446.328 Mechanical System Analysis
기계시스템해석

- lecture 15-

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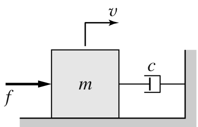
First Order System Examples

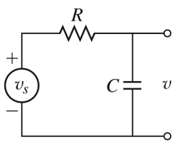


$$m\dot{v} + cv = f$$

$$\tau\dot{v} + v = f/c$$

$$\tau = m/c$$

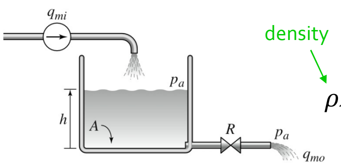
time constant



$$v_s = RC\frac{dv}{dt} + v$$

$$\tau\dot{v} + v = v_s$$

$$\tau = RC$$



density

$$\rho A\frac{dh}{dt} = \rho q_i - \frac{1}{R}\rho g h$$

volume rate

resistance

$$\tau\dot{h} + h = \frac{R}{g}q_i$$

$$\tau = AR/g$$

First Order Systems

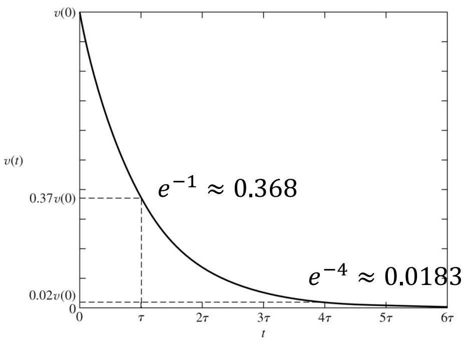
first order system (standard form)

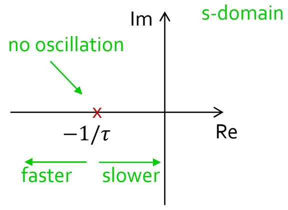
$$\tau\dot{y} + y = u \text{ with } y(0)$$

$$Y(s) = \frac{1}{\tau s + 1}[\tau y(0)] + \frac{1}{\tau s + 1}(s)$$

free response

$$Y(s) = \frac{1}{s + 1/\tau}y(0) \qquad y(t) = y(0)e^{-\left(\frac{1}{\tau}\right)t}$$





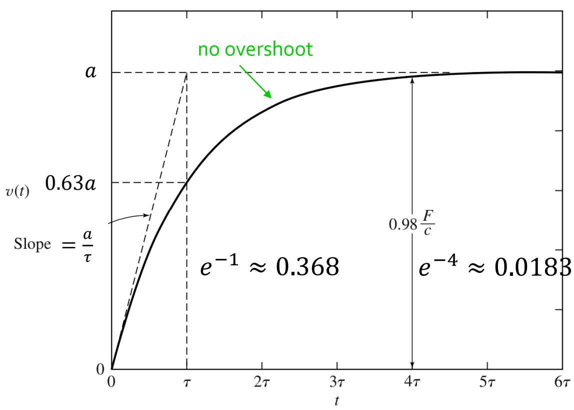
τ: time constant [sec]
smaller τ : faster response
larger τ : slower response

First Order System: Step Response

step response

$$Y(s) = \frac{1}{\tau s + 1}[\tau y(0)] + \frac{1}{\tau s + 1}U(s) \qquad u(t) = a1(t), y(0) = 0$$

$$\boxed{y(t) = a[1(t) - e^{-\left(\frac{1}{\tau}\right)t}]}$$



FVT

transient & steady-state

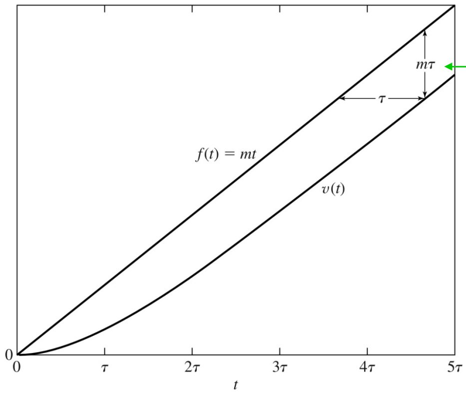
$$y(t) = [y(0) - a]e^{-\frac{1}{\tau}t} + a1(t)$$

