar{k} z)}$$

PHASE VELOCITY OF THE SUPERPOSED WAVE:

Phase velocity Vp can be concluded by keeping the phase a constant:

$$\bar{\omega} t-\bar{k} z=constant$$

$$z=\frac{\bar{\omega} t}{\bar{k}}-\frac{constant}{\bar{k}}$$

Then by doing derivative of z we get the Phase Velocity **Vp** of the superposed wave:

$$\boxed{V_p=\frac{dz}{dt}=\frac{\bar{\omega}}{\bar{k}}}$$

$$\boxed{A=2\,a\cos(\omega_m t-k_m z)}$$

GROUP VELOCITY OF THE SUPERPOSED WAVE:

Similarly we can get the Group Velocity **Vg** by keeping the amplitude a constant:

$$\omega_m t-k_m z=constant$$

Following the same steps, we get the Group Velocity of the superposed wave:

$$V_g=\frac{dz}{dt}=\frac{\omega_m}{k_m}=\frac{\omega_1-\omega_2}{k_1-k_2}=\frac{\Delta\omega}{\Delta k}$$

when **Δω** is very small, we then get:

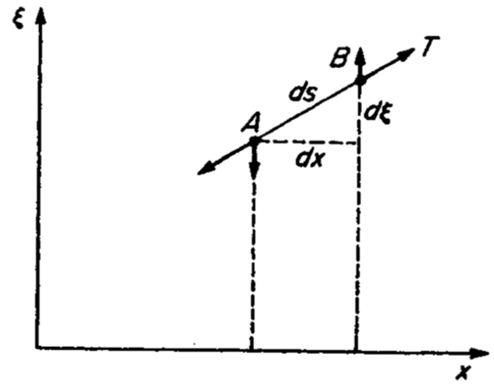
$$V_g=\frac{\partial\omega}{\partial k}$$

So **Vg** is the partial derivative of **ω**.

Transverse waves in an infinite string

- Transverse waves in an infinite string consider string under tension.

Element of the string



T: tension
ξ: displacement
L: length of string

If $d\xi$ is very small,

$$ds = \sqrt{(dx)^2 + (d\xi)^2}$$
$$\approx dx$$

Upward force at A

$$F_{up} = -T\sin\theta$$
$$= -T\frac{d\xi}{ds} \approx -T\frac{d\xi}{dx}$$

Upward force at B

$$F_{up} = T\left\{\frac{d\xi}{dx} + \frac{d}{dx}\left(\frac{d\xi}{dx}\right)dx\right\}$$

Net upward force $F_{up} = T\frac{d}{dx}\left(\frac{d\xi}{dx}\right)dx = T\frac{d^2\xi}{dx^2}dx = \rho\, dx\,\frac{d^2\xi}{dt^2}$ ρ: linear density

Transverse waves in an infinite string

∴ Force balance

$$\underbrace{T\frac{d^2\xi}{dx^2}dx}_{F_{up}} = \underbrace{\rho\, dx}_{m}\,\underbrace{\frac{d^2\xi}{dt^2}}_{a}\,, \qquad \rho\text{:linear density}$$

$$\boxed{\frac{d^2\xi}{dt^2} = \frac{T}{\rho}\frac{d^2\xi}{dx^2}}$$: wave equation

① Solutions are the harmonic waves

② Determine the dispersion relation, $\omega(k)$

③ Apply the boundary conditions

④ Determine the allowed ω_n

Transverse waves in an infinite string

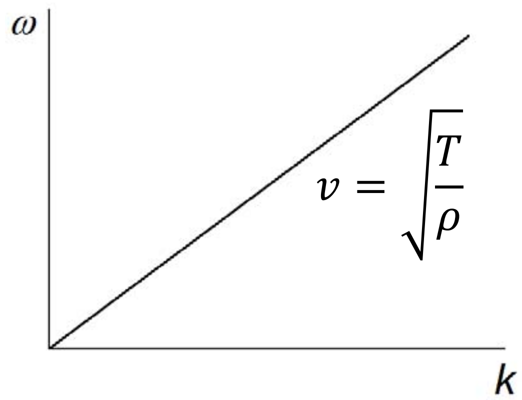
- Assume

$$\xi(x,t) = A\exp\{i(kx - wt)\} + B\exp\{i(kx + wt)\}$$

- Then,

$$\frac{d^2\xi}{dx^2} = -k^2\xi, \qquad \frac{d^2\xi}{dt^2} = -\omega^2\xi$$

$$\Rightarrow\ v = \frac{\omega}{k} = \left(\frac{T}{\rho}\right)^{1/2}$$



$$\frac{d^2\xi}{dt^2} = v^2\frac{d^2\xi}{dx^2}$$

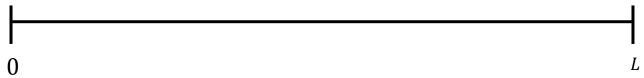
$$v_{ph} = const. = v_{gr} \Rightarrow \text{non−dispersive system}$$

$$\omega = \sqrt{T/\rho}\,k \quad \text{for an infinite string}$$

- All frequencies and all wavelengths are allowed.

