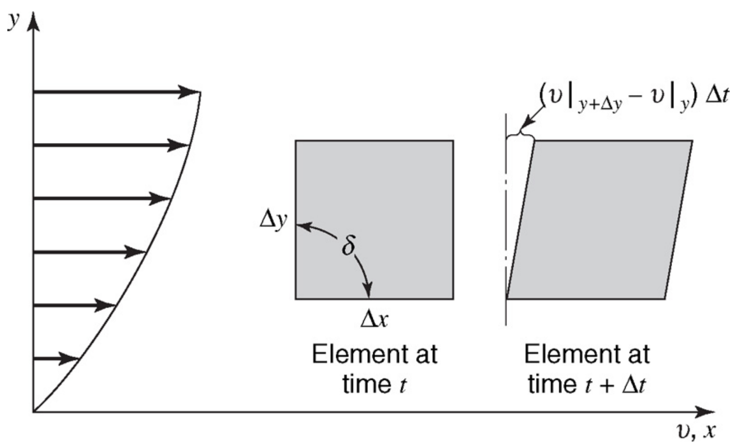
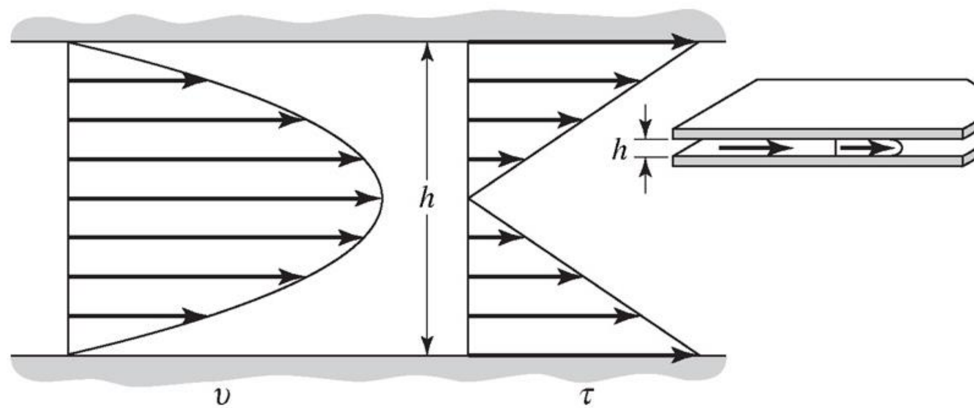
Shear stress in laminar flow

elastic solid $\tau = G\gamma$

Newtonian fluid $\tau = \mu \dfrac{dv}{dy}$

viscosity is the property of a fluid to resist the rate at which deformation takes place when the fluid is acted upon by shear forces