First Order System: Ramp Response

step response

$$Y(s) = \frac{1}{\tau s + 1}[\tau y(0)] + \frac{1}{\tau s + 1}F(s) \qquad f(t) = at, y(0) = 0$$

$$\boxed{y(t) = at - a\tau + a\tau e^{-\frac{1}{\tau}t} \;\text{->}\; y(t) - f(t) = -a\tau(1 - e^{-\frac{1}{\tau}t})}$$

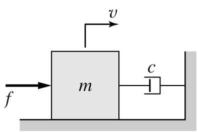


smaller τ => faster response
w/ less tracking error

FVT

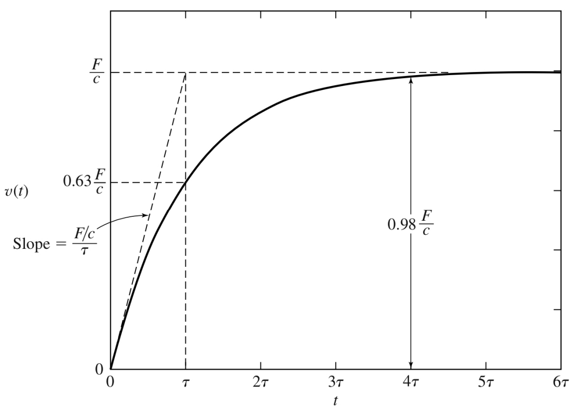
$$\lim_{t\to\infty}[y(t) - f(t)] = -a\tau$$

First Order System: Parameter Estimation



$$m\dot{v} + cv = f \qquad \tau = m/c,\; u = f/c,\; f(t) = F$$

$$\boxed{y(t) = F/c[1(t) - e^{-\left(\frac{1}{\tau}\right)t}]}$$



FVT

$$y(\infty) = F/c \longrightarrow c$$

IVT

$$\lim_{t\to 0_+} x(t) = \lim_{s\to\infty} sX(s)$$

$$\dot{y}(0+) = F/(\tau c)$$

$$\downarrow$$

m

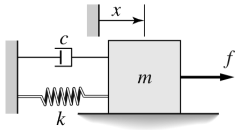
Second Order Systems

second order system (standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

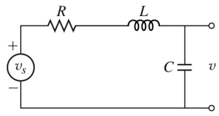
input

damping ratio natural freq. to make $x \to u$ in steady-state



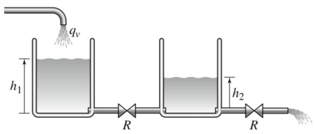
$$m\ddot{x} + c\dot{x} + kx = f \qquad u = f/k$$

$$w_n = \sqrt{k/m}, \ \ 2\zeta w_n = c/m \ \text{->} \ \zeta = \frac{c}{2\sqrt{mk}}$$



$$LC\ddot{v} + RC\dot{v} + v = v_s \qquad u = v_s$$

$$w_n = \sqrt{1/LC}, \ \zeta = \frac{RC}{2\sqrt{LC}}$$



$$AR\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_i$$

$$AR\frac{dh_2}{dt} + g(h_2 - h_1) = -gh_2$$

$$u = \frac{R}{g}q_i, \ \ w_n = \frac{g}{RA}, \ \ \zeta = 1.5$$

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Second Order System: Free Response

second order system (standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

damping ratio natural freq. input

$$\boxed{X(s) = \frac{1}{s^2 + 2\zeta w_n s + w_n^2}[sx(0) + \dot{x}(0) + 2\zeta w_n x(0)] + \frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2}U(s)}$$

free response forced response

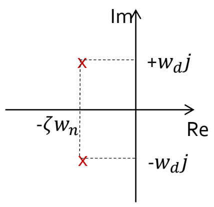
characteristic equation: $s^2 + 2\zeta w_n s + w_n^2 = 0 \Rightarrow s = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$

1. $\zeta = 0$ (un-damped): $s = \pm w_n j$

$w_d := w_n\sqrt{1-\zeta^2}$: damped-frequency

2. $0 < \zeta < 1$ (under-damped): $s = -\zeta w_n \pm w_n\sqrt{1-\zeta^2}j$

3. $\zeta = 1$ (critically-damped): $s = -w_n$ -> no oscillation



4. $\zeta > 0$ (over-damped): $s = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$

dominant root: $s = -w_n(\zeta - \sqrt{\zeta^2 - 1})$

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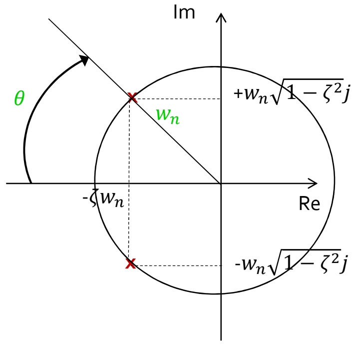
Second Order System: Pole Locations

second order system
(standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

damping ratio natural freq. input



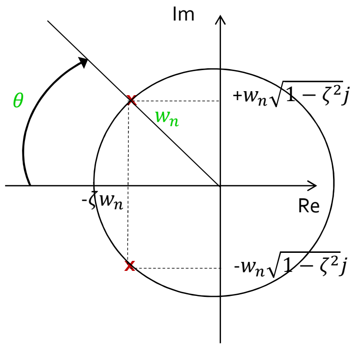
- ζw_n: how fast is the convergence ($\tau = \frac{1}{\zeta w_n}$)
- $w_d = w_n\sqrt{1-\zeta^2}$: damped oscillation frequency
- $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$: depends only on ζ
- radius $= w_n$

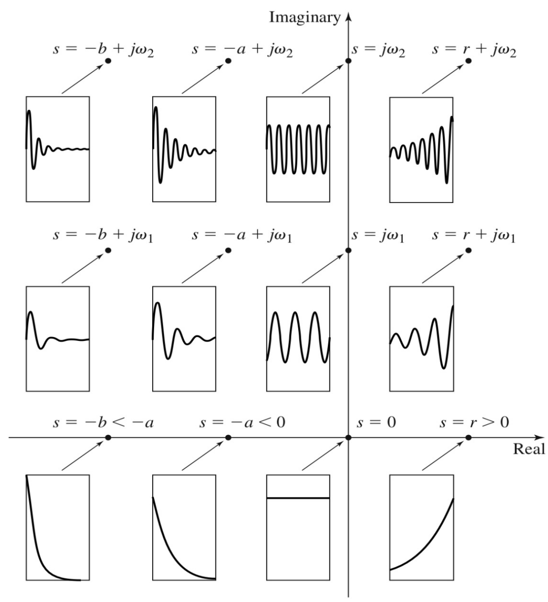
Second Order System: Pole Locations

second order system
(standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

damping ratio natural freq. input





Second Order System: Step Response

second order system (standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

damping ratio natural freq. input

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)} \Rightarrow \boxed{X(s) = \frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2} U(s)}$$

$U(s) = 1/s, x(0) = \dot{x}(0) = 0$

1. $\zeta = 0$ (un-damped): $x(t) = 1 - \cos w_n t$

2. $0 < \zeta < 1$ (under-damped): $x(t) = 1 - e^{-\zeta w_n t}\left[\cos w_d t + \frac{\zeta w_n}{w_d} \sin w_d t\right]$

3. $\zeta = 1$ (critically-damped): $x(t) = 1 - e^{-w_n t} - w_n t e^{-w_n t}$

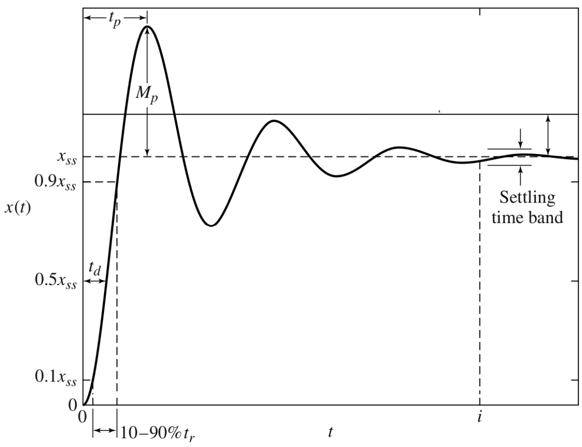
4. $\zeta > 0$ (over-damped): $x(t) = 2 + \alpha e^{-w_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)t} + \beta e^{-w_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)t}$

Step Response of Under-Damped Systems

second order system (standard form)

$$\boxed{\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)}$$

2. $0 < \zeta < 1$ (under-damped): $x(t) = 1 - e^{-\zeta w_n t}\left[\cos w_d t + \frac{\zeta w_n}{w_d} \sin w_d t\right]$



M_p: maximum overshoot

t_p: peak time

t_r: rise time

t_s: settling time

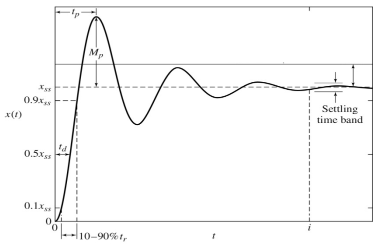
t_d: 50% delay time

Step Response of Under-Damped Systems

$$x(t) = 1 - e^{-\zeta w_n t}\left[\cos w_d t + \frac{\zeta w_n}{w_d}\sin w_d t\right]$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t}\sin(w_d t + \alpha) \qquad 0 \le \alpha = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \le \frac{\pi}{2}$$

$$= 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t}\sin(w_d t + \phi) \qquad \pi \le \phi = \alpha + \pi \le \frac{3\pi}{2}$$



M_p: maximum overshoot

t_p: peak time

t_r: rise time

t_s: settling time

t_d: 50% delay time

Maximum Overshoot & Peak Time

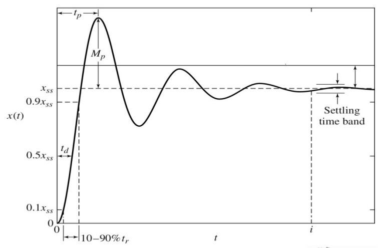
$$x(t) = 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t}\sin(w_d t + \phi)$$

$$\frac{dx}{dt} = 0 \;\rightarrow\; \tan(w_d t_p + \phi) = \tan\alpha \;\rightarrow\; w_d t_p = k\pi, k = 0,1,..$$

$$t_p = \frac{\pi}{w_d} = \frac{\pi}{w_n\sqrt{1-\zeta^2}} = \frac{\pi m}{\sqrt{4mk - c^2}}$$

$$M_p = x(t_p) - 1 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{-\frac{\pi c}{\sqrt{4mk-c^2}}} \le 1$$

1. c++ => t_p ++, M_p --
2. M_p depends only on ζ:
   can be used to estimate ζ
3. k++ => t_p --, M_p ++
4. m++ => M_p ++, t_p ++



Rise, Settling, and Delay Time

$$x(t) = 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \sin(w_d t + \phi)$$

100% rise time

$$x(t_r) = 1 \rightarrow \sin(w_d t + \phi) = 0 \qquad \pi \le \phi = \alpha + \pi \le 3\pi/2$$

$$w_d t + \phi = 2\pi \rightarrow w_d t = \pi - \alpha \rightarrow t_r = \frac{\pi - \alpha}{w_d}$$

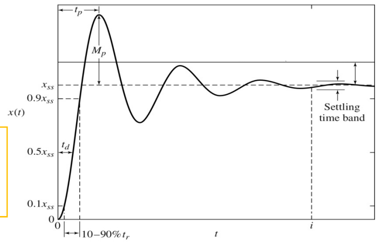
w_n ++ => t_r--

$$\alpha = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

2% settling time

$$t_s = \frac{4}{\zeta w_n} = \frac{2m}{c}$$

c ++ or m-- => t_s--



50% delay time

$$x(t_d) \ge 0.5$$

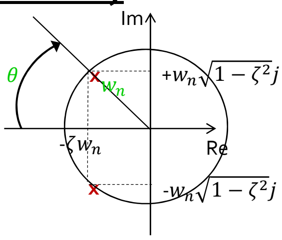
$$t_d \approx \frac{1 + 0.7\zeta}{w_n} = \frac{1}{\sqrt{k}} + \frac{0.7c}{2k}$$

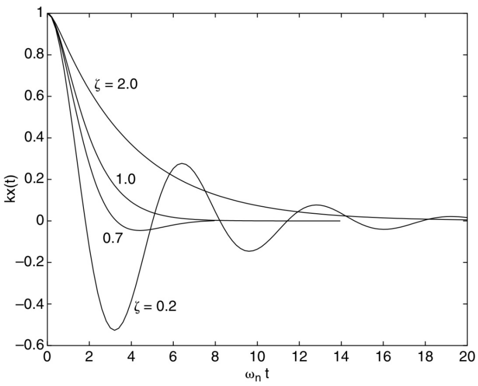
c -- or k++ => t_d++

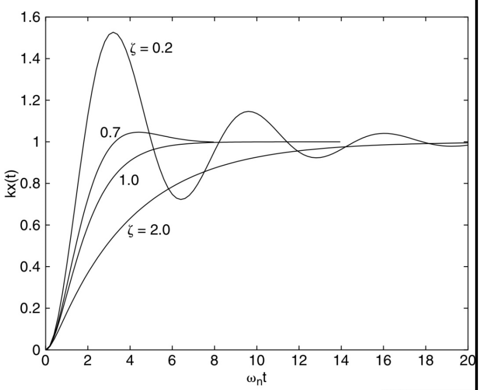
Second Order System: Effect of ζ

second order system (standard form)

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = w_n^2 u(t)$$

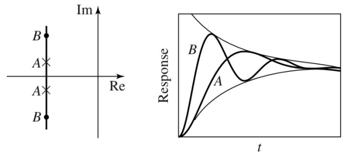






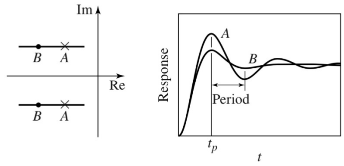
Second Order System: Effect of Pole Locations

Models A and B have the same real part, the same time constant, and the same decay time.



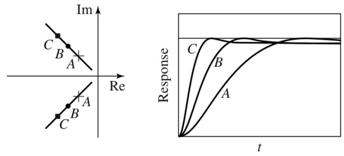
(a)

Models A and B have the same imaginary part, the same period, and the same peak time.

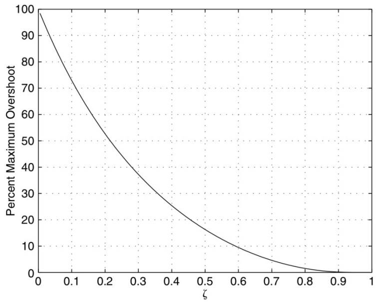


(b)

Models A, B, and C have the same damping ratio and the same overshoot.



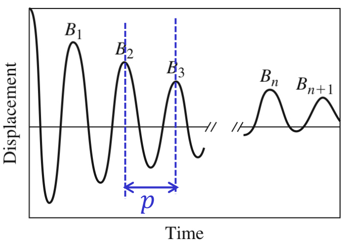
(c)



(a)

Second Order System: Parameter Estimation

$$\boxed{m\ddot{x} + c\dot{x} + kx = f} \qquad w_n = \sqrt{k/m}\,, \zeta = \frac{c}{2\sqrt{mk}}\,, w_d = w_n\sqrt{1-\zeta^2}$$



$$\boxed{x(t) = 1 + Ae^{-\zeta w_n t}\sin(w_d t + \phi)}$$

$$B_i = Ae^{-\zeta w_n t_i}$$

$$B_{i+1} = Ae^{-\zeta w_n (t_i + p)}$$

$$\delta = e^{\zeta w_n p} = \frac{B_i}{B_{i+1}} \to \delta^n = \frac{B_1 B_2 \dots B_n}{B_2 B_3 \dots B_{n+1}}$$

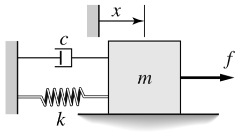
$$\boxed{\delta = \frac{1}{n}\ln\frac{B_1}{B_{n+1}}, \qquad p = \frac{2\pi}{w_d}}$$

MatLab Example

first-order system

$$m\dot{v} + cv = f, \qquad \tau = \frac{m}{c}$$

$$v \to \frac{a}{c}, \qquad \dot{v}(0+) = \frac{a}{\tau}$$



second-order system

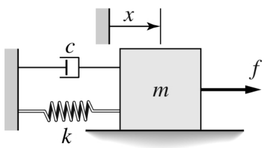
$$m\ddot{x} + c\dot{x} + kx = f$$

$$w_n = \sqrt{k/m}, \zeta = \frac{c}{2\sqrt{mk}}, \quad w_d = w_n\sqrt{1-\zeta^2}$$

$$\boxed{t_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}}, \quad M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}} \qquad \boxed{t_s = \frac{4}{\zeta w_n} = \frac{2m}{c}}$$

with zero

Second Order System: Example



$$m\ddot{x} + c\dot{x} + kx = f \qquad u = f/k$$

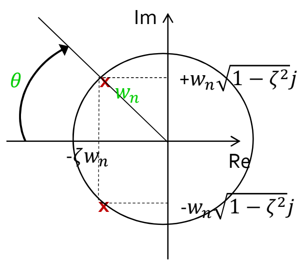
$$w_n = \sqrt{k/m}, \ 2\zeta w_n = c/m \ \text{->} \ \zeta = \frac{c}{2\sqrt{mk}}$$

$$w_d = w_n\sqrt{1-\zeta^2} = \sqrt{4mk - c^2}/m$$

1. changing c

2. changing m

3. changing k



Next Lecture