Transverse waves in a finite string

- Fix the ends of the string.



- Boundary conditions $\xi(0) = 0 \quad \& \quad \xi(L) = 0$
- From

$$\xi(0) = 0 = A + B \qquad - (1)$$

$$\xi(L) = 0 = Ae^{ikL} + Be^{-ikL} \qquad - (2)$$

$$0 = e^{ikL} - e^{-ikL}$$

$$0 = 2i\sin(kL) \qquad k = \frac{2\pi}{\lambda} \qquad n\frac{\lambda}{2} = L$$

$$\therefore k = n\pi/L, \qquad n = 1, 2, \cdots$$

$\therefore$ Due to boundary conditions (confinement of the wave between 0 and L on the x-axis), frequencies that can exist in the string are limited.

Transverse waves in a finite string

$$\xi_n = A\exp\left\{i\left(\frac{n\pi}{L}x - \frac{n\pi}{L}\sqrt{T/\rho}\,t\right)\right\} - A\exp\left\{-i\left(\frac{n\pi}{L}x + \frac{n\pi}{L}\sqrt{T/\rho}\,t\right)\right\}$$

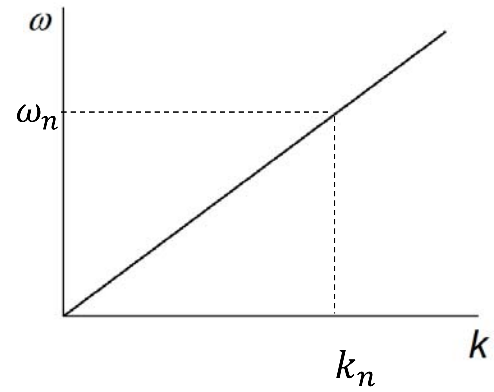
$$\text{or } \xi_n = C\sin\frac{n\pi}{L}x\exp\left\{-i\left(\frac{n\pi}{L}\sqrt{T/\rho}\,t\right)\right\}$$

"a standing wave", induced by the boundary condition

$$\omega_n = \frac{n\pi}{L}\sqrt{T/\rho}$$: allowed frequencies

"normal modes"

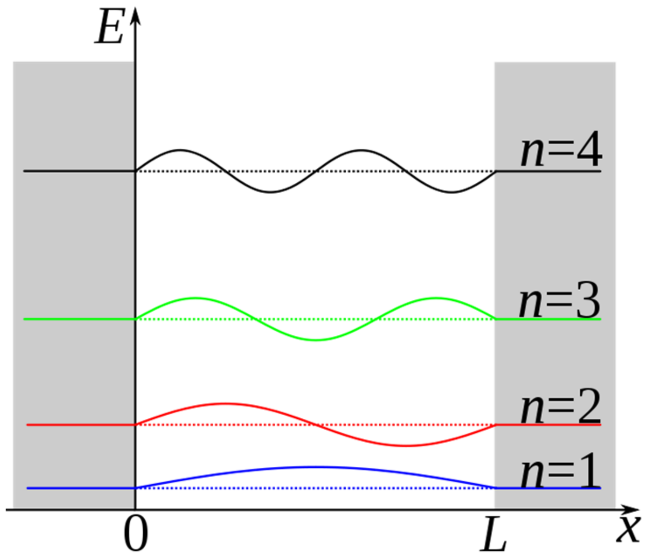
- Confinement of waves induces the "quantization" of frequency.
- General solution $= \sum_n \xi_n(x,t)$



Simple Applications of $n(\lambda/2) = L$

1. Particle in a box

- An electron in a one-dimensional box of the length, L (confined within the box with high potential walls)
- What energy levels are allowed for the electrons if it exhibits wave-like properties?