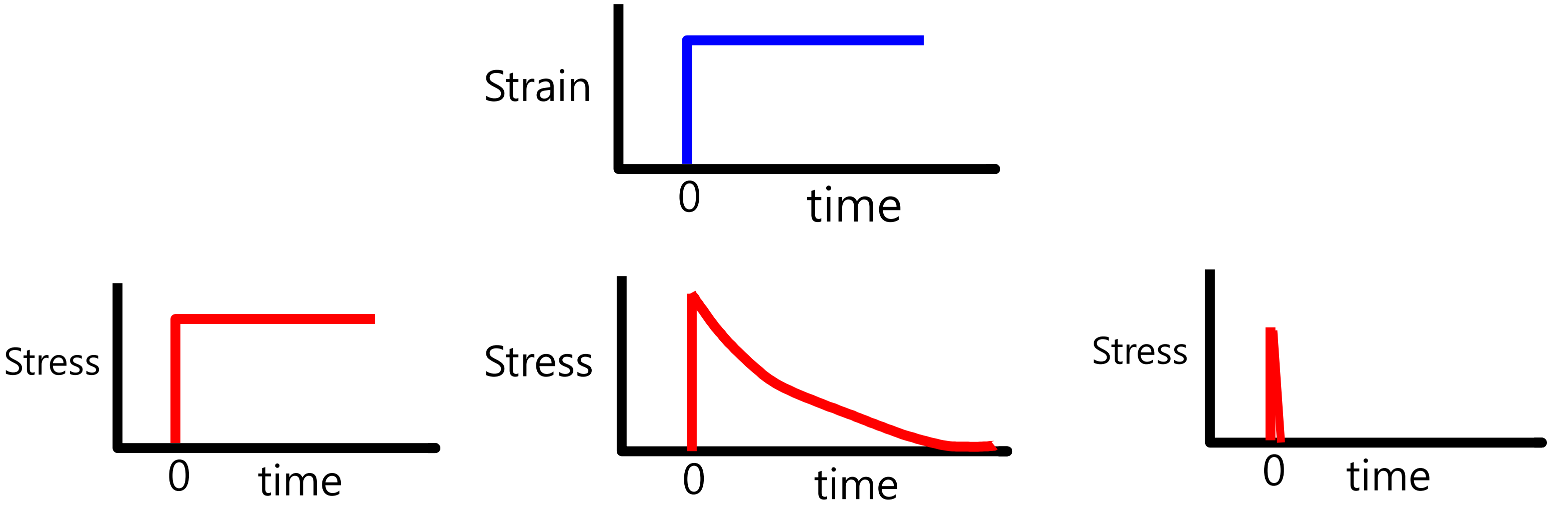
Solid Like ---------- Liquid Like

Ideal Solid ----- Most Materials ----- *Ideal Fluid*

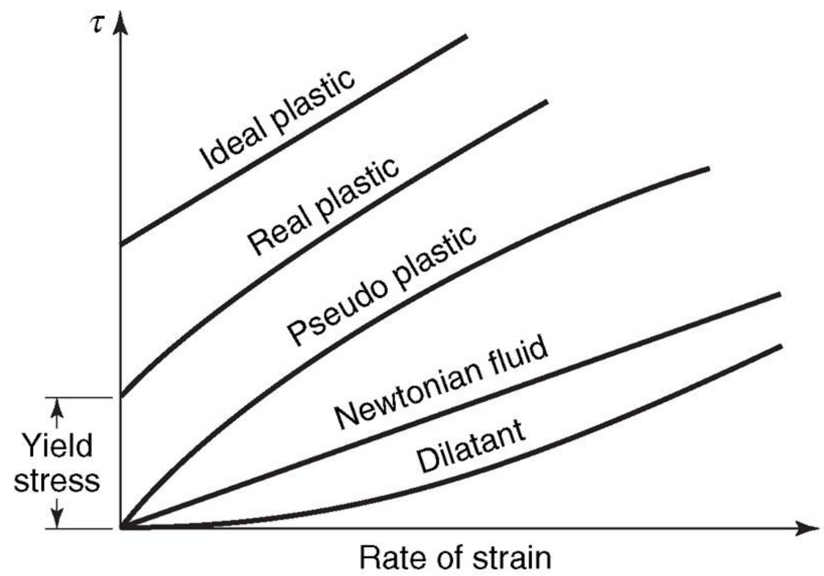
Purely Elastic ----- *Viscoelastic* ----- *Purely Viscous*

$$\tau = G\gamma \qquad\qquad \tau = \eta\dot{\gamma}$$

$$\tau_p + We\left(\frac{\partial \tau_p}{\partial t} + u\cdot\nabla\tau_p - \left(\nabla u\right)^T\cdot\tau_p - \tau_p\cdot\nabla u\right) = \beta\left(\nabla u + \left(\nabla u\right)^T\right)$$



non-Newtonian fluid



no-slip condition;

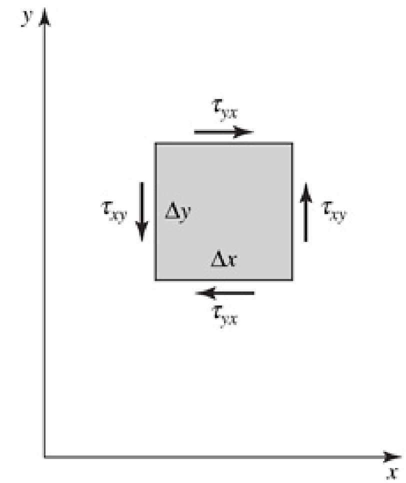
when the boundary is a stationary wall, the layer of fluid next to the wall is at rest
if the boundary is moving, the fluid moves at the velocity of the boundary
when viscous effects are neglected (inviscid fluid), the velocity component normal to the boundary is zero
; result of experimental observation

viscosity; a resistance to deformation rate $$\tau = \mu \frac{dv}{dy}$$

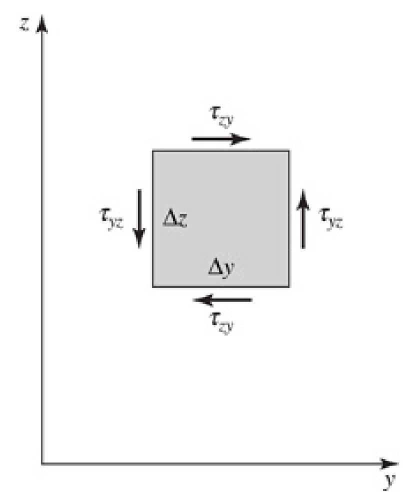
kinematic viscosity $$\nu = \frac{\mu}{\rho}$$

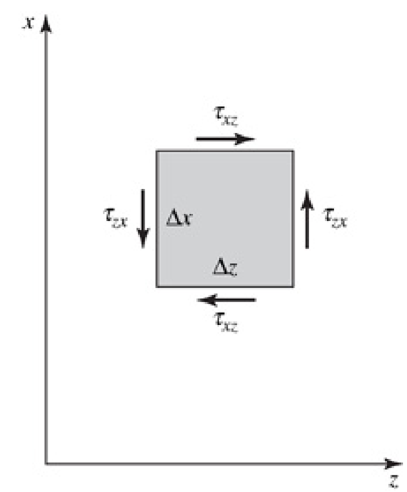
shear stress τ_{ij}

i ; direction of plane
j ; direction of force
positive when both
positive or both negative



(a)





shear stress

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)$$

normal stress

$$\sigma_{xx} = \mu\left(2\frac{\partial v_x}{\partial x} - \frac{2}{3}\nabla\cdot\mathrm{v}\right) - P$$

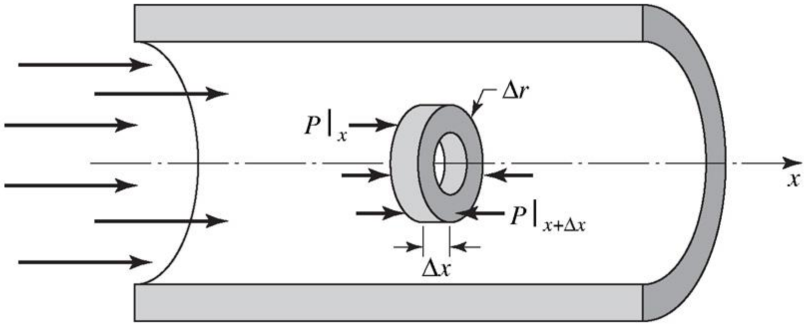
$$\sigma_{yy} = \mu\left(2\frac{\partial v_y}{\partial y} - \frac{2}{3}\nabla\cdot\mathrm{v}\right) - P$$

$$\sigma_{zz} = \mu\left(2\frac{\partial v_z}{\partial z} - \frac{2}{3}\nabla\cdot\mathrm{v}\right) - P$$

analysis of a differential fluid element

flow in a circular conduit

incompressible,
laminar,
fully developed,



apply Newton's second law to the control volume

$$\sum F_x = \iint_{\text{c.s.}} \rho v_x (\mathbf{v}\cdot\mathbf{n})\,dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_x\,dV$$

$$\sum F_x = P(2\pi r\Delta r)\big|_x - P(2\pi r\Delta r)\big|_{x+\Delta x} + \tau_{rx}(2\pi r\Delta x)\big|_{r+\Delta r} - \tau_{rx}(2\pi r\Delta x)\big|_r$$

$$\iint_{\text{c.s.}} v_x\rho(\mathbf{v}\cdot\mathbf{n})\,dA = (\rho v_x)(2\pi r\Delta r v_x)\big|_{x+\Delta x} - (\rho v_x)(2\pi r\Delta r v_x)\big|_x$$

$$dA = (dr)(r d\theta) = 2\pi r dr$$

$$-\left[P(2\pi r\Delta r)\big|_{x+\Delta x} - P(2\pi r\Delta r)\big|_x\right] + \tau_{rx}(2\pi r\Delta x)\big|_{r+\Delta r} - \tau_{rx}(2\pi r\Delta x)\big|_r = 0$$

$$-r\frac{P|_{x+\Delta x} - P|_x}{\Delta x} + \frac{(r\tau_{rx})\big|_{r+\Delta r} - (r\tau_{rx})\big|_r}{\Delta r} = 0$$

$$-r\frac{dP}{dx} + \frac{d}{dr}(r\tau_{rx}) = 0 \qquad \boxed{-\frac{dP}{dx} + \frac{1}{r}\frac{d}{dr}(r\tau_{rx}) = 0}$$

$$\boxed{-\frac{dP}{dx}+\frac{1}{r}\frac{d}{dr}\left(r\tau_{rx}\right)=0}$$

$$\tau_{rx}=\left(\frac{dP}{dx}\right)\frac{r}{2}+\frac{C_1}{r}$$

$$\tau_{rx}=\mu\frac{dv_x}{dr}=\left(\frac{dP}{dx}\right)\frac{r}{2}$$

$$v_x=\left(\frac{dP}{dx}\right)\frac{r^2}{4\mu}+C_2$$

no-slip BC

$$v_x=-\left(\frac{dP}{dx}\right)\frac{R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$$

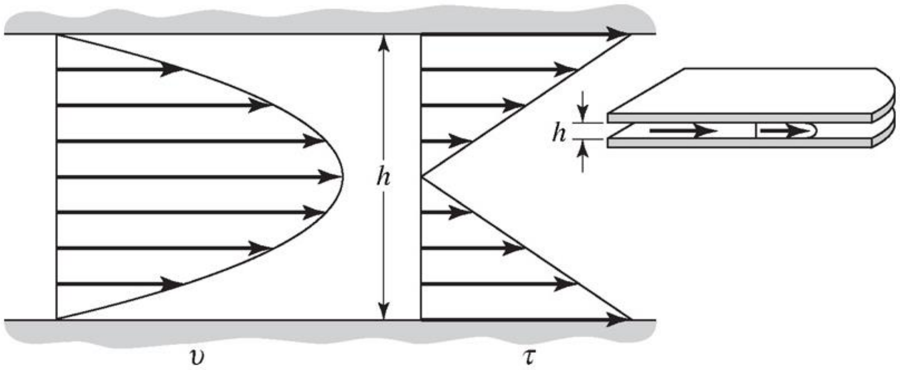
$$\boxed{v_x=v_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]}$$

$$v_{ave}=\frac{1}{A}\iint_A vdA=\frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R v_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]rdrd\theta$$

$$\boxed{Q=\int v_z dA=\int_0^R\int_0^{2\pi} v_z r\,drd\theta \;=\frac{\pi R^4}{8\mu}\frac{\Delta p}{L}}$$

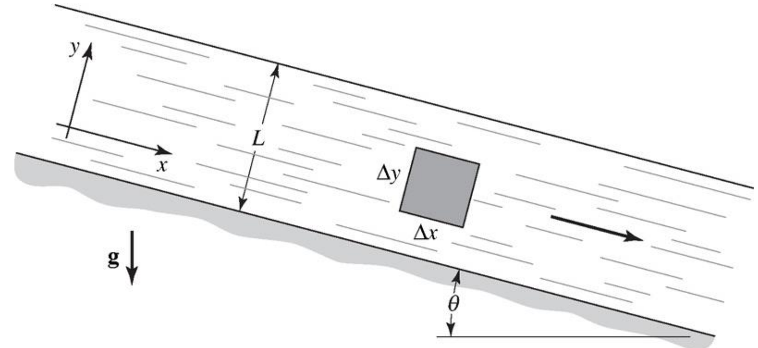
Hagen-Poiseuille equation

$$\Delta p=p_o-p_L$$



flow down an inclined plane surface

incompressible,
laminar,
fully developed,



$$\sum F_x = \iint_{\text{c.s.}} v_x \rho (\mathrm{v}\cdot\mathrm{n})\,dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_x\,dV$$

$$\sum F_x = P\Delta y\Delta z\big|_x - P\Delta y\Delta z\big|_{x+\Delta x} + \tau_{yx}\Delta x\Delta z\big|_{y+\Delta y} - \tau_{yx}\Delta x\Delta z\big|_y + \rho g\Delta x\Delta y\Delta z\sin\theta$$

$$\tau_{yx}\Delta x\Delta z\big|_{y+\Delta y} - \tau_{yx}\Delta x\Delta z\big|_y + \rho g\Delta x\Delta y\Delta z\sin\theta = 0$$

$$\frac{\tau_{yx}\big|_{y+\Delta y} - \tau_{yx}\big|_y}{\Delta y} + \rho g\sin\theta = 0$$

$$\boxed{\frac{d}{dy}\tau_{yx} + \rho g\sin\theta = 0}$$

$$\tau_{yx} = -\rho g\sin\theta\, y + C_1$$

BC; shear stress is zero at free surface

$$\tau_{yx} = \mu\frac{dv_x}{dr} = \rho g L\sin\theta\left[1 - \frac{y}{L}\right]$$

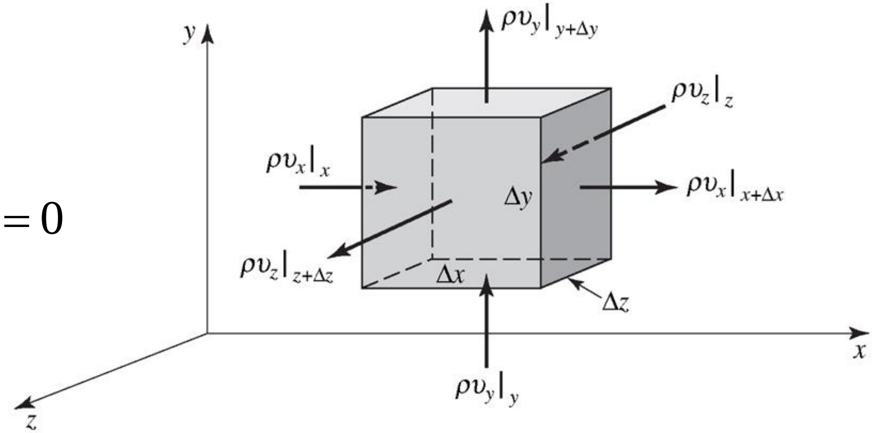
BC; no slip at surface

$$\boxed{v_x = \frac{\rho g L^2\sin\theta}{\mu}\left[\frac{y}{L} - \frac{1}{2}\left(\frac{y}{L}\right)^2\right]}$$

differential equations of fluid flow

conservation of mass

$$\left\{\begin{matrix}\text{net rate of mass}\\ \text{flux out of}\\ \text{control volume}\end{matrix}\right\}+\left\{\begin{matrix}\text{rate of accumulation}\\ \text{of mass within}\\ \text{controlvolume}\end{matrix}\right\}=0$$



mass flux through a differential control volume

$$\left(\rho v_x\big|_{x+\Delta x}-\rho v_x\big|_x\right)\Delta y\Delta z+\left(\rho v_y\big|_{y+\Delta y}-\rho v_y\big|_y\right)\Delta x\Delta z$$

$$+\left(\rho v_z\big|_{z+\Delta z}-\rho v_z\big|_z\right)\Delta x\Delta y+\frac{\partial}{\partial t}\left(\rho\Delta x\Delta y\Delta z\right)=0$$

$$\frac{\partial}{\partial x}\left(\rho v_x\right)+\frac{\partial}{\partial y}\left(\rho v_y\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)+\frac{\partial\rho}{\partial t}=0$$

$$\boxed{\nabla\cdot\rho\mathbf{v}+\mathbf{v}\frac{\partial\rho}{\partial t}=0}$$

$$\frac{\partial\rho}{\partial t}+v_x\frac{\partial\rho}{\partial x}+v_y\frac{\partial\rho}{\partial y}+v_z\frac{\partial\rho}{\partial z}+\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)=0$$

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\mathbf{v}=0}$$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}$$

Navier-Stokes equations

$$\left\{\begin{matrix}\text{sum of the external}\\ \text{forces acting}\\ \text{on the c.v.}\end{matrix}\right\} = \left\{\begin{matrix}\text{net rate of linear}\\ \text{momentum}\\ \text{efflux}\end{matrix}\right\} + \left\{\begin{matrix}\text{time rate of change}\\ \text{of linear momentum}\\ \text{within the c.v.}\end{matrix}\right\}$$

$$\boxed{\rho\frac{D\mathbf{v}}{Dt} = \nabla\bullet\boldsymbol{\sigma} + \rho\mathbf{g}}$$

$$\boxed{\rho\frac{D\mathbf{v}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}}$$

$$\rho\frac{D v_x}{Dt} = \rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}$$

$$\rho\frac{D v_y}{Dt} = \rho g_y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}$$

$$\rho\frac{D v_z}{Dt} = \rho g_z + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}$$

$$\rho\frac{D v_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\rho\frac{D v_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\rho\frac{D v_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Bernoulli equation

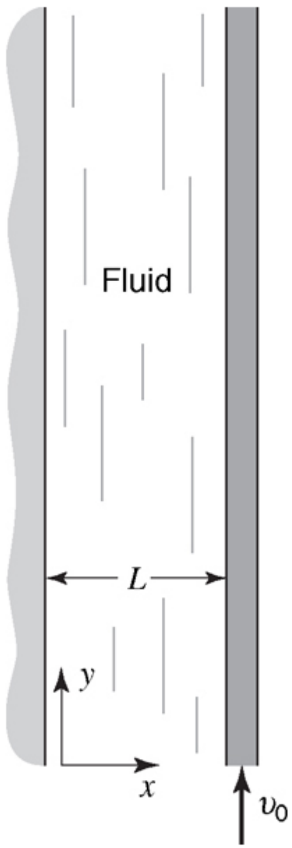
$$\boxed{\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{Dt}} = -\nabla \mathrm{P} + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}}$$

$$\boxed{\frac{v^2}{2} + \mathrm{gy} + \frac{\mathrm{P}}{\rho} = \mathrm{constant}}$$

1. inviscid flow
2. steady flow
3. incompressible flow
4. the equation applies along a streamline

flow between two vertical plates

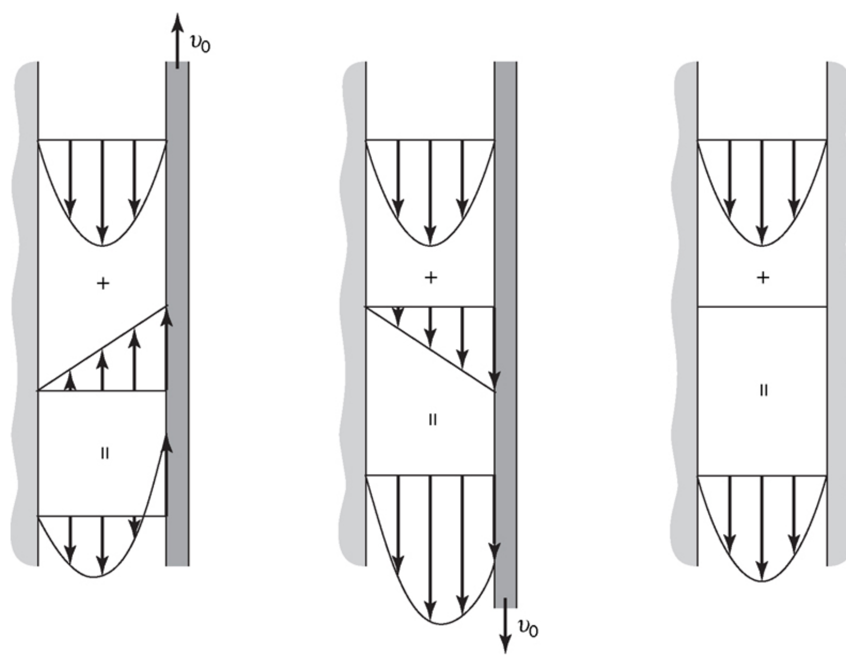
$$\boxed{\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}}$$



$$\rho \frac{D v_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right)$$

$$0 = -\rho g \quad - \frac{dP}{dy} + \mu \frac{\partial^2 v_y}{\partial x^2}$$

$$v_y = \frac{1}{2\mu} \left\{ -\rho g - \frac{dP}{dy} \right\} \left\{ Lx - x^2 \right\} + v_0 \frac{x}{L}$$



inviscid fluid flow

vorticity (rotation at a point)

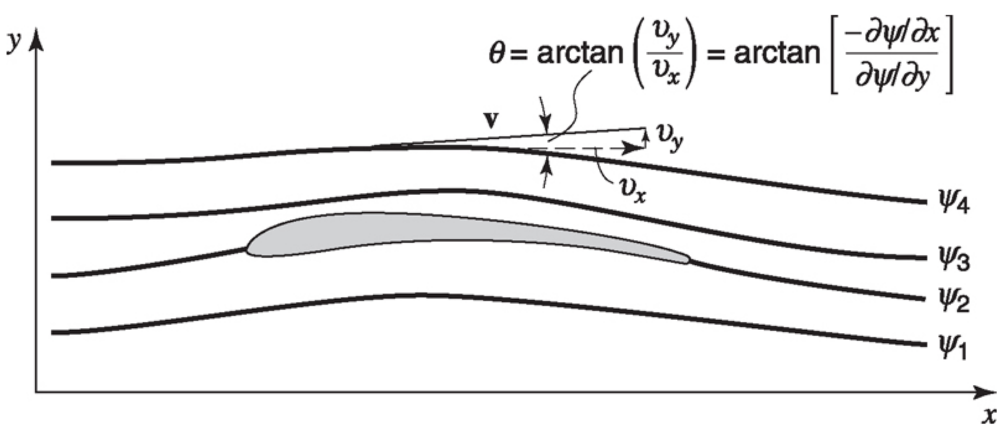
$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega} \qquad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z$$

irrotational flow; when the rotation at a point is zero

stream function; for two dimensional incompressible flow

$$\frac{\partial \Psi}{\partial x} = -v_y \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = v_x$$

$$d\Psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy = -v_y dx + v_x dy \qquad \left.\frac{dy}{dx}\right|_{\Psi=\text{constant}} = \frac{v_y}{v_x}$$

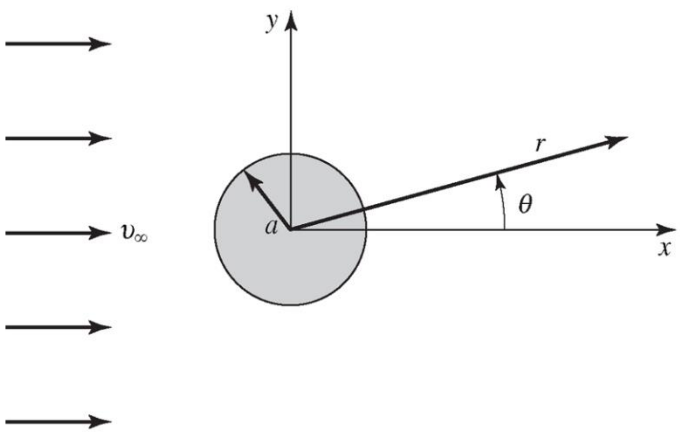


$$-2\omega_z = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

for irrotational flow

irrotational flow around a cylinder



$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = 0$$

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \qquad v_\theta = -\frac{\partial\Psi}{\partial r}$$

4 boundary conditions;

$$\Psi(r,\theta) = v_\infty r\sin\theta\left[1-\frac{a^2}{r^2}\right]$$

$$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = v_\infty\cos\theta\left[1-\frac{a^2}{r^2}\right]$$

$$v_\theta = -\frac{\partial\Psi}{\partial r} = -v_\theta\sin\theta\left[1+\frac{a^2}{r^2}\right]$$

for irrotational flow $\mathbf{v} = \nabla \phi$ velocity potential

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy \qquad \left.\frac{dy}{dx}\right|_{\Psi=\text{constant}} = \frac{v_y}{v_x}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \qquad \left.\frac{dy}{dx}\right|_{d\phi=0} = -\frac{v_x}{v_y}$$

$$\left. dy/dx \right|_{\phi=\text{constant}} = -\left.\frac{1}{dy/dx}\right|_{\Psi=\text{constant}}$$

orthogonal