- Since

$$n \cdot \frac{\lambda}{2} = L$$

$$\lambda = \frac{2L}{n}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

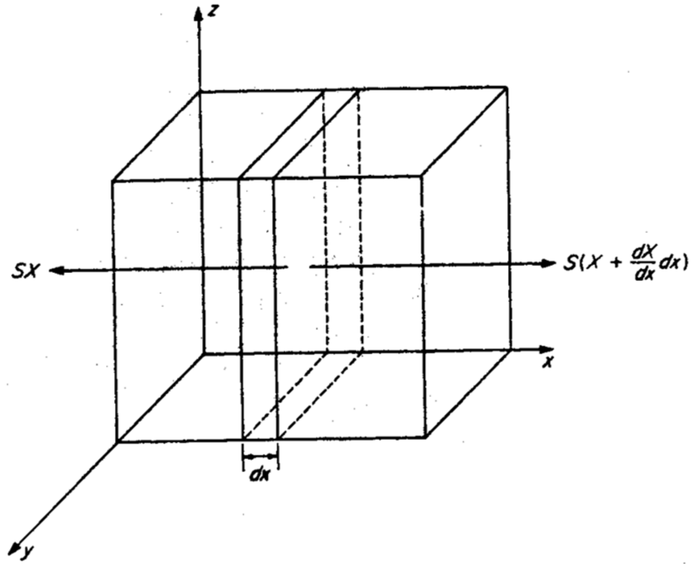
$$E_n = \frac{n^2 h^2}{8mL^2},$$ identical with the actual solution

❖ A discrete set of energy levels is allowed.

❖ These energy levels are spaced according to the square of the integers.

❖ The spacing between energy levels decreases as *L* increases.

Longitudinal waves in a rod



S : cross−sectional area

- Consider the rod in the figure with a tensile stress, X (force per unit area) acting along the x axis.
- Force balance
  - Net force in the +x direction on an element of width dx and cross-sectional area S is

$$\left(SX + S\frac{dX}{dx}dx\right) - SX = S\frac{dX}{dx}dx$$

Longitudinal waves in a rod

- Define
  - ρ: volume density of the rod
  - ξ: displacement of any cross-section in the *x*-direction
  - Y: Young's modulus = $X/(d\,\xi/dx)$
- Then,

$$S\frac{dX}{dx}dx = \underbrace{\rho(dx \cdot S)}_{\text{m}}\underbrace{\frac{d^2\xi}{dt^2}}_{\text{a}}$$

$$\frac{d(Y \cdot d\xi/dx)}{dx} = \rho\frac{d^2\xi}{dt^2}$$

$$\boxed{\begin{gathered}\frac{d^2\xi}{dx^2} = \frac{\rho}{Y}\frac{d^2\xi}{dt^2} \\ v = \frac{\omega}{k} = \left(\frac{Y}{\rho}\right)^{1/2}\end{gathered}}$$

(wavefunction of longitudinal waves)

cf) transverse waves

$$\frac{d^2\xi}{dt^2} = \frac{T}{\rho}\frac{d^2\xi}{dx^2}$$

$$v = \frac{\omega}{k} = \left(\frac{T}{\rho}\right)^{1/2}$$

Summary of wave systems

STRING	$\frac{\partial^2 \xi}{\partial x^2} = [1/(T/\rho)] \frac{\partial^2 \xi}{\partial t^2}$ displacement	$\xi = A \exp[i(kx - \omega t)]$ $v = \omega/k = (T/\rho)^{1/2}$
LIGHT Chap. 4	$\nabla^2 \mathcal{E} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathcal{E}}{\partial t^2} + \mu_r \mu_0 \sigma \frac{\partial \mathcal{E}}{\partial t}$ electric and magnetic fields	$\mathcal{E} = \mathcal{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ $1/v^{*2} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 + i\mu_r \mu_0 \sigma/\omega$
LATTICE WAVES Chap. 3	$\eta \xi_{r-1} - 2\eta \xi_r + \eta \xi_{r+1} = \frac{\partial^2 \xi}{\partial t^2}$ displacement	$\xi_r = A \exp[i(kra - \omega t)]$ $\omega = 2\eta^{1/2} \lvert \sin ka/2 \rvert$
ELECTRON WAVES Chap. 5	$\frac{\partial^2 \Psi}{\partial x^2} = (2m/\hbar^2) V \Psi - i(2m/\hbar) \frac{\partial \Psi}{\partial t}$ Just mathematical function	$\Psi = A \exp[i(kx - \omega t)]$ If $V = 0$, $\omega = \hbar k^2/2